Generalized N-metric Spaces

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Abstract

We have introduced the notion of generalized N-metric spaces, taking inspiration from the idea of path integral in physics. Many cases for different values of N are explicitly verified, mostly giving strong hints to actually be generalized N-metric spaces. Some open questions are proposed.

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1. Introductions and definitions

We introduce the metric space with N-metric, inspired by the path integral in physics, defined as follows.

Definition 1.1. *Let X be a nonempty set, and let $d(\cdot,\cdot;N): X \times X \to [0,+\infty)$ be a mapping such that for all $x_0 = x, x_N = y \in X$ and all points x_i from X, $x_i \neq x_j$ for $i \neq j$, $i, j = 0 \dots N$:*

- $\mathbf{N}_1$ $d(x_i, x_j; N) = 0$ iff $i = j$
- $\mathbf{N}_2$ $d(x_i, x_j; N) = d(x_j, x_i; N)$
- $\mathbf{N}_3$ $d(x, y; N) \leqslant \sum_{i=0}^{N-1} d(x_i, x_{i+1}; N)$,

($\mathbf{N}_3$ being the generalization of triangular inequality for a standard metric space). Such $(X, d(\cdot,\cdot;N))$ space is called a generalized N-metric space (gNms for short).

Using this notation, a metric with triangular inequality could be written as $d(x, y; 2)$. The underlying idea is that $d(x, y; N)$ implies $d(x, y; N+1)$, as every triangular metric is a rectangular one, but not vice versa.

The definition (1.1) draws inspiration from the expression of quantum mechanical amplitude for a particle to go from the initial point x to the final point y:

$$\int \mathrm{D}q\, e^{(i/\hbar)S(q)}\,, \qquad (1)$$

where q is the position of the particle, $\int \mathrm{D}q$ the sum of all possible paths between x and y, $S(q) = \int_0^T dt\, L(q, \dot{q})$ the classical action, $\hbar$ the Planck's constant [1][2][3][4]; for a review of the subject, see for instance[5]. The result for quantum mechanics, loosely speaking,
is that the classical path between the two points has the largest weights, and quantum effects give fluctuations around it.

Our approach is to modify the measure weight in (1) by taking the simple case of $S(q) = 0$ and proceeding with (1.1). This generalization extends and improves the idea of [6]
for a rectangular metric.

It has to be stressed out that this method is based purely on an analogy, and there is no other connection of any kind between the path integral and the particular metric space we have introduced.

Consider first the case of a 4-metric (a pentagonal metric), that is

$$d(x, y; 4) \leqslant d(x, u; 4) + d(u, v; 4) + d(v, z; 4) + d(z, y; 4) \qquad (2)$$

and explicitly define the metric in the following way[7],[8].

$B = \left\{\frac{1}{n}\right\}_{n \geqslant 1}$, $n \in \mathbb{N}$, $X = A \cup B$ and $d(\cdot\,,\,\cdot\,; N) : X \times X \to [0, +\infty)$, $N \in \mathbb{N}$ and $N \geqslant 2$ as follows:

$$d(x, y; N) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } \{x, y\} \subset A \text{ or } \{x, y\} \subset B \\ y, & \text{if } x \in A, y \in B \\ x, & \text{if } x \in B, y \in A\,. \end{cases} \qquad (3)$$

This particular metric has been first adopted to prove that the analogies between triangular $d(x, y; 2)$ and rectangular metric $d(x, y; 3)$ were wrong. In fact, the case $N = 3$ is not a standard metric space, as there exists at least a convergent sequence that is not Cauchy, and there exists a sequence converging to two different points. Moreover, the distance d is not a continuous function. The rectangular metric is a generalized metric space [9]. We shall verify which of the cases $N > 3$ are generalized N-metric spaces. In order to do so, we shall verify explicitely the properties N_1, N_2, N_3 of gNms for various cases.

2. Main results

It is readily apparent that the most complex property of (1.1) to verify is N_3.

For the metric defined in (1.1) and (3), for the sets $A = \{0, 2\}$, $B = \left\{\frac{1}{n}\right\}_{n \geqslant 1}$, $n \in \mathrm{N}$, we have obtained the following results for $N = 4, \ldots, 9$.

- $d(x, y; 4)$, pentagonal metric: it fails to obey to property N_3 of (1.1), for instance in the case $\{x, y, u, v, z\} = \left\{\frac{1}{14}, \frac{1}{13}, 0, \frac{1}{12}, 2\right\}$, where it gives the relation $1 \leqslant \frac{86}{273}$. It is not a generalized N-metric space.
- $d(x, y; 5)$, hexagonal metric: more than 5 million permutations of points (x_i, x_j) have been checked, without contradiction. It is possibly a generalized N-metric space.
- $d(x, y; 6)$, heptagonal metric: more than 57 million permutations have been checked, without contradiction. It is possibly a generalized N-metric space.
- $d(x, y; 7)$, octagonal metric: more than 121 million permutations have been checked, without contradiction. It is possibly a generalized N-metric space.
- $d(x, y; 8)$, nonagonal metric: more than 847 million permutations have been checked, without contradiction. It is possibly a generalized N-metric space.
- $d(x, y; 9)$, decagon metric: more than 3.6 billion permutations have been checked, without contradiction. It is possibly a generalized N-metric space.

Other permutations have been also checked for different sets A, B giving same results as in the previous case.

Let $A = \{0, 1, 2, 3\}$, $B = \left\{\frac{1}{n}\right\}_{n \geqslant 4}$, $n \in \mathrm{N}$, then it is found, for instance, that applying eq. (2)

$$d\left(\frac{1}{6}, \frac{1}{7}; 4\right) = 1 \leqslant d\left(\frac{1}{6}, 1; 4\right) + d\left(1, \frac{1}{8}; 4\right) + d\left(\frac{1}{8}, 0; 4\right) + d\left(0, \frac{1}{7}; 4\right) = \frac{47}{84}$$

which is a contradiction.

The same happens when $A = \{0, 1, 2, 3, 4\}$, $B = \left\{\frac{1}{n}\right\}_{n \geqslant 5}$, $n \in \mathrm{N}$, that is when A has 5 elements, giving for instance $1 \leqslant \frac{47}{84}$ for $\{x, y, u, v, z\} = \left\{\frac{1}{6}, \frac{1}{7}, 1, \frac{1}{8}, 0\right\}$, or $A = \{0, 1, 2, 3, 4, 5\}$, $B = \left\{\frac{1}{n}\right\}_{n \geqslant 6}$, $n \in \mathrm{N}$, that is when A has 6 elements, giving $1 \leqslant \frac{157}{360}$ for $\left\{\frac{1}{9}, \frac{1}{8}, 1, \frac{1}{10}, 0\right\}$. Therefore, the $d(x, y; 4)$ metric is not a generalized N-metric space.

On the other hand, $d(x, y; 5)$ appears to be a generalized N-metric space, for the metric (3) defined also on sets $A = \{0, 1, 2, 3, 4\}$, $B = \left\{\frac{1}{n}\right\}_{n \geqslant 5}$, $n \in \mathrm{N}$ as no contradiction have been met in more than 151200 inequality permutations, and checked in the case $A = \{0, 1, 2, 3, 4\}$, $B = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}\right\}$, and for sets $A = \{0, 2\}$, $B = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\right\}$ as well.

One must notice that all hints given for those N-metrics to be generalized N-metrics are not definitive proofs (apart from the case $d(x, y; 4)$, that furnishes counterexamples). However, due to the nature of the metric defined in (1.1), (3), there are no other possible techniques but to check every single case, be it by hand or via software.

3. Conclusions and outlook

A new concept, the generalized N-metric space, has been given. It has been explicitly verified that, at least for the specific case $N = 4$, it is not a generalized N-metric space. The values of $N = 5, \ldots, 9$ have been tested for many cases, and appear to be all generalized N-metric spaces. The problem itself does not seem to offer a different method of solution except for an explicit verification of all possible permutation of variables.

We are leaving the reader with some open questions on the subject.

- Does actually exist a counterexample for the cases $N = 5 \ldots 9$, or they are genuine generalized N-metric spaces?
- If so, why only the isolated case $N = 4$ is not a generalized N-metric space?
  And does this result have a deeper meaning?
- How generalized N-metric space are distributed when varying the value of N?
- When is the distance of a gNms continuous?
- Could the topology of a gNms be Hausdorff?

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