Research Article

On a Certain Inequality for the Sum of Norms and Reverse Uncertainty Relations

Krzysztof Urbanowski[1]

1. Institute of Physics, University of Zielona Góra, Zielona Góra, Poland

We prove a simple inequality for a sum of squares of norms of two vectors in an inner product space. Next, using this inequality we derive the so–called "reverse uncertainty relation" and analyze its properties.

Corresponding author: K. Urbanowski, K.Urbanowski@if.uz.zgora.pl

1. Introduction

In linear spaces with the inner product, one can prove many inequalities satisfied by vectors belonging to these spaces. They have a number of important applications not only in mathematics, but also in mathematical physics, and in particular in quantum mechanics. An example here is the Schwartz inequality used in the derivation of Heisenberg's quantum uncertainty principle. Other inequalities are used to derive the so–called "sum uncertainty relations" (see, e.g.[1][2]). In some applications it is important to know the upper bound on the sum of norms. Such a bound can be found using, for example, the Dunkle–Wiliams inequality[3]. This upper bound also finds various applications. An example of this is its use in mathematical physics to derive the so–called "reverse uncertainy relation"[4][5][6]. Here we present a simple inequality, (much simpler than that following from the Dunkle–Wiliams inequality), which seems to be a new, that can be used in the derivation of the above mentioned reverse uncertainty relation, as well as in other cases.

2. A certain simple inequality

There is a simple inequality, which may be useful in some applications:

Theorem. Consider the linear vector space with the inner product. Vectors, $|\psi_1\rangle, |\psi_2\rangle$, belonging to such a space satisfy the inequality

$$\||\psi_1\rangle\|^2 + \||\psi_2\rangle\|^2 \leq \||\psi_1\rangle - |\psi_2\rangle\|^2 + 2\,|\langle\psi_1|\psi_2\rangle| \quad (1)$$
$$\leq \||\psi_1\rangle - |\psi_2\rangle\|^2 + 2\,\||\psi_1\rangle\|\,\||\psi_2\rangle\|. \quad (2)$$

Proofs.

1) Let's use the identity

$$\||\psi_1\rangle\|^2 + \||\psi_2\rangle\|^2 = \||\psi_1\rangle - |\psi_2\rangle\|^2 + 2\Re(\langle\psi_1|\psi_2\rangle), \quad (3)$$

(where $\Re(z)$ is the real part of a complex number z), and apply the Cauchy–Schwartz inequality to $\Re(\langle\psi_1|\psi_2\rangle)$. There is

$$\Re(\langle\psi_1|\psi_2\rangle) \leq |\langle\psi_1|\psi_2\rangle|, and\,|\langle\psi_1|\psi_2\rangle| \leq \||\psi_1\rangle\|\,\||\psi_2\rangle\|. \quad (4)$$

Replacing $\Re(\langle\psi_1|\psi_2\rangle)$ in (3) with the the results (4) we obtain the inequalities (1) and (2). □

2) Let's use the triangle inequality

$$\||\psi_1\rangle\| + \||\psi_2\rangle\| \geq \||\psi_1\rangle + |\psi_2\rangle\|, \quad (5)$$

and take the squares of its both sides. Then we get that

$$\||\psi_1\rangle\|^2 + \||\psi_2\rangle\|^2 + 2\,\||\psi_1\rangle\|\,\||\psi_2\rangle\| \geq \||\psi_1\rangle + |\psi_2\rangle\|^2. \quad (6)$$

In the next step we find $\||\psi_1\rangle + |\psi_2\rangle\|^2$ from the parallelogram law: $2\left(\||\psi_1\rangle\|^2 + \||\psi_2\rangle\|^2\right) = \||\psi_1\rangle + |\psi_2\rangle\|^2 + \||\psi_1\rangle - |\psi_2\rangle\|^2$, and replace the right hand–side of (6) with the $\||\psi_1\rangle + |\psi_2\rangle\|^2$ calculated in this way, which leads to the inequality (2). □

3. Applications: A reverse uncertainty relation

The inequality (2) is simple and may be useful in some applications: It seems that it should be of interest to, among others, physicists studying the so–called "reverse uncertainty relations", see e.g.[4][5][6]. In these papers an upper bound for a sum of norms was applied to define the reverse uncertainty relation for the sum of variances and properties of such a relation were analyzed. The upper bound mentioned and having more complicated form then that resulting from (2) was found in[4][5][6] using the Dunkl–Williams inequality[3]. Now let us try to derive the "reverse uncertainty relation" using inequality (2).

In a general case, the variance $(\Delta_\phi F)^2$ of an observable F, when the quantum system is in the state $|\phi\rangle$, is defined as follows

$$(\Delta_\phi F)^2 = \|\delta_\phi F|\phi\rangle\|^2, \quad (7)$$

where $\delta_\phi F = F - \langle F\rangle_\phi \mathbb{I}$, and $\langle F\rangle_\phi \stackrel{def}{=} \langle\phi|F|\phi\rangle$ is the expected value of an observable F in a system that is in the state $|\phi\rangle$, (where $|\phi\rangle \in \mathcal{H}$ is the normalized vector and $\mathcal{H}$ is the Hilbert space of states of the quantum system under considerations), provided that $|\langle\phi|F|\phi\rangle| < \infty$. Equivalently: $(\Delta_\phi F)^2 \equiv \langle F^2\rangle_\phi - \langle F\rangle_\phi^2$. The observable F is represented by hermitian operator F acting in $\mathcal{H}$. Here $\Delta_\phi F \geq 0$ is the standard deviation. Let us consider two observables, A and B, represented by non–commuting hermitian operators A and B acting in $\mathcal{H}$, such that $[A, B]$ exists and $|\phi\rangle \in \mathcal{D}(AB)\bigcap\mathcal{D}(BA)$, ($\mathcal{D}(\mathcal{O})$ denotes the domain of an operator $\mathcal{O}$ or of a product of operators). Let

$$|\psi_1\rangle = \delta_\phi A|\phi\rangle, \text{ and } |\psi_2\rangle = \delta_\phi B|\phi\rangle. \tag{8}$$

If to insert (8) into (1) then we obtain

$$\|\delta_\phi A|\phi\rangle\|^2 + \|\delta_\phi B|\phi\rangle\|^2 \leq \|\delta_\phi A|\phi\rangle - \delta_\phi B|\phi\rangle\|^2 + 2\,|\langle\phi|\delta_\phi A\delta_\phi B|\phi\rangle|. \tag{9}$$

There is $(\delta_\phi A|\phi\rangle - \delta_\phi B|\phi\rangle) \equiv \delta_\phi(A - B)|\phi\rangle$. This and the definition (7) means that the inequality (9) takes the following form,

$$(\Delta_\phi A)^2 + (\Delta_\phi B)^2 \leq [\Delta_\phi(A - B)]^2 + 2\,|\mathcal{C}_\phi(A, B)|, \tag{10}$$

where

$$\mathcal{C}_\phi(A, B) = \langle\phi|\delta_\phi A\delta_\phi B|\phi\rangle \equiv \langle AB\rangle_\phi - \langle A\rangle_\phi\langle B\rangle_\phi, \tag{11}$$

is a quantum version of the covariance (or, of the correlation function) of the observables A and B in quantum state $|\phi\rangle$. Here defining the correlation function $\mathcal{C}_\phi(A, B)$ we follow, e. g.[7][8] and others. The inequality (9) is a simple variant of the "reverse uncertainty relation".

Another simple variant of the "reverse uncertainty relation" can be obtained using (2). Namely, applying the method used to derive the inequality (9) to (2) and keeping in mind all steps leading to (9) we obtain that,

$$(\Delta_\phi A)^2 + (\Delta_\phi B)^2 \leq [\Delta_\phi(A - B)]^2 + 2\Delta_\phi A \cdot \Delta_\phi B, \tag{12}$$

which is another, less restrictive, variant of the "reverse uncertainty relation".

4. Final remarks

The Dunkl–Williams inequality for vectors $|\psi_1\rangle, |\psi_2\rangle$ from a real or complex inner product space has the following form[3][9],

$$\||\psi_1\rangle - |\psi_2\rangle\| \geq \frac{1}{2}(\||\psi_1\rangle\| + \||\psi_2\rangle\|)\left\|\frac{|\psi_1\rangle}{\||\psi_1\rangle\|} - \frac{|\psi_2\rangle}{\||\psi_2\rangle\|}\right\|, \tag{13}$$

doi.org/10.32388/3Z5SY1

where the condition that $|\psi_1\rangle, |\psi_2\rangle$ are nonzero vectors must be satisfied[3][9]. This inequality was used in[4] to find the "reverse uncertainty relation". Indeed, replacing $|\psi_1\rangle, |\psi_2\rangle$ in (13) by (8) and using the definition (7) after some algebra authors of[4] obtain that

$$(\Delta_\phi A)^2 + (\Delta_\phi B)^2 \leq \; 2\frac{[\Delta_\phi(A-B)]^2}{\left[1 - \frac{cov_\phi(A,B)}{\Delta_\phi A \cdot \Delta_\phi B}\right]} - 2\Delta_\phi A \cdot \Delta_\phi B, \tag{14}$$

where $cov_\phi(A,B) = \Re[\mathcal{C}_\phi(A,B)]$. The inequality (14) is the *reverse uncertainty relation* derived in[4].

As can be seen from the inequality (14) this reverse uncertainty relation has rather complicated form and is undefined if $|\phi\rangle$ is an eigenvector of A (or of B). This is because then $|\psi_1\rangle = \delta_\phi A|\phi\rangle = 0$ (or $|\psi_2\rangle = \delta_\phi B|\phi\rangle = 0$) and the inequality (13) does not hold. These kinds of weaknesses are absent in inequalities (9) and (12), i.e., in our simpler reverse uncertainty relations. Although inequalities (9) and (12) do not provide any useful information about the upper bound for the sum of two variances, $(\Delta_\phi A)^2 + (\Delta_\phi B)^2$, if $|\phi\rangle$ is an eigenvector of A (or B), but even in such a case the left and right sides of these inequalities are finite and well–defined, which cannot be said about inequality (14). Moreover, inequalities (9) and (12) seem to be simpler in applications than inequality (14).

Reverse uncertainty relations, (9), (12) and (14) have another non–obvious property that is worth mentioning. Namely, if the system is in such a state $|\phi\rangle$ that $|\psi_1\rangle = \delta_\phi A|\phi\rangle \perp |\psi_2\rangle = \delta_\phi B|\phi\rangle$, and simultaneously $\Delta_\phi A > 0$ and $\Delta_\phi B > 0$, then it is not possible to obtain any useful information about the upper bound for the sum of variances from these relations. Indeed, in this case $\mathcal{C}_\phi(A,B) = 0$, so also $\Re[\mathcal{C}_\phi(A,B)] = cor_\phi(A,B) = 0$, and $[\Delta_\phi(A \pm B)]^2 \equiv (\Delta_\phi A)^2 + (\Delta_\phi B)^2$. The first observation is that in such a case observables A and B are uncorelated in this state. Further observations are that in the situation under consideration the inequality (9) takes the form $(\Delta_\phi A)^2 + (\Delta_\phi B)^2 \leq (\Delta_\phi A)^2 + (\Delta_\phi B)^2$, and the inequality (12) looks as follows, $0 \leq \Delta_\phi A \cdot \Delta_\phi B$, and finally, inequality (14) takes the form: $0 \leq (\Delta_\phi A - \Delta_\phi B)^2$. Neither of these results says anything about the upper bound for the sum $(\Delta_\phi A)^2 + (\Delta_\phi B)^2$. So if observables A and B are uncorrelated in state $|\phi\rangle$ then using only inequalities (9), (12) and (14) nothing can be said about the upper bound on the sum of their variances.

To sum up: inequalities (9), (12) and also (14) (i.e. all reverse uncertainty relations presented here) are worth further investigation, both theoretical and experimental. The experiment should decide which of them better describe the real properties of quantum systems.

doi.org/10.32388/3Z5SY1

References

1. ^Pati AK, Sahu PK. "Sum uncertainty relation in quantum theory". Physics Letters A. 367: 177 (2007). doi:10.1016/j.physleta.2007.03.005.
2. ^Maccone L, Pati AK. "Stronger Uncertainty Relations for All Incompatible Observables". Phys. Rev. Lett.. 113: 260401 (2014). doi:10.1103/PhysRevLett.113.260401.
3. a, b, c, dC. F. Dunkl and K. S. Williams, A Simple Norm Inequality, The American Mathematical Monthly, 71, No. 1, 53 — 54, 1964,
4. a, b, c, d, e, fMondal D, Bagchi S, Pati AK. "Tighter uncertainty and reverse uncertainty relations". Physical Review A. 95: 052117 (2017). doi:10.1103/PhysRevA.95.052117.
5. a, b, cXiao L, Fan B, Wang K, Pati AK, Xue P. "Direct experimental test of forward and reverse uncertainty relations". Physical Review Research. 2: 023106 (2020). doi:10.1103/PhysRevResearch.2.023106.
6. a, b, cZheng X, Ji AL, Zhang GF. "Stronger reverse uncertainty relation for multiple incompatible observables". Physica Scripta. 98: 065113 (2023). doi:10.1088/1402-4896/acd484.
7. ^Pozsgay V, Hirsch F, Branciard C, Brunner N. "Covariance Bell inequalities". Physical Review A. 96: 062128 (2017). doi:10.1103/PhysRevA.96.062128.
8. ^Khrennikov A, Basieva I. "Entanglement of Observables: Quantum Conditional Probability Approach". Foundations of Physics. (2023) 53:84. doi:10.1007/s10701-023-00725-7.
9. a, bP. Cerone and S. S. Dragomir, Mathematical Inequalities (Chapman and Hall/CRC, London, 2011).

Declarations

Funding: No specific funding was received for this work.

Potential competing interests: No potential competing interests to declare.