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# On the Critical Distance in Rotating Frames and Its Analogies to the Schwarzschild Radius

**George Lavranos**

E-mail: giorgoslavranos@gmail.com

**Abstract.** This paper explores the concept of a critical distance in rotating frames where objects appear to move at the speed of light. Using principles from special relativity and rotational dynamics, we derive a critical distance $d = \frac{c^2}{2a}$. We discuss the implications of this critical distance, drawing analogies to the Schwarzschild radius of black holes. This analysis highlights the interconnectedness of relativistic effects in different contexts and the role of the equivalence principle in bridging kinematic and gravitational phenomena.

## 1. Introduction

The effects of relativistic speeds in rotating frames have long been a subject of interest in both theoretical and experimental physics. While the relativistic time dilation and length contraction effects in linear motion are well-studied, rotational dynamics present unique challenges and insights. This paper aims to elucidate the concept of a critical distance in rotating frames where objects appear to move at the speed of light, analogous to the Schwarzschild radius in the context of black holes.

Rotating frames are common in various physical systems, from everyday occurrences like spinning wheels to complex astrophysical phenomena such as rotating neutron stars and black holes. Understanding the limits imposed by relativistic effects in these systems is crucial for both theoretical investigations and practical applications.

## 2. Theoretical Background

### *2.1. Special Relativity and Rotational Dynamics*

Special relativity, formulated by Albert Einstein in 1905, revolutionized our understanding of space and time. It describes how measurements of various quantities differ for observers in different inertial frames of reference. One of the cornerstones of special relativity is that the speed of light $c$ is constant in all inertial frames [1].

The equivalence principle, a key idea in general relativity, posits that the effects of acceleration are locally indistinguishable from those of a gravitational field [2]. This principle allows us to draw analogies between accelerating frames and gravitational fields.

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In the context of rotating frames, the equivalence principle suggests that the effects of rotation can be compared to those of a gravitational field. This analogy helps in understanding the limits imposed by relativistic effects on the observation of objects in rotating systems [3].

## 3. Derivation of the Critical Distance

Consider a rotating frame of reference. The tangential velocity $v$ at a distance $d$ from the axis of rotation is given by:

$$v = \omega d$$

where $\omega$ is the angular velocity.

The critical distance $d$ is defined as the maximum distance at which objects can be observed without appearing to exceed the speed of light $c$. This is given by the relationship:

$$v = \omega d \Rightarrow c = \omega d \Rightarrow d = \frac{c}{\omega}$$

To understand this further, we need to analyze the proper time interval $t'$ experienced by the rotating observer. For the rotating observer, the time it takes for light to travel the critical distance $d$ is:

$$t = \frac{d}{c} = \frac{1}{\omega}$$

Thus, the proper time interval $t'$ for the rotating observer is:

$$t' = \frac{1}{\omega}$$

This proper time interval is the time it takes for light to reach the observer at the critical distance, considering the relativistic effects of rotation.

At this critical distance, the rotational speed equals the speed of light, leading to significant relativistic time dilation and length contraction effects.

## 4. Relativistic Time Dilation

Using the time dilation factor $\gamma$:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Since $v^2 = ad$, we get:

$$\gamma = \frac{1}{\sqrt{1 - \frac{ad}{c^2}}}$$

Relating the proper time interval $t'$ and coordinate time $t$ as $t' = \gamma t$ and $d = ct$:

$$t' = \gamma t$$

Given $t = \frac{d}{c}$, we substitute to get:

$$\frac{1}{\omega} = \frac{1}{\sqrt{1 - \frac{ad}{c^2}}} \cdot \frac{d}{c}$$

Solving for $\omega$:

$$\frac{1}{\omega} = \frac{d}{c\sqrt{1 - \frac{ad}{c^2}}} \Rightarrow \omega = \frac{\sqrt{1 - \frac{ad}{c^2}} \cdot c}{d}$$

Squaring both sides:

$$\omega^2 d^2 = (1 - \frac{ad}{c^2})c^2 \Rightarrow \omega^2 d^2 = c^2 - ad \Rightarrow \omega^2 d = \frac{c^2}{d} - a \Rightarrow a = \frac{c^2}{d} - \omega^2 d \Rightarrow a = \frac{c^2}{2d}$$

This derivation shows the interplay between rotational dynamics and relativistic effects, leading to the conclusion that there is a maximum distance at which objects can be observed without violating the principles of relativity.

## 5. Analogy to the Schwarzschild Radius

The Schwarzschild radius $R_s$ defines the event horizon for a black hole:

$$R_s = \frac{2GM}{c^2}$$

Using the equivalence principle, we relate proper acceleration $a$ to gravitational acceleration $g$. The farthest distance $d$ we can observe is given by:

$$d = \frac{c^2}{2g}$$

The gravitational acceleration $g$ is given by:

$$g = \frac{GM}{d^2}$$

Substituting this into the equation for $d$:

$$d = \frac{c^2}{2} \cdot \frac{d^2}{GM} \Rightarrow d = \frac{d^2 c^2}{2GM} \Rightarrow d = \frac{2GM}{c^2}$$

This is the Schwarzschild radius, which defines the event horizon beyond which nothing, not even light, can escape from a black hole.

The analogy between the critical distance in rotating frames and the Schwarzschild radius in black holes highlights the fundamental limits imposed by relativistic physics. Both distances represent boundaries beyond which certain relativistic effects dominate, leading to significant implications for observation and information transmission.

## 6. Calculations and Observations

### *6.1. Schwarzschild Radius*

The Schwarzschild radius $R_s$ is given by:

$$R_s = \frac{2GM}{c^2}$$

*6.2. Angular Frequency*

The angular frequency $\omega$ is given by:

$$\omega = \frac{c}{\sqrt{2}R_s}$$

*6.3. Constants*

- $G \approx 6.67430 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$
- $c \approx 2.998 \times 10^8\,\mathrm{m/s}$
- Mass of the sun $M_\odot \approx 1.989 \times 10^{30}\,\mathrm{kg}$

*6.4. Stellar-Mass Black Hole (10 $M_\odot$)*

$$M_{\text{stellar}} = 10 \times M_\odot$$

*6.5. Supermassive Black Hole ( $10^6$ $M_\odot$)*

$$M_{\text{supermassive}} = 10^6 \times M_\odot$$

*6.6. Calculations*

- **Stellar-Mass Black Hole:**

$$R_s(\text{stellar}) = \frac{2 \times 6.67430 \times 10^{-11} \times 10 \times 1.989 \times 10^{30}}{(2.998 \times 10^8)^2}$$

$$R_s(\text{stellar}) \approx 2.953 \times 10^4\,\mathrm{m}$$

$$\omega(\text{stellar}) = \frac{2.998 \times 10^8}{\sqrt{2} \times 2.953 \times 10^4}$$

$$\omega(\text{stellar}) \approx 7.178 \times 10^3\,\mathrm{rad/s}$$

- **Supermassive Black Hole:**

$$R_s(\text{supermassive}) = \frac{2 \times 6.67430 \times 10^{-11} \times 10^6 \times 1.989 \times 10^{30}}{(2.998 \times 10^8)^2}$$

$$R_s(\text{supermassive}) \approx 2.953 \times 10^7\,\mathrm{m}$$

$$\omega(\text{supermassive}) = \frac{2.998 \times 10^8}{\sqrt{2} \times 2.953 \times 10^7}$$

$$\omega(\text{supermassive}) \approx 7.178\,\mathrm{rad/s}$$

### *6.7. Summary*

- **Stellar-Mass Black Hole:**
  - Schwarzschild Radius: $R_s(\text{stellar}) \approx 2.953 \times 10^4\,\text{m}$
  - Angular Frequency: $\omega(\text{stellar}) \approx 7.178 \times 10^3\,\text{rad/s}$
- **Supermassive Black Hole:**
  - Schwarzschild Radius: $R_s(\text{supermassive}) \approx 2.953 \times 10^7\,\text{m}$
  - Angular Frequency: $\omega(\text{supermassive}) \approx 7.178\,\text{rad/s}$

These calculations show that the angular frequency decreases significantly with increasing mass, corresponding to larger Schwarzschild radii for supermassive black holes.

## 7. Comparison with Observed Data

To understand whether the theoretical calculations correlate with measured quantities, we compare them with observed data from astrophysical sources.

### *7.1. Stellar-Mass Black Holes*

Observations of X-ray binaries and stellar-mass black holes in our galaxy provide data for rotational frequencies. For example, the black hole in the binary system Cygnus X-1 has a mass of about $14.8M_\odot$ and shows high-frequency quasi-periodic oscillations (QPOs) in the range of a few hundred Hz, corresponding to angular frequencies on the order of $10^3$ to $10^4\,\text{rad/s}$ [6].

### *7.2. Supermassive Black Holes*

Observations of supermassive black holes, such as those in active galactic nuclei (AGN) or the supermassive black hole at the center of the Milky Way (Sgr A*), show rotational periods of several hours to days. For example, Sgr A* with a mass of about $4 \times 10^6 M_\odot$ has inferred rotational periods corresponding to angular frequencies on the order of $10^{-5}\,\text{rad/s}$ [7].

### *7.3. Comparison*

- **Stellar-Mass Black Holes:**
  - **Theoretical Frequency:** $\omega(\text{stellar}) \approx 7.178 \times 10^3\,\text{rad/s}$
  - **Observed Frequencies:** High-frequency QPOs in the range of $10^3$ to $10^4\,\text{rad/s}$

  The theoretical angular frequency derived from the Schwarzschild radius for a stellar-mass black hole is consistent with the observed high-frequency QPOs.
- **Supermassive Black Holes:**

  - **Theoretical Frequency:** $\omega(\text{supermassive}) \approx 7.178\,\text{rad/s}$
  - **Observed Frequencies:** Rotational frequencies on the order of $10^{-5}\,\text{rad/s}$

  The theoretical angular frequency for a supermassive black hole is significantly higher than observed rotational frequencies. This discrepancy can be attributed to the fact that the actual rotational dynamics of supermassive black holes are influenced by more complex factors, including the distribution of mass, the presence of accretion disks, and relativistic frame-dragging effects not accounted for in the simplified model.

## 8. Discussion

The critical distance derived from special relativity and rotational dynamics highlights the interplay between kinematic limits and relativistic effects. Both the critical distance $d = \frac{c^2}{2a}$ and the Schwarzschild radius $R_s = \frac{2GM}{c^2}$ define boundaries beyond which certain relativistic effects dominate.

- **Observation Limits**: Both distances represent a boundary beyond which objects cannot be observed due to relativistic constraints.
- **Relativistic Speeds**: At the critical distance, the tangential velocity reaches the speed of light, analogous to the escape velocity reaching the speed of light at the Schwarzschild radius.
- **Role of the Equivalence Principle**: The analogy draws on the equivalence principle, which suggests that the effects of acceleration in a rotating frame can be compared to gravitational effects in a static frame.
- **Implications for Astrophysics**: Understanding these limits is crucial for studying extreme astrophysical objects like neutron stars and black holes, where relativistic effects are significant.

## 9. Conclusion

We have derived a critical distance in a rotating frame using principles from special relativity and rotational dynamics. This critical distance, beyond which objects would appear to move faster than light, is analogous to the Schwarzschild radius in the context of general relativity. Both concepts highlight the fundamental limits imposed by relativistic physics on observation and information transmission.

Understanding these limits is crucial for both theoretical investigations and practical applications, ranging from astrophysics to advanced technological systems. Future research could explore further implications of these relativistic limits and their applications in various fields of science and technology.

## References

[1] Einstein A 1905 Zur Elektrodynamik bewegter Körper *Annalen der Physik* **322** 891–921

[2] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco, CA: W. H. Freeman)
[3] Wald R M 1984 *General Relativity* (Chicago, IL: University of Chicago Press)
[4] Schwarzschild K 1916 Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften* 189-196
[5] Kerr R P 1963 Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics *Physical Review Letters* **11** 237-238
[6] Nowak M A, Wilms J, Vaughan B A, Dove J B, Begelman M C 1999 Rossi X-ray Timing Explorer Observations of Cygnus X-1. II. Timing Analysis *Astrophysical Journal* **510** 874
[7] Ghez A M, Salim S, Weinberg N N, Lu J R, Do T, Dunn J K, Matthews K, Morris M R, Yelda S, Becklin E E, Kremenek T, Milosavljević M, and Naiman J 2008 Measuring Distance and Properties of the Milky Way's Central Supermassive Black Hole with Stellar Orbits *Astrophysical Journal* **689** 1044-1062