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Abstract:
Many sophisticated real world science and engineering problems after formulation simply reduce to a problem of finding a solution of partial differential equations (PDEs) with relevant boundary conditions over a domain. Numerical methods like FEM, FDM and BEM are most used and popular methods to solve these real-world PDEs. However, in last few decades considerable amount of research has been dedicated to develop meshless methods which don't involve tedious and time consuming process of generating high quality mesh for the domain. Many of these meshless methods have difficulty in handling point boundary conditions which are quite frequent in engineering applications. Hence, in this paper, a Taylor series based domain collocation PDE solution methodology is proposed. The proposed methodology is well suited to handle multiple boundary conditions including point boundary conditions. The main idea of the method is to formulate a function which satisfies all the boundary conditions and then generalize the function to a family of functions by using Taylor series. Since the family of functions already satisfies the boundary conditions, the PDE solution can be determined by finding the values of unknown Taylor coefficients for which the residual of the PDE over the domain is closest to zero. Using domain collocation method, the linear PDE problem transforms into a linear regression problem. The proposed method is extended by using multi-point Taylor series to solve problems with point boundary conditions. The proposed method has been successfully applied to solve homogenous/non-homogenous Helmholtz and Poisson's PDEs in the paper. The proposed methodology has also been shown to solve complex PDEs efficiently with less number of degrees of freedom (DOFs) as compared to Taylor meshless method (TMM). The proposed method is illustrated for both problems with Dirichlet and Neumann boundary conditions. Moreover, the method has been also presented to solve a problem where the boundary is defined using a set of points instead of an analytical function.

Keywords: Meshless method, Taylor Series, Multiple Boundary Conditions, Point Boundary Conditions, Domain Collocation, linear regression, Multi-point Taylor Series

1. Introduction:

Physical laws which govern the universe are generally in the form of partial differential equations (PDEs) and they dictate the change of physical quantities with time and space. Earlier, partial differential equations and their solution were mostly studied in mathematically-oriented fields like physics and engineering. Nowadays; however with significant technological advances in the computational hardware and capability, the solution of partial differential equations is considered as a fundamental research tool in several multi-disciplinary areas. Some of these areas are biology, quantum mechanics, cell biology, physiology, chemistry, geological exploration, chemical physics, etc... The process of converting a real world problem into mathematical PDE problem consists of the following steps. First step is to understand the physics of the problem, which lead to the formation of mathematical equations in the form of PDE. Next step is to figure out the boundary and initial condition of the PDE which detail the known behaviour of the real world system at certain spatial co-ordinates and/or at certain time. Most of these PDEs applied to real world problem cannot be solved analytical and hence numerical methods are used in order to find solutions to the PDE [1]. In summary, a governing PDE is solved over a domain with given boundary conditions and/or initial conditions to study and understand the behaviour of a real world system.

Over the years, efficient numerical methods to solve PDEs have been a significant research topic. Among numerous PDE solution methods finite element methods (FEM), boundary element methods (BEM) and finite difference methods (FDM) are the most popular. Solutions from any numerical methods will always have some error either in satisfying boundary conditions or governing PDE or both. For example, FEM solution satisfies the boundary condition exactly but some residual exists in satisfying the governing equation. FEM solves the PDE by using the principle of minimizing the weighted residual of the PDE at discrete points over the domain [2-3]. FEM uses equivalent weak form of the PDE and hence reduces its continuity requirement of the solution. BEM [4] transforms the PDE into boundary integral equation by employing the fundamental solutions and weighted residual approach. The boundary integral equation deals only with the boundary and hence are solved on discrete points on the boundary.FDM replaces derivatives in the PDE by their difference quotients, which then can be solved at discrete points in the domain [5]. All of these three numerical methods (FDM, FEM and BEM) convert the PDE into a system of algebraic equation over discrete points on the domain or the boundary and then solve these algebraic equations with the given boundary condition. These methods are widely used for solving many real world problems, especially finite element methods. However, despite of widespread applications, each method has its own shortcomings and limitations. Some of drawbacks of FEM are creation of a mesh for a complicated domain could be time-consuming. Moreover, by principle FEM solution is only C^0 continuous and hence higher order solution differentials are discontinuous and require some sort of averaging as seen in the case in stress calculations.

Most of the drawbacks associated with FEM, BEM and FDM mainly arise from the requirement of a mesh in which predefined connections between neighbour points are required. Therefore concept of numerical methods without mesh has been researched upon, in which the domain of the problem is represented by a set of arbitrarily distributed nodes. The meshless framework alleviates some of the problems with mesh, such as mesh distortion, crack propagation, high velocity impact or explosive mechanics. Gingold and Monaghan [6] proposed smooth particle hydrodynamics which is considered as one of earliest meshless method. After that many methods based on Galerking technique were proposed. Some of those methods include diffuse element method (DEM) proposed by Nayroles et al. [7] and Element Free Galerkin Method (EFGM) proposed by Belytschko [8]. Majority of the meshless PDE solution methods can be classified into two categories namely (i) domain type meshless methods and (ii) boundary type meshless methods.

There are two main steps in domain type meshless methods: a) approximation of unknown function by using interpolation function, b) discretization of governing PDE. The discretized equations obtained by using selected interpolation functions are solved at arbitrary and irregular scattered points in the domain with the given boundary conditions to get the solution of the PDE. Some of interpolation functions used in literature are kernel particle approximation [5], reproducing kernel particle [9], moving least square [10], partition of unity [11], radial basis function [12], etc.. Regarding the method to discretize the governing PDE, Galerkin-based methods [13, 14] use background integration in the whole domain whereas the local Petrov-Galerkin method [15, 16] takes account of the integration in a rather small local sub-domain and no background mesh is required. Another method to discretize the governing PDE is the collocation technique [17, 18]. No background mesh and no integration are required, that makes it very efficient but with large scale problems the ill-conditioned matrices may lead to numerical instability and low accuracy.

Boundary-type meshless methods are based on families of interpolation functions that are exact solutions of governing PDE. Since the PDEs are explicitly satisfied, only the discretization of boundary is needed to satisfy the given boundary conditions. This can be either done using boundary integration or boundary collocation method. Boundary integral meshless methods are similar to BEM that is based on the boundary integral equation. Some of these meshless methods include boundary node method [19] (BNM), boundary element-free method [20] (BEFM), local boundary integral equation method [21] (LBIEM), etc. Although the boundary integral type meshless method benefits from the reduction in the number of degrees of freedom, it is difficult to obtain the fundamental solutions of complex PDEs and to compute singular boundary integration. Boundary-type collocation meshless methods are also based on fundamental solutions but have higher computational efficiency. Some of these methods include method of fundamental solution [22] (MFS) proposed by Kupradze and Aleksidze, boundary knot method [23] (BKM) proposed by Chen et al., singular boundary method [24] (SBM) proposed by Chen, etc. Recently, Yang et al, [25] presented a generalized method of fundamental solution (GMFS) which uses a bilinear combination of fundamental solutions rather the linear combination as used in MFS.

Zézé et al. [26], proposed Taylor meshless method (TMM) which is also a boundary meshless method but uses Taylor polynomial as a solution to any governing PDE instead of using different fundamental solutions for different PDEs. This can be done because Taylor series has property of approximating any continuous and differentiable function. In TMM, residual of governing PDE upto n^{th} degree are made zero to find relationship between Taylor series coefficients. The boundary is discretized using collocation technique and the boundary conditions are applied to solve for Taylor coefficients. Yang et al. [27] presented the successful implementation of TMM to large scale problems. For problems with singularities or rapid change in function, accurate results with TMM can only be obtained by splitting the main domain into several sub-domains. Additional collocation points are added on the interfaces of sub-domains to satisfy C^0 and C^1 continuity between sub-domains. Yang et al. [28] also studied different combinations of least square collocation and Lagrange multiplier techniques to solve the boundary and interface conditions based on TMM method. TMM was also applied to solve non-linear PDEs by using Newton linearization method and automatic differentiation [29]. Recently, Zézé et al.[30] proposed a multi point Taylor method along with Hermite–Birkhoff interpolation to solve PDE with singularities.

Even though several meshless methods have been proposed in literature, solution of problem with multiple boundary especially point boundary conditions is still a challenge to these methods. Some of the practical usages of point boundary condition are in stress analysis in form of point loads and in crack growth simulations. Another practical application of point boundary conditions is in solution of journal bearing problem where oil holes are very small in dimensions as compared to bearing dimensions and oil holes are usually modelled as point boundary conditions [31]. The journal bearing problem is generally solved using custom FEM code with relevant physics [32, 33]. In the boundary meshless methods, multiple grid points are assigned on the each boundary but for point boundary condition only one point can be assigned so the algorithms generally give preferences to minimize residual for boundary with many points rather than satisfying the point boundary condition. The main aim of the paper is present a meshless method that can solve problems with multiple boundary conditions including point boundary conditions. The basic idea behind the present method is to formulate a function that satisfies all the given boundary conditions including point boundary conditions and use that function as solution for the governing PDE. The formulated function will have a Taylor series term to generalize the function to a family to functions. The unknown Taylor coefficients are found out by minimizing residual of governing PDE using domain collocation method. Section 2 presents detailed formulation and section 3 presents usage of methods on some practical PDEs.

2. Taylor based Domain Collocation Meshless Methodology:

There are two major steps in the proposed methodology i.e. (i) development of generalized family of functions which satisfy the boundary and initial conditions and (ii) computing the Taylor coefficients of the generalized function by using domain collocation

Initially a function is formulated that satisfies all the systems boundary and initial conditions. The formulated function would then be converted into family of functions which satisfy the boundary and initial conditions by adding generalizing term in the form of Taylor series. Since the formulated family of functions satisfy the boundary conditions explicitly, the coefficients of specific Taylor series are required to found out which satisfy the given governing PDE. This can be done by minimizing the sum of squares of residual of governing PDE at collocation points either distributed randomly or uniformly throughout the domain. For linear homogeneous or non-homogenous PDE, minimization of sum of least square residual at collocation points yields solution of Taylor series coefficients in form of a linear regression problem which can be easily solved and many software have optimized code to solve the linear regression problem.

2.1 Development of generalized family of functions which satisfy given boundary conditions including point boundary conditions:

The procedure for formulation of family of functions that satisfies multiple boundary conditions including point boundary conditions is explained below. To understand the principle of formulation, consider a simple single variable function u(x) with value of function u(x) known at n points

$u(x_1)= k_1$, $u(x_2)= k_2$, $u(x_3)= k_3$, $u(x_4)= k_4$, $u(x_5)= k_5$…………………….. ..$u(x_n)=k_n$

The Newton polynomial [34, 35] is the least order polynomial possible which passes though n points and can be given as:

$$u(x) = [k_1] + [k_1, k_2] * (x - x_1) + [k_1, k_2, k_3] * (x - x_1)(x - x_2) + [k_1, k_2, k_3, k_4] * (x - x_1)(x - x_2)(x - x_3) \dots\dots\dots\dots + [k_1, k_2, k_3, k_4 \dots\dots, k_n] * (x - x_1)(x - x_2)(x - x_3) \dots\dots (x - x_{n-1}) \quad (1)$$

where $[k_1, k_2, k_3, k_4 \dots\dots, k_n]$ is the notation for divided difference

Addition of a generalizing term in the form of Taylor series and slight modification of the polynomial yields a general function $u(x)$ which is a polynomial having order greater than or equal to minimum order to satisfy boundary conditions.

$$u(x) = k_1 + v_1 * l_1(x) + v_2 * l_2(x) + v_3 * l_3(x) \dots\dots\dots\dots + v_{n-1} * l_{n-1}(x) + (x - x_1)(x - x_2)(x - x_3) \dots\dots (x - x_n) \sum_{i=0}^{i=\infty} c_i x^i \quad (2)$$

where $l_1(x) = \frac{(x-x_1)}{(x_2-x_1)}$, $l_2(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$, $l_3(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}$ ……

$$l_{n-1}(x) = \frac{(x - x_1)(x - x_2)(x - x_3) \dots\dots (x - x_{n-1})}{(x_n - x_1)(x_n - x_2)(x_n - x_2) \dots\dots (x_n - x_{n-1})}$$

$$v_1 = k_2 - k_1, v_2 = k_3 - k_1 - v_1 * l_1(x_3), v_3 = k_4 - k_1 - v_1 * l_1(x_4) - v_2 * l_2(x_4), \dots$$

$$v_{n-1} = k_n - k_1 - v_1 * l_1(x_n) - v_2 * l_2(x_n) - v_3 * l_3(x_n) \dots\dots\dots\dots\dots - v_{n-2} * l_{n-2}(x_n)$$

The function formulation will be illustrated by taking a simple example. Let's formulate family of function $u(x)$ with u(1)= 2 , u(2)= 3 and u(4)=5. The generalized family of functions for this cased will be

$$u(x) = 2 + (3 - 2) * \frac{(x - 1)}{(2 - 1)} + (5 - 2 - 3) * \frac{(x - 1)(x - 2)}{(4 - 1)(4 - 2)} + (x - 1)(x - 2)(x - 3) \sum_{i=0}^{i=\infty} c_i x^i$$

$$= (x + 1) + (x - 1)(x - 2)(x - 3) \sum_{i=0}^{i=\infty} c_i x^i$$

The family of functions using function boundary conditions can also be formulated using similar procedure. For physical problem in two dimensional spatial co-ordinates, x and y or physical problem with single spatial co-ordinate x and time t, the formulation of function would be similar. Here the formulation is presented for problem in x and y only. Consider a physical problem with m boundary conditions on m boundaries $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \ldots\ldots\ldots\ldots.. \Gamma_{m-1}, \Gamma_m$ where each boundary is represented by a function $\Psi_1(x, y) = 0$, $\Psi_2(x, y) = 0$, $\Psi_3(x, y) \ldots\ldots\ldots\ldots \Psi_m(x, y) = 0$ respectively. The value of the function $u(x, y)$ at each boundary is given as $\varphi_1(x, y)$, $\varphi_2(x, y)$, $\varphi_3(x, y) \ldots\ldots \varphi_m(x, y)$.Function that satisfy these m boundary conditions can be given as

$$u(x,y) = \varphi_1(x,y) + p_1(x,y) * h_1(x,y) + p_2(x,y) * h_2(x,y) + p_3(x,y) * h_3(x,y) \ldots\ldots\ldots\ldots + p_{m-1}(x,y) * h_{m-1}(x,y) \qquad (3)$$

where $h_1(x,y) = \frac{\Psi_1(x,y)}{\Psi_1(x,y) + \Psi_2(x,y)}$, $h_2(x,y) = \frac{\Psi_1(x,y) * \Psi_2(x,y)}{\Psi_3(x,y) + \Psi_1(x,y) * \Psi_2(x,y)}$,

$h_3(x,y) = \frac{\Psi_1(x,y) * \Psi_2(x,y) * \Psi_3(x,y)}{\Psi_4(x,y) + \Psi_1(x,y) * \Psi_2(x,y) \Psi_3(x,y)}$, ...

$$h_{m-1}(x,y) = \frac{\Psi_1(x,y) * \Psi_2(x,y) * \Psi_3(x,y) * \ldots\ldots\ldots * \Psi_{m-1}(x,y)}{\Psi_m(x,y) + \Psi_1(x,y) * \Psi_2(x,y) * \Psi_3(x,y) * \ldots\ldots\ldots * \Psi_{m-1}(x,y)}$$

and $p_1(x,y) = \varphi_2(x,y) - \varphi_1(x,y)$, $p_2(x,y) = \varphi_3(x,y) - \varphi_1(x,y) - p_1(x,y) * h_1(x,y)$

$p_3(x,y) = \varphi_3(x,y) - \varphi_1(x,y) - p_1(x,y) * h_1(x,y) - p_2(x,y) * h_2(x,y)$

....

$$p_{m-1}(x,y) = \varphi_n(x,y) - \varphi_1(x,y) - p_1(x,y) * h_1(x,y) - p_2(x,y) * h_2(x,y) \ldots\ldots.. p_{m-2}(x,y) * h_{m-2}(x,y)$$

The above function can be converted into general family of functions which satisfy the boundary conditions by adding the generalization term in form of Taylor series

$$u(x,y) = \varphi_1(x,y) + p_1(x,y) * h_1(x,y) + p_2(x,y) * h_2(x,y) + p_3(x,y) * h_3(x,y) \ldots\ldots\ldots\ldots + p(x,y) * h_{m-1}(x,y) + \Psi_1(x,y) * \Psi_2(x,y) * \Psi_3(x,y) * \ldots\ldots * \Psi_m(x,y) * \sum_{i=0}^{i=\infty} \sum_{j=0}^{j=\infty} c_{ij} x^i y^j \qquad (4)$$

It should be noted that the above formulations of function $h_1(x,y), h_2(x,y), h_3(x,y), \ldots\ldots\ldots\ldots\ldots\ldots.. \ldots\ldots.. h_{m-1}(x,y)$ are only valid when denominator of these functions is non-zero for any point inside the domain otherwise the function $u(x,y)$ would have an infinite value at those points. If the limits of h functions can be estimated beforehand and it is well know that these functions have non-zero values in the domain then above formulation can be used. Otherwise, one solution is to slightly modify formulations of h functions by using squares of $\Psi(x, y)$ as shown below

$h_1(x,y) = \frac{\Psi_1(x,y)^2}{\Psi_1(x,y)^2 + \Psi_2(x,y)^2}$, $h_2(x,y) = \frac{\Psi_1(x,y)^2 * \Psi_2(x,y)^2}{\Psi_3(x,y)^2 + \Psi_1(x,y)^2 * \Psi_2(x,y)^2}$,

$h_3(x,y) = \frac{\Psi_1(x,y)^2 * \Psi_2(x,y)^2 * \Psi_3(x,y)^2}{\Psi_4(x,y)^2 + \Psi_1(x,y)^2 * \Psi_2(x,y)^2 * \Psi_3(x,y)^2}$, ...

$$h_{m-1}(x,y) = \frac{\Psi_1(x,y)^2 * \Psi_2(x,y)^2 * \Psi_3(x,y)^2 * \dots\dots\dots * \Psi_{m-1}(x,y)^2}{\Psi_m(x,y)^2 + \Psi_1(x,y)^2 * \Psi_2(x,y)^2 * \Psi_3(x,y)^2 * \dots\dots\dots * \Psi_{m-1}(x,y)^2}$$

The modified formulation of h functions would always have non-zero values for any real value point as long as m boundaries are non-intersecting. For intersecting boundaries, the intersecting boundaries conditions can be combined to remove redundant conditions to find h functions as shown for problem in section 3.1. If instead of Dirichlet boundary condition, Neumann boundary condition specified at the boundary in form of $\frac{\partial^{a+b}u(x,y)}{\partial x^a \partial y^b}$ =Δ(x,y), then Δ(x,y) can be integrated to find out $\varphi(x,y)$ at boundary which include the integration constants. For the formulation of $u(x,y)$ the h functions can be modified by using $\Psi(x,y)^{a+b+1}$ instead of $\Psi(x,y)$ since the $\frac{\partial^{a+b}(f(x,y)*\Psi(x,y)^{a+b+1})}{\partial x^a \partial y^b}$ and its lower derivatives would always be zero for any arbitrary function $f(x,y)$ and for any x, y which satisfy $\Psi(x,y) = 0.0$. A problem with Neumann boundary is solved in section 3.2.

If only points (x,y) are specified at the boundary instead of the function Ψ(x,y) or if there is a difficultly in integrating Δ(x,y) then surrogate Taylor polynomial functions can be found out and can be used in formulation. An example for this surrogate Taylor function is given section 3.4 for amoeba shaped boundary. If instead of 2 variables the physical problem contains 3 variables either x,y,z or x,y,t then the general Taylor family of functions satisfying boundary condition can be formulated in a similar way

$$u(x,y,z) = \varphi_1(x,y,z) + p_1(x,y,z) * h_1(x,y,z) + p_2(x,y,z) * h_2(x,y,z) + p_3(x,y,z) * h_3(x,y,z)\dots\dots\dots\dots + p_{m-1}(x,y,z) * h_{m-1}(x,y,z) + \Psi_1(x,y,z) * \Psi_2(x,y,z) * \Psi_3(x,y,z) * \dots\dots\dots * \Psi_m(x,y,z) * \sum_{i=0}^{i=\infty}\sum_{j=0}^{j=\infty}\sum_{k=1}^{k=\infty} c_{ijk}\, x^i y^j z^k \qquad (5)$$

where $h_1(x,y,z) = \frac{\Psi_1(x,y,z)}{\Psi_1(x,y,z)+ \Psi_2(x,y,z)}$, $h_2(x,y,z) = \frac{\Psi_1(x,y,z) * \Psi_2(x,y,z)}{\Psi_3(x,y,z)+ \Psi_1(x,y) * \Psi_2(x,y,z)}$,

$h_3(x,y) = \frac{\Psi_1(x,y,z) * \Psi_2(x,y,z) * \Psi_3(x,y,z)}{\Psi_4(x,y,z)+ \Psi_1(x,y,z) * \Psi_2(x,y,z)\Psi_3(x,y,z)}$, …………

$$h_{m-1}(x,y,z) = \frac{\Psi_1(x,y,z) * \Psi_2(x,y,z) * \Psi_3(x,y,z) * \dots\dots\dots * \Psi_{m-1}(x,y,z)}{\Psi_m(x,y,z) + \Psi_1(x,y,z) * \Psi_2(x,y,z) * \Psi_3(x,y,z) * \dots\dots\dots * \Psi_{m-1}(x,y,z)}$$

and $p_1(x,y,z) = \varphi_2(x,y,z) - \varphi_1(x,y,z)$, $p_2(x,y,z) = \varphi_3(x,y,z) - \varphi_1(x,y,z) - p(x,y,z) * h_1(x,y,z)$, $p_3(x,y,z) = \varphi_3(x,y,z) - \varphi_1(x,y,z) - p_1(x,y,z) * h_1(x,y,z) - p_2(x,y,z) * h_2(x,y,z)$

………

$$p_{m-1}(x,y,z) = \varphi_n(x,y,z) - \varphi_1(x,y,z) - p_1(x,y,z) * h_1(x,y,z) - p_2(x,y,z) * h_2(x,y,z)\dots\dots\dots\dots.\, p_{m-2}(x,y,z) * h_{m-2}(x,y,z)$$

In addition to m boundary conditions, if e point boundary conditions specifying the value of function at certain points in the domain are required to be satisfied.

$$u(x_1,y_1) = u_1, u(x_2,y_2) = u_2, u(x_3,y_3) = u_3 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots u(x_e,y_e) = u_e$$

Using simple function representation for functions in Eq.3 as

$\alpha(x,y) = \varphi_1(x,y) + p_1(x,y) * h_1(x,y) + p_2(x,y) * h_2(x,y) + p_3(x,y) * h_3(x,y) \ldots \ldots \ldots \ldots + p_{m-1}(x,y) * h_{m-1}(x,y)$

and

$\omega(x,y) = \Psi_1(x,y) * \Psi_2(x,y) * \Psi_3(x,y) * \ldots \ldots \ldots * \Psi_{m-1}(x,y) * \Psi_m(x,y)$

The function satisfying the m boundary conditions and 1 point boundary conditions can be obtained by using co-ordinates of point boundary (x_1, y_1) as the development point for Taylor series and separating the constant term

$$u(x,y) = \alpha(x,y) + w_1\omega(x,y) + \omega(x,y)\sum_{i=0}^{i=\infty}\sum_{j=0}^{j=\infty} c_{ij}(x-x_1)^i(y-y_1)^j \quad (i,j) \neq (0,0) \qquad (6)$$

where $w_1 = \frac{u_1-\alpha(x_1,y_1)}{\omega(x_1,y_1)}$

For formulation of functions with more than 1 point boundary conditions requires multi-point Taylor series[30,36] expansions, a function formulation which satisfies the e point along with m boundary conditions is given below

$$\begin{aligned} u(x,y) = \alpha(x,y) + \omega(x,y) * norm(x,y)\{w_1 * \theta_1(x,y) + w_2 * \theta_2(x,y) + w_3 * \theta_3(x,y) \ldots \ldots \ldots \ldots + w_e \\ * \theta_e(x,y)\} + norm(x,y)\{\omega(x,y)\prod_{k=2}^{e}\theta_k(x,y)[\sum_{i=0}^{i=\infty}\sum_{j=0}^{j=\infty} C1_{ij}(x-x_1)^i(y-y_1)^j] \\ + \omega(x,y)\prod_{k=1,k\neq 2}^{e}\theta_k(x,y)[\sum_{i=0}^{i=\infty}\sum_{j=0}^{j=\infty} C2_{ij}(x-x_2)^i(y-y_2)^j] \\ + \omega(x,y)\prod_{k=1,k\neq 3}^{e}\theta_k(x,y)[\sum_{i=0}^{i=\infty}\sum_{j=0}^{j=\infty} C3_{ij}(x-x_3)^i(y-y_3)^j] \ldots \ldots \ldots \ldots \ldots \ldots \\ + \omega(x,y)\prod_{k=1,k\neq e-1}^{e}\theta_k(x,y)[[\sum_{i=0}^{i=\infty}\sum_{j=0}^{j=\infty} C(e-1)_{ij}(x-x_{e-1})^i(y-y_{e-1})^j] \\ + \omega(x,y)\prod_{k=1}^{e-1}\theta_k(x,y)[[\sum_{i=0}^{i=\infty}\sum_{j=0}^{j=\infty} Ce_{ij}(x-x_e)^i(y-y_{e-1})^j\}\ (i,j) \neq (0,0) \qquad (7) \end{aligned}$$

where $norm(x,y)$ is a normalization function and is given by

$$norm(x,y) = \frac{1}{\sum_{r=1}^{e}\prod_{k=1,k\neq r}^{e}\theta_k(x,y)}$$

and $w_1 = \frac{u_1-\alpha(x_1,y_1)}{\omega(x_1,y_1)}$, $w_2 = \frac{u_2-\alpha(x_2,y_2)}{\omega(x_2,y_2)}$, $w_3 = \frac{u_3-\alpha(x_3,y_3)}{\omega(x_3,y_3)}$,………………… $w_e = \frac{u_e-\alpha(x_e,y_e)}{\omega(x_e,y_e)}$

and $\theta_k(x,y)$ are the functions such that their value should be zero at (x_k, y_k) and non-zero at other points describing point boundary conditions i.e. $(x_i, y_i), i = 1,2 \ldots, e, i \neq k$.

Initially $\theta_k(x,y) = (x-x_k)(y-y_k)$ was tried but for some cases the denominator of $norm(x,y)$ was zero for some points on the domain and function $u(x,y)$ became infinite for those points. After several iterations with several different functions $\theta_k(x,y) = (x-x_k)^2 + (y-y_k)^2$ was found to be suitable for all problems as denominator of $norm(x,y)$ can never be zero for the this $\theta_k(x,y)$ as $norm(x,y)$ now represents the inverse of sum of square of distances for any point to all the boundary condition points. Following same procedure similar equation to Eq.7 can be derived for 3 variables but is not presented for the sake of brevity.

2.2 Computation of Taylor coefficients using domain collocation:

For approximation of real world functions, the Taylor series is truncated to have polynomial upto a maximum degree d [26-29]. Physical problems with m boundary conditions and Taylor truncated series upto maximum degree of d $(i + j \leq d)$ would have $(d + 1)(d + 2)/2$ Taylor coefficients or DOFs (degree of freedoms). For physical problem with additional e point boundary conditions the total DOFs should be $(e(d + 1)(d + 2)/2) - e)$. However, it can be observed that all the terms of multi-point Taylor series in Eq.7 are not independent and some Taylor coefficients are linear combinations of other Taylor coefficients. Almost all of the linear regression solvers highlight and remove these dependent coefficients and hence the formulation doesn't have any issue. For sake of simplicity of notations, let Taylor coefficients and integration constants be represented as t and total number of DOF be represented as z. After substituting the formulated function $u(x, y)$ in any given linear governing PDE would finally result in some form linear equation of Taylor coefficients as shown in Eq.8 where $\beta_1(x, y), \beta_2(x, y) \ldots\ldots \beta_z(x, y), \gamma(x, y)$ are function of (x,y) based on the PDE. The differentiation of $u(x, y)$ may look cumbersome and tedious in some cases but these can easily computed using numerical differentiation. A simple finite difference based numerical differentiation is used in the paper but more sophisticated numerical differentiation methods may also be used to get more accurate results.

$$[\, t_1\, t_2\, t_3\, t_4\, t_5\, t_6\, t_7\, t_8 \ldots\ldots\ldots\ldots t_z] \, [\beta_1(x, y) \;\; \beta_2(x, y) \;\; \beta_3(x, y) \;\; \beta_4(x, y) \ldots\ldots\ldots\ldots\ldots \beta_z(x, y)]^T + \gamma(x,y) = 0 \quad (8)$$

Taking n collocation points on the domain of interest $(x_{c1}, y_{c1}), (x_{c2}, y_{c2}), (x_{c3}, y_{c3}) \ldots\ldots\ldots\ldots, (x_{cn}, y_{cn})$ and using a least square minimization of the residual according of Eq.8 for the n collocations give values of unknown Taylor coefficients as

$$[T]_{zx1} = ([A]_{nxz}{}^T [A]_{nxz})^{-1} [A]_{nxz}{}^T [B]_{nx1} \qquad (9)$$

where $[T] = [\, t_1\, t_2\, t_3\, t_4\, t_5\, t_6\, t_7\, t_8 \ldots\ldots t_z]^T$, $[B] = -[\gamma(x_{c1}, y_{c1}) \;\; \gamma(x_{c2}, y_{c2}) \; \gamma(x_{c3}, y_{c3}) \ldots\ldots \gamma(x_{cn}, y_{cn})]^T$

and $$[A] = \begin{bmatrix} \beta_1(x_{c1}, y_{c1}) & \beta_2(x_{c1}, y_{c1}) & \beta_3(x_{c1}, y_{c1}) & \beta_4(x_{c1}, y_{c1}) & \ldots\ldots\ldots\ldots & \beta_z(x_{c1}, y_{c1}) \\ \beta_1(x_{c2}, y_{c2}) & \beta_2(x_{c2}, y_{c2}) & \beta_3(x_{c2}, y_{c2}) & \beta_4(x_{c2}, y_{c2}) & \ldots\ldots\ldots\ldots & \beta_z(x_{c2}, y_{c2}) \\ & & & \vdots & & \\ & & & \vdots & & \\ \beta_1(x_{cn}, y_{cn}) & \beta_2(x_{cn}, y_{cn}) & \beta_3(x_{cn}, y_{cn}) & \beta_4(x_{cn}, y_{cn}) & \ldots\ldots\ldots\ldots & \beta_z(x_{cn}, y_{cn}) \end{bmatrix}$$

Thing to note for the proposed algorithm is that collocation points need not match with the point boundary co-ordinates or even be close to them as the formulation of functions explicitly satisfies the point boundary condition. The linear regression solver of freeware software R [37] has been used in the paper to apply the proposed methodology to solve the PDEs.

3. Application of Solution Methodology:

In this section, the proposed methodology is applied to solve different linear homogenous and non-homogenous governing PDE over different shaped domains.

3.1 Helmholtz equation in rectangular domain:

Helmholtz equation is considered as the fundamental governing PDE in solving heat and the inverse heat conduction problem, the wave propagation problem, and the scattering problem. The Helmholtz equation with same boundary conditions as in Zeze et.al [26] is considered in the present study and is given below

$$u - \Delta u = 0 \qquad (10)$$
$$u(x,0) = u(x,4) = 0,\ \ u(5,y) = u(-5,y) = \sin(\pi y/4)$$

The analytical solution for Eq.10 is $u(x,y) = \frac{\cosh(x\sqrt{(1+\frac{\pi^2}{16}})}{\cosh(5\sqrt{(1+\frac{\pi^2}{16}})} \sin(\frac{\pi y}{4})$

The generalized formulation of $u(x,y)$ which satisfies the boundary conditions can be given as

$$u(x,y) = \sin\left(\frac{\pi y}{4}\right) + (x^2-25)(y^2-4y)\sum_{i=0}^{i=d}\sum_{j=0}^{j=d} C_{ij}x^i y^j \qquad i+j \le d$$

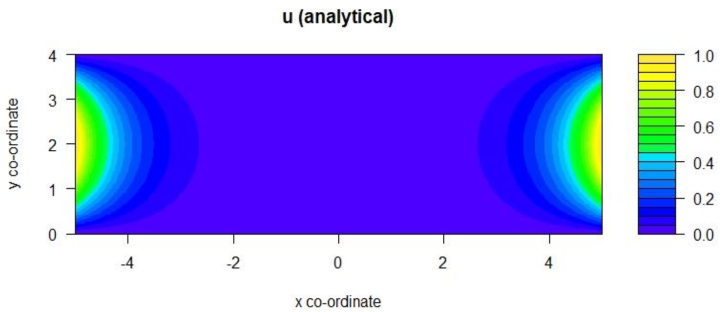
Substituting function for $u(x,y)$ in governing PDE, Eq.10 and simplifying we get

$$\beta_{ij}(x,y) = (x^2-25)(y^2-4y)x^i y^j - (y^2-4y)y^j * \left((i+2)(i+1)x^i - 25 * i(i-1)x^{i-2}\right) - (x^2-25)x^i * \left((j+2)(j+1)y^j - 4j(j+1)y^{j-1}\right)$$

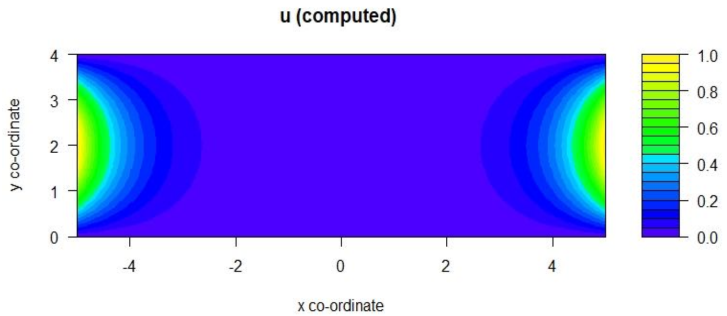
$for\ any\ i\ and\ j\ and\ for\ i \le 1\ and\ j \le 1$ negative power terms are replaced with zero

$$\gamma(x,y) = \left(-\left(\frac{\pi}{4}\right)^2 - 1\right) * \sin\left(\frac{\pi y}{4}\right)$$

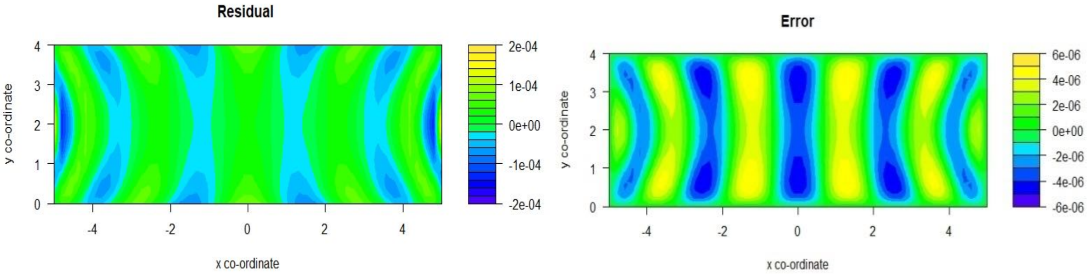
Forming the [A] and [B] matrices for a given grid of collocation points and solving the linear regression problem yields the Taylor coefficients. In the present study, uniform grids of 3 sizes (50x50, 100x100 and 200x200) are used to study the effect of collocation grid size on computed function $u(x,y)$ with degree of Taylor series being taken as 10. Results are shown in Fig 1.



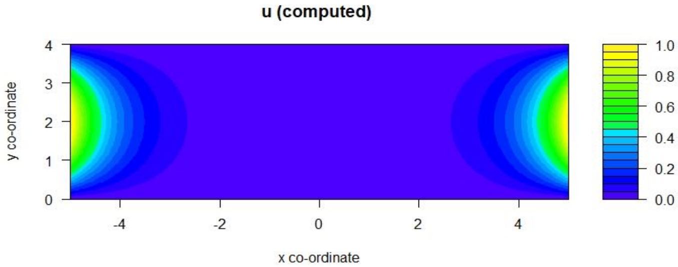
a. Analytical value of function u(x,y)



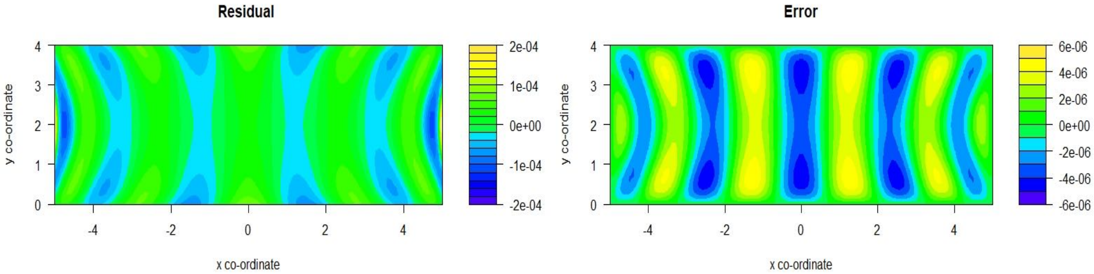
b. Computed value of function u(x,y) for grid 50x50



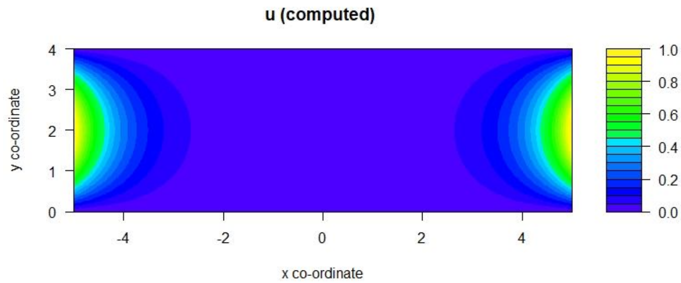
c. Residual of PDE and error between u (computed) and u (analytical) for grid 50x50



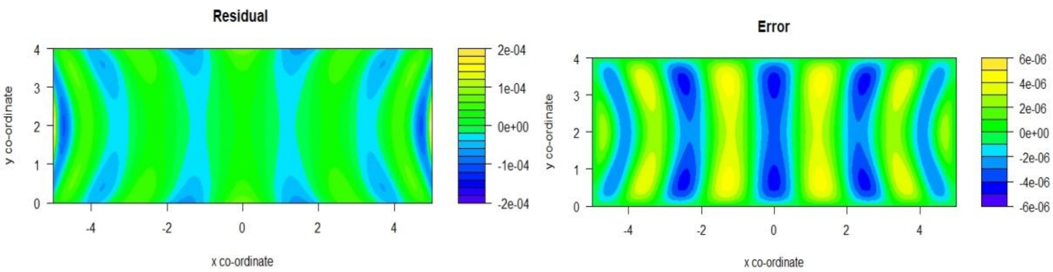
d. Computed value of function u(x,y)for grid 100x100



e. Residual of PDE and error between u (computed) and u (analytical) for grid 100x100



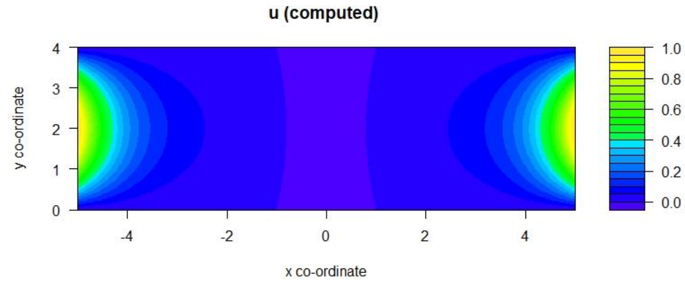
f. Computed value of function u(x,y) for grid 200x200



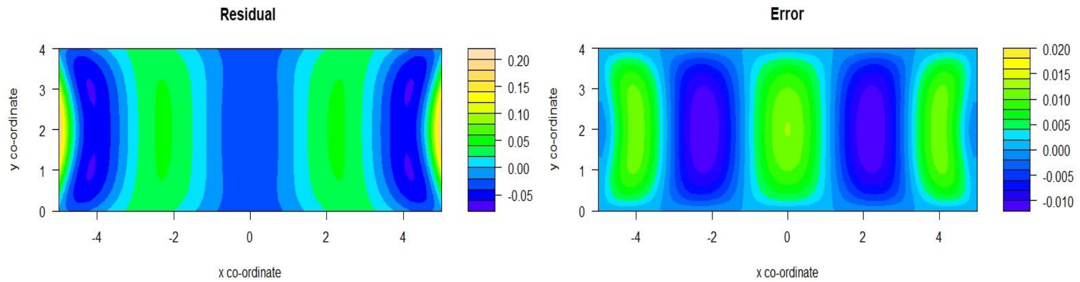
g. Residual of PDE and error between u (computed) and u (analytical) for grid 200x200

Figure.1 Study of collocation grid size on computed solution of Helmholtz equation on rectangular domain

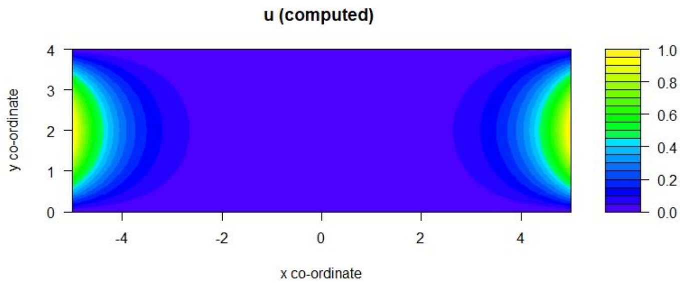
It can be observed from Fig.1 that there is only a marginal improvement in solution when collocation grid is increased from 50x50 to 200x200. Moreover, it can be seen that residual of the Helmholtz equation is couple of orders higher that the order of error of $u(x, y)$. Next, the effect of degree of Taylor polynomial on solution is studied for 3 different values (5, 10 and 15) and the collocation grid size is kept constant at 100x100.



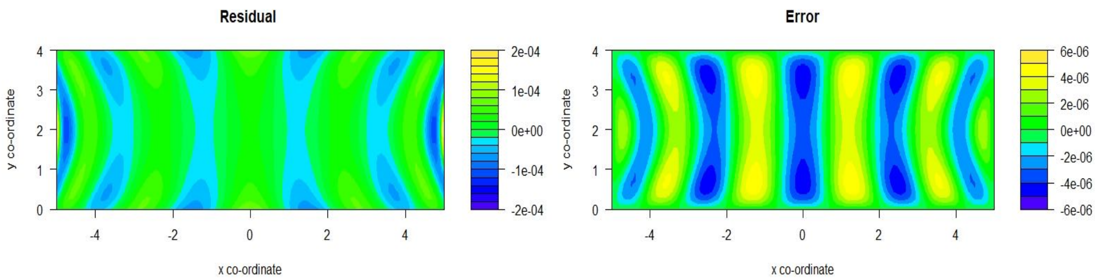
a. Computed value of function u(x,y) for degree 5



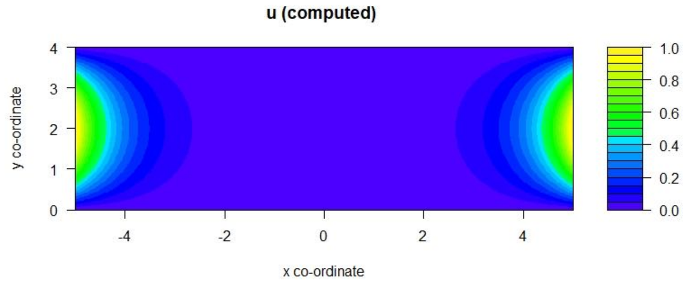
b. Residual of PDE and error between u (computed) and u (analytical) for degree 5



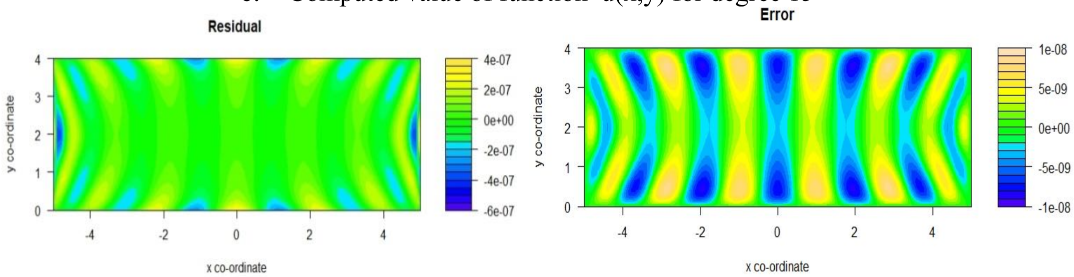
c. Computed value of function u(x,y) for degree 10



d. Residual of PDE and error between u (computed) and u (analytical) for degree 10



e. Computed value of function u(x,y) for degree 15



f. Residual of PDE and error between u (computed) and u (analytical) for degree 15

Figure.2 Study of Taylor polynomial degree on computed solution of Helmholtz equation on rectangular domain

It can be seen from Fig.2 that the selected degree of polynomial plays a crucial role in accuracy of the computed $u(x, y)$ and increasing the degree of Taylor polynomial significantly increases the accuracy of the computed result. On first glance it might look like the TMM [26-29] requires less DOFs as compared to proposed method i.e. for maximum degree d, $2d + 1$ DOFs is required in TMM as compared to $(d + 1)(d + 2)/2$ DOFs in the proposed method. However, for many of the physical problems like Helmholtz equation, the domain is required to be split into several sub-domains to get accurate with TMM which increases DOFs significantly whereas a single domain is used in the proposed method. Table.1 compares the number of DOFs and accuracy of TMM [26] and proposed method for the Helmholtz equation on rectangular domain

Method	Degree	NDOF	Maximal error (log10)
TMM with 5 sub-domains	4	35	-1.4570
	8	65	-2.7181
	12	95	-2.4173
	20	165	-2.3726
	25	205	-2.3726
Proposed	5	21	-0.001522
	10	66	-5.3331
	15	136	-7.4830

Table.1 Comparison between Taylor Meshless Method (TMM) [26] and proposed method for Helmholtz equation on rectangular domain

It can be seen from Table.1 that proposed method achieves higher accuracy with less number of DOFs as compared TMM [26]. However, it should be noted that the proposed method requires significantly higher number of collocation points as compared to TMM since TMM is a boundary collocation whereas proposed method is domain collocation.

In addition to existing boundary conditions, if there is a point boundary condition at a particular (x, y) which is required to be satisfied i.e. $u(x_1, y_1) = u_1$ then the formulation of function u would change as

$$u(x,y) = \sin\left(\frac{\pi y}{4}\right) + k(x^2-25)(y^2-4y) + (x^2-25)(y^2-4y)\sum_{i=0}^{i=d}\sum_{j=0}^{j=d} C_{ij}(x-x_1)^i(y-y_1)^j$$
where $i+j \le d, (i,j) \ne (0,0)$

$$k = \frac{u_1 - \sin(\pi y_1/4)}{({x_1}^2-25)({y_1}^2-4y_1)}$$

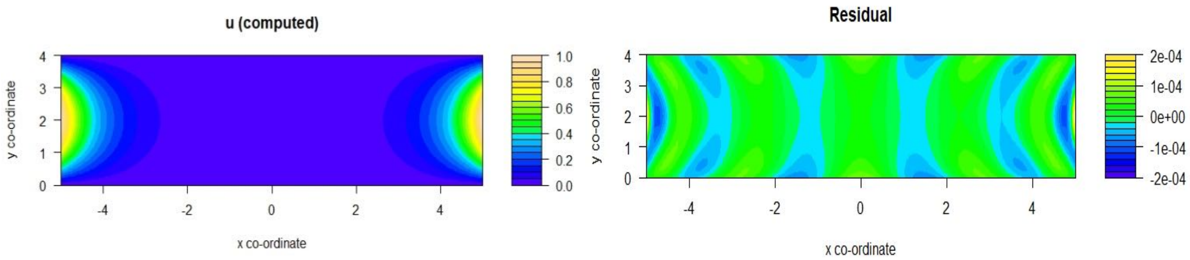
The β and γ functions can be obtained for the modified $u(x,y)$ as

$$\begin{aligned}\beta_{ij}(x,y) &= (x^2-25)(y^2-4y)(x-x_1)^i(y-y_1)^j - (y^2-4y)(y-y_1)^j \\ &* \left((x^2-25)(i)(i-1)(x-x_1)^{i-2} + 2(x-x_1)^i + 4x(i)(x-x_1)^{i-1}\right) - (x^2-25)(x-x_1)^i \\ &* \left((y^2-4y)(j)(j-1)(y-y_1)^{j-2} + 2(y-y_1)^j + 2(2y-4)(j)(y-y_1)^{j-1}\right)\end{aligned}$$

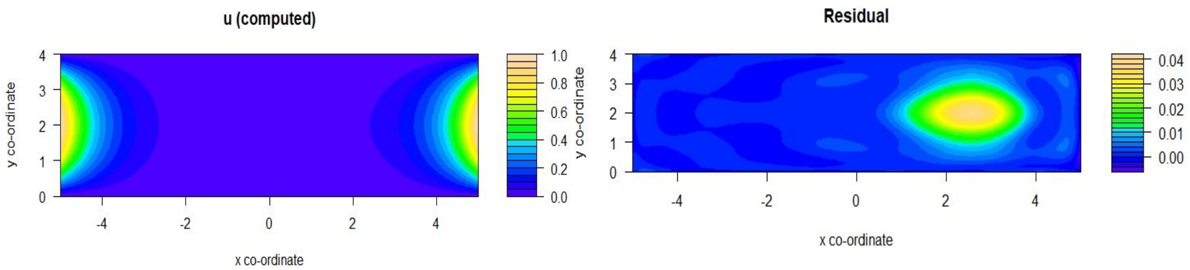
$for\ any\ i\ and\ j\ and\ for\ i \le 1\ and\ j \le 1$ negative power terms are replaced with zero

$$\gamma(x,y) = \left(-\left(\frac{\pi}{4}\right)^2 - 1\right) * \sin\left(\frac{\pi y}{4}\right) - k(x^2-25)(y^2-4y) + 2k(x^2-25+y^2-4y)$$

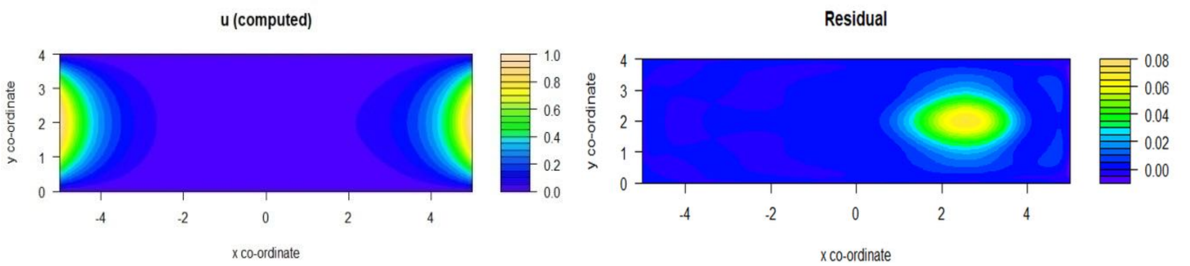
The results are presented by considering $x_1 = 2.5, y_1 = 2$ and for 3 values of u_1. The values of u_1 are taken as 0.04017 which is obtained from analytical solution and 0.055 and 0.065 which consist a slight error from analytical solution. The degree of Taylor polynomial is taken as 10 and collocation grid of 100x100 is used.



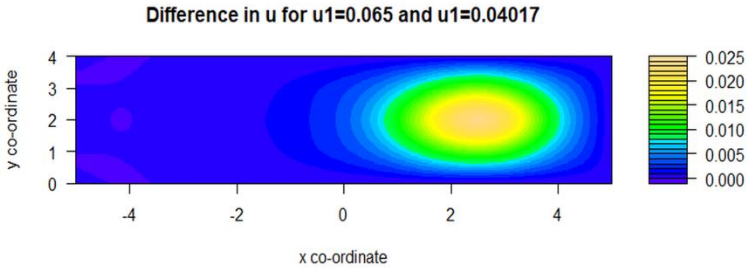
a. Computed value of function u(x,y) and Residual for $u_1 = 0.04017$



b. Computed value of function u(x,y) and Residual for $u_1 = 0.055$



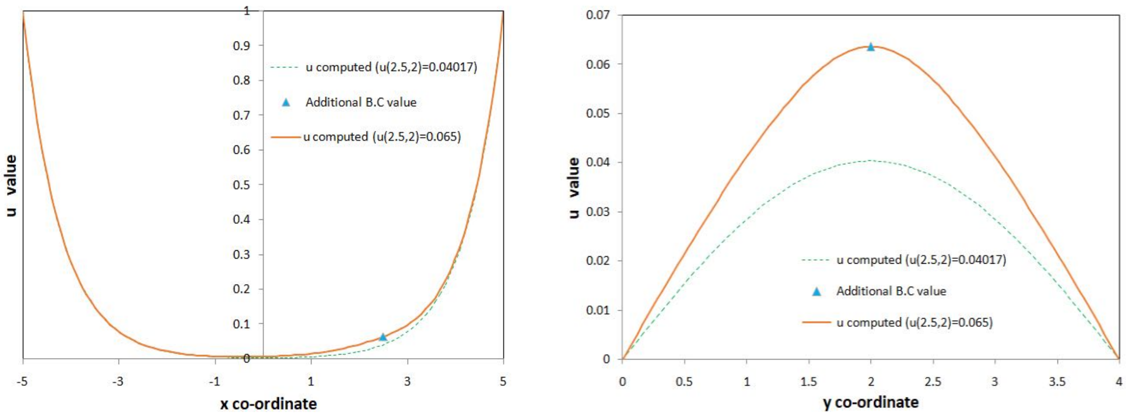
c. Computed value of function u(x,y) and Residual for $u_1 = 0.065$



d. Difference in computed $u(x, y)$ for $u_1 = 0.04017$ and $u_1 = 0.065$

Figure.3 Results of Helmholtz equation on rectangular domain with one additional point boundary condition

It can observed from Fig.3 (a) that when u_1 value satisfies the analytical solution, then the residual value is very similar to the residual value without point boundary condition but shifting the value of u_1 slightly away from analytical solution increases the value of residual as seen in Fig. 3(b),(c).Furthermore, it can be seen from Fig.3(d) that the difference in $u(x, y)$ for $u_1 = 0.04017$ and $u_1 = 0.065$ becomes smaller and smaller when the solution is far away from point boundary condition. This behaviour is generally desired for PDE solution methods. Fig.4 (a) and Fig 4(b) depicts the variation of function $u(x, y)$ predicted by the method at $y = 2$ and $x = 2.5$ for $u_1 = 0.04017$ and $u_1 = 0.065$. It can be seen from 4 that the proposed method predicts smooth, continuous and differentiable function even with the addition of point boundary condition.



a. Variation $u(x, y)$ in x direction for $y = 2$ b. Variation $u(x, y)$ in y direction for $x = 2.5$

Figure.4 Variation of $u(x, y)$ in x and y directions for Helmholtz equation on rectangular domain with and without one additional point boundary condition

If there are 2 point boundary conditions which are required to be satisfied instead of one i.e. $u(x_1, y_1) = u_1$ and $u(x_2, y_2) = u_2$ then the formulation of function $u(x, y)$ would change as

$$u(x,y) = \sin\left(\frac{\pi y}{4}\right) + k_1(x^2-25)(y^2-4y)\frac{\theta_1(x,y)}{\theta_1(x,y)+\theta_2(x,y)} + k_2(x^2-25)(y^2-4y)\frac{\theta_2(x,y)}{\theta_1(x,y)+\theta_2(x,y)} + (x^2-25)(y^2-4y)\frac{\theta_2(x,y)}{\theta_1(x,y)+\theta_2(x,y)}\sum_{i=0}^{i=d}\sum_{j=0}^{j=d} C1_{ij}(x-x_1)^i(y-y_1)^j + (x^2-25)(y^2-4y)\frac{\theta_1(x,y)}{\theta_1(x,y)+\theta_2(x,y)}\sum_{i=0}^{i=d}\sum_{j=0}^{j=2} C2_{ij}(x-x_2)^i(y-y_2)^j$$

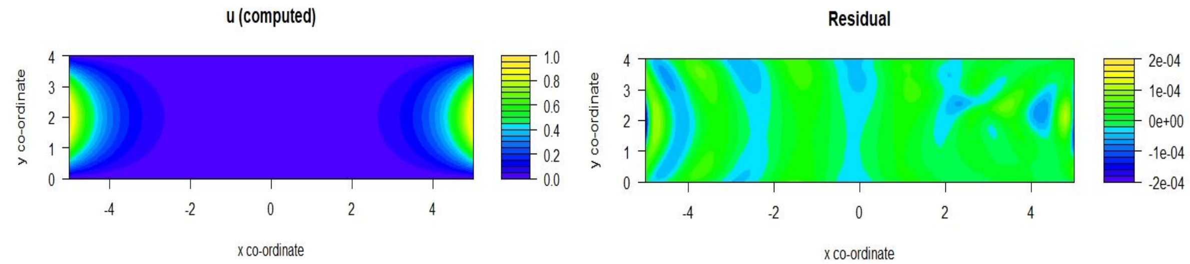
$$i + j \le d, (i,j) \neq (0,0)$$

where $\theta_1(x,y) = (x-x_1)^2 + (y-y_1)^2$ and $\theta_2(x,y) = (x-x_2)^2 + (y-y_2)^2$

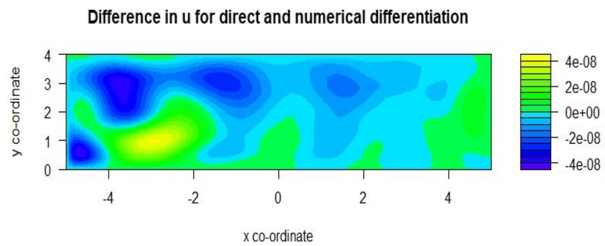
$$k_1 = \frac{u_1 - \sin(\pi y_1/4)}{({x_1}^2 - 25)({y_1}^2 - 4y_1)}$$

$$k_2 = \frac{u_2 - \sin(\pi y_2/4)}{({x_2}^2 - 25)({y_2}^2 - 4y_2)}$$

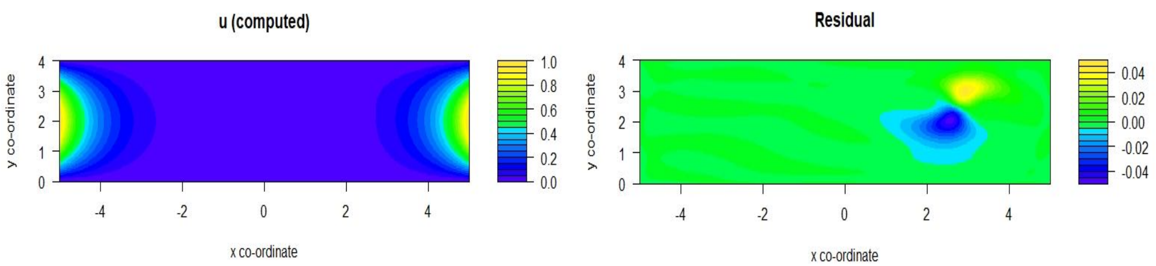
The β and γ functions can be obtained in a similar procedure as before but for the sake of brevity they are not presented here. The results are presented by considering $x_1 = 2.5, y_1 = 2$ and $x_2 = 3\ , y_2 = 3$. The analytical values for u_1 and u_2 can be computed as 0.04017 and 0.05562 respectively. The computed function $u(x, y)$ and residual of Eq.10 for analytical values of u_1 and u_2 is shown in Fig.5 (a). The degree of Taylor polynomial taken as 10 and collocation grid of 100x100 was used.



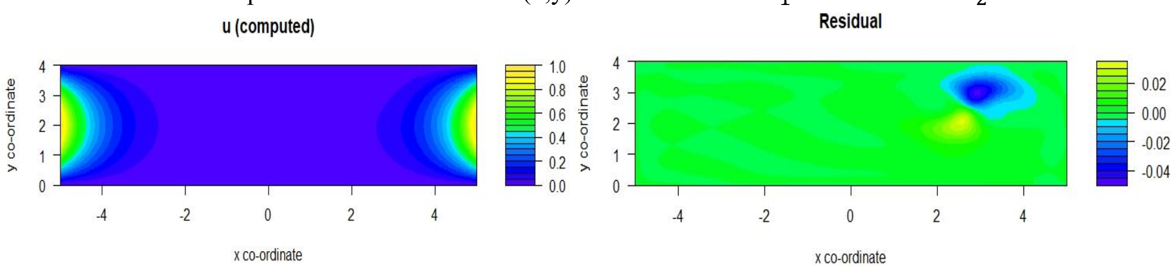
a. Computed value of function u(x,y) and Residual for $u_1 = 0.04017$ and $u_2 = 0.05562$



b. Difference in computed u(x,y) for direct and numerical differentiation



c. Computed value of function u(x,y) and Residual for $u_1 = 0.035$ and $u_2 = 0.060$



d. Computed value of function u(x,y) and Residual for $u_1 = 0.045$ and $u_2 = 0.050$

Figure.5 Results of Helmholtz equation on rectangular domain with two point boundary conditions

It can be seen from Fig.5 (a) that the residual from 2 point boundary function formulation is similar to function formulation without point boundary conditions. Moreover, it can be observed that the function formulation can become quite complex with addition of many point boundary conditions and therefore the direct differentiation can become quite cumbersome and tedious. A simple numerical differentiation scheme based on finite difference was used to compute the β and γ functions and results are compared between direct and numerical differentiation. The differential were computed numerically as $\frac{\partial^2 u(x,y)}{\partial x^2} = \frac{u(x+\Delta x,y)-2u(x,y)+u(x-\Delta x,y)}{(\Delta x)^2}$ and $\frac{\partial^2 u(x,y)}{\partial y^2} = \frac{u(x,y+\Delta y)-2u(x,y)+u(x,y-\Delta y)}{(\Delta y)^2}$ where Δx and Δy were taken as 10^{-6}.The difference between $u(x, y)$ using direct and numerical differentiation is shown in Fig. 5(b). It can be seen that difference in $u(x, y)$ is very small in order of 10^{-8} and it can be concluded that numerical differentiation can be used in cases where function is complex without much loss in accuracy. Fig. 5(c) and 5(d) depicts the $u(x, y)$ and residual for two different sets of value of u_1 and u_2.It can observed from Fig. 5(c) and (d) that the function $u(x, y)$ morphs itself to satisfy the additional 2 point boundary conditions with as much minimum residual as possible.

3.2 Poisson's and Helmholtz equation on circular domain

Laplace equation is another fundamental governing PDE which is used in computation of fluid flows, electrostatics and gravitation fields. Poisson's equation is a generalization of Laplace equation with a non-zero function on the right hand side of equation. In the present study, the solution of Poisson's equation on a circular domain was studied. The Poisson's PDE along with boundary conditions is given as

$$\Delta u = -4(x^2 + y^2)\sin(x^2 + y^2 - 1) + 4\cos(x^2 + y^2 - 1) \quad \text{(11) on } \Omega$$

where Ω represents a circular domain given by $x^2 + y^2 - 1 \leq 0$

$$u(x, y) = 0 \; for \; x, y \, \epsilon \, \Gamma \text{ where } \Gamma \text{ is a circle defined by } x^2 + y^2 - 1 = 0$$

The analytical solution for Eq.11 is $u(x, y) = \sin(x^2 + y^2 - 1)$

The generalized formulation of u which satisfies the boundary conditions can be given as

$$u(x, y) = \left(x^2 + y^2 - 1\right)\sum_{i=0}^{i=d}\sum_{j=0}^{j=d} C_{ij}x^i y^j \qquad i + j \leq d$$

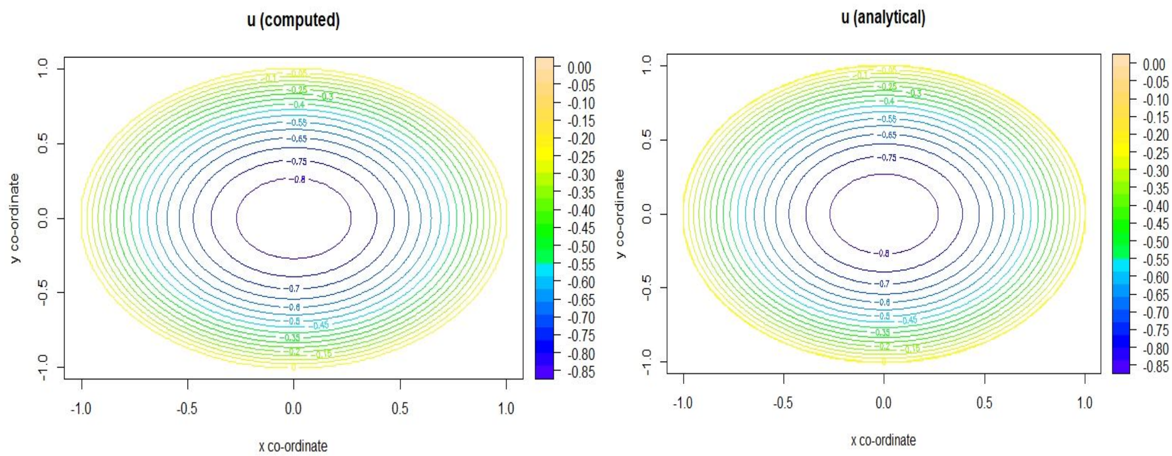
Substituting function for u in Eq.11 and simplifying

$$\beta_{ij}(x, y) = x^i\left((x^2 - 1) * j * (j - 1) * y^{j-2} + (j + 2) * (j + 3) * y^j\right) + y^j((y^2 - 1) * i * (i - 1) * x^{i-2} + (i + 2) * (i + 1) * x^i)$$

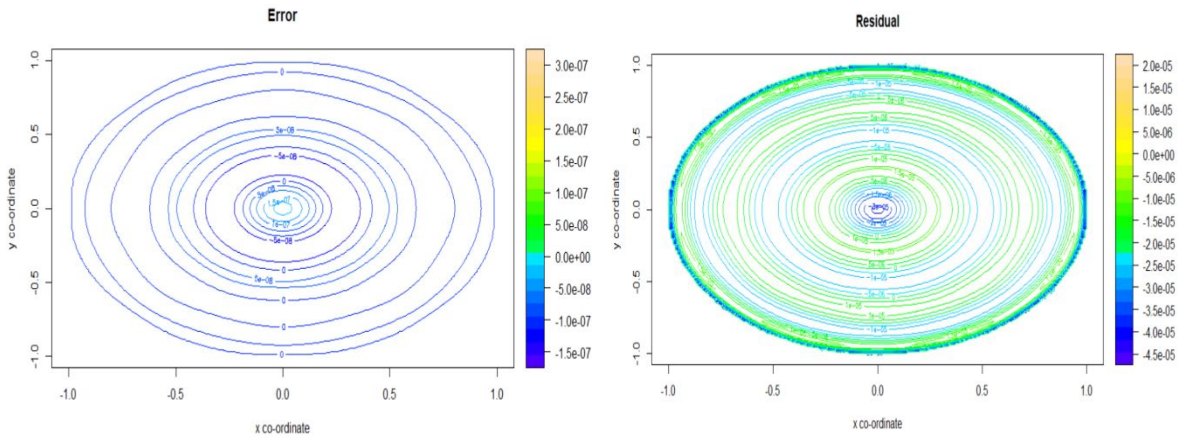
$$for \; any \; i \; and \; j \; and \; for \; i \leq 1 \; and \; j \leq 1 \text{ negative power terms are replaced with zero}$$

$$\gamma(x, y) = -4\left(x^2 + y^2\right)\sin\left(x^2 + y^2 - 1\right) + 4\cos\left(x^2 + y^2 - 1\right)$$

Collocation grid of 100x100 are uniformly distributed in a rectangular space of [-1, 1] but computations were only done for collocations points which are on or inside the circular domain. Forming the [A] and [B] matrices for the collocation points the Taylor coefficients can be solved using linear regression. Results for Poisson's equation on circular domain are shown in Fig 6.



a. Analytical value and computed value of function u(x,y)



b. Residual of PDE and error between u (computed) and u (analytical)

Figure.6 Results of Poisson's equation on circular domain

It can be observed from Fig.6 that $u(x,y)$ computed from present methodology is very close to analytical solution with error in the order of 10^{-7}.

For the same circular domain, the proposed methodology was used to solve Helmholtz equation with Neumann boundary conditions. The governing PDE and boundary conditions are given as

$$\Delta u - u = (4x^2 + 4y^2 + 1)\sin(x^2 + y^2 - 1) - 4\cos(x^2 + y^2 - 1) \quad \text{(12) on } \Omega$$

where Ω represents a circular domain given by $x^2 + y^2 - 1 \leq 0$

$$\frac{\partial u(x,y)}{\partial x} = 2x \; for \; x, y \, \epsilon \, \Gamma \text{ where } \Gamma \text{ is a circle defined by } x^2 + y^2 - 1 = 0$$

The analytical solution for Eq.12 is $u(x,y) = \sin(x^2 + y^2 - 1)$

The generalized formulation of $u(x,y)$ which satisfies the boundary conditions can be given as

$$u(x,y) = g(y) + x^2 + \left(x^2 + y^2 - 1\right)^2 \sum_{i=0}^{i=d} \sum_{j=0}^{j=d} C_{ij} x^i y^j \qquad i + j \leq d$$

where g(y) is the integration constant and can be any function of y. Hence, the Taylor series expansion can be used for $g(y)$ and $u(x,y)$ can be written as

$$u(x,y) = \sum_{k=0}^{k=d} d_k y^k + x^2 + \left(x^2 + y^2 - 1\right)^2 \sum_{i=0}^{i=d} \sum_{j=0}^{j=d} C_{ij} x^i y^j \qquad i + j \le d$$

Substituting function for $u(x,y)$ in Eq.12 and simplifying

$$\beta_k(x,y) = y^k - k * (k-1) y^{k-2}$$

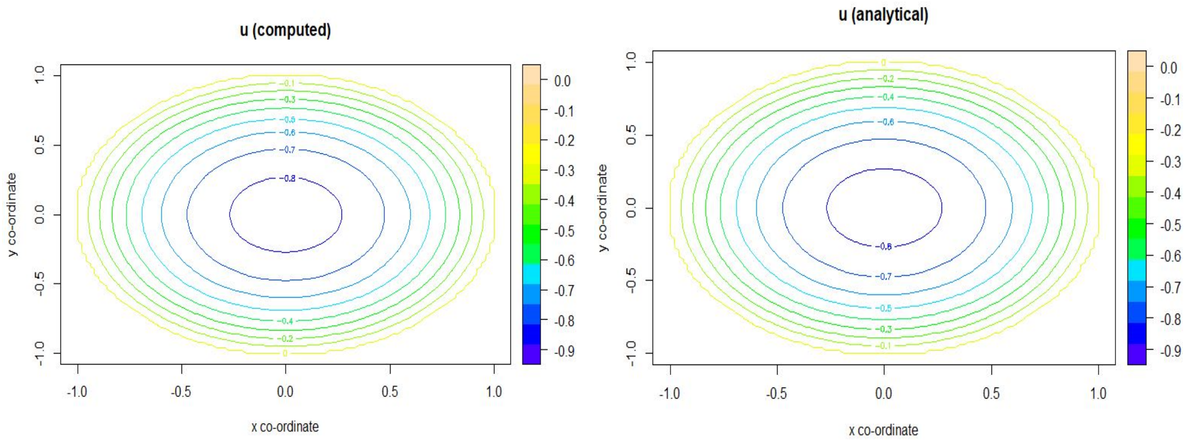
$$for\ any\ k\ and\ for\ k \le 1 \text{ negative power terms are replaced with zero}$$

$$\beta_{ij}(x,y) = x^i y^j (x^2 + y^2 - 1)^2 - x^i \Big((4+j) * (3+j) * y^{2+j} + (-2 + 2x^2) * (j+2) * (j+1) * y^j + (x^4 - 2x^2 + 1) * j * (j-1) * y^{j-2} \Big) - y^j \Big((4+i) * (3+i) * x^{2+i} + (-2 + 2y^2) * (i+2) * (i+1) * x^i + (y^4 - 2y^2 + 1) * j * (j-1) * x^{x-2} \Big)$$

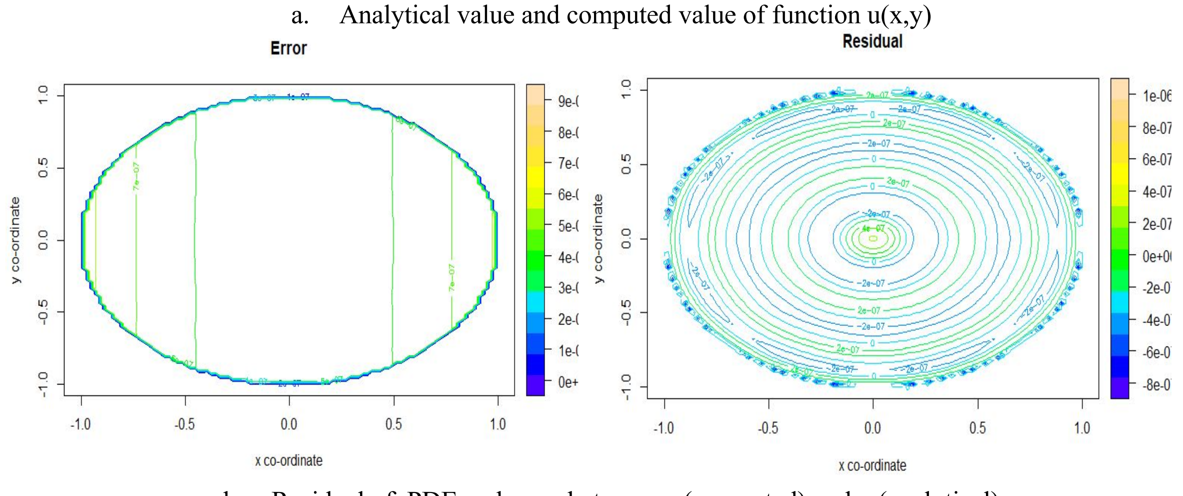
$$for\ any\ i\ and\ j\ for\ i \le 1\ and\ j \le 1 \text{ negative power terms are replaced with zero}$$

$$\gamma(x,y) = \left(4x^2 + 4y^2 + 1\right) \sin\left(x^2 + y^2 - 1\right) - 4\cos\left(x^2 + y^2 - 1\right)$$

Same collocation grid was used as was used for earlier Poisson's equation and maximum degree of both Taylor polynomials was taken as 10 to compute the function $u(x,y)$. It can be observed from Fig.7 that the proposed methodology can be successfully used for problems with Neumann boundary conditions. Moreover, the maximum error between computed function $u(x,y)$ and analytical solution is 8.312×10^{-7} and maximum residual in the computational domain is 1.1×10^{-6}.



a. Analytical value and computed value of function u(x,y)



b. Residual of PDE and error between u (computed) and u (analytical)

Figure.7 Results of Helmholtz equation on circular domain with Neumann boundary condition

Poisson's equation on circular disk is also solved using the proposed methodology. The problem contains singularity at the centre of domain and is presented by Yang [28]. The governing PDE along with boundary conditions can be given as

$$\Delta u = \frac{4}{(x^2+y^2)^2} \qquad (13)$$

On the interior circle with radius 0.8 ($x^2 + y^2 - 0.64 = 0$) the function $u(x, y) = 1.5625$
On the exterior circle with radius 1.0 ($x^2 + y^2 - 1.00 = 0$) the function $u(x, y) = 1.0$
The analytical solution for Eq.13 is $u(x, y) = \frac{1}{(x^2+y^2)}$

It should be noted that function $u(x, y)$ has an infinite value or a singularity when $(x, y) = (0,0)$ and hence results of TMM with single domain were always completely wrong and multi domain approach must be used to solve the problem with TMM[28].

The generalized formulation of $u(x, y)$ which satisfies the boundary conditions can be given as

$$u(x,y) = 1 + 0.5625\frac{(x^2+y^2-1)}{(x^2+y^2-1)-(x^2+y^2-0.64)} + (x^2+y^2 - 0.64)(x^2+y^2-1)\sum_{i=0}^{i=d}\sum_{j=0}^{j=d} C_{ij}x^i y^j \qquad i+j \le d$$

It can be observed that in the denominator, subtraction of boundaries was used instead of addition as addition would results in denominator value being zero on the circle $(x^2 + y^2 - 0.82 = 0.0)$ which is inside the domain. The sum of squares of boundaries i.e. $(x^2 + y^2 - 1)^2 + (x^2 + y^2 - 0.64)^2$ was not used in denominator as subtraction gives a much simpler function.

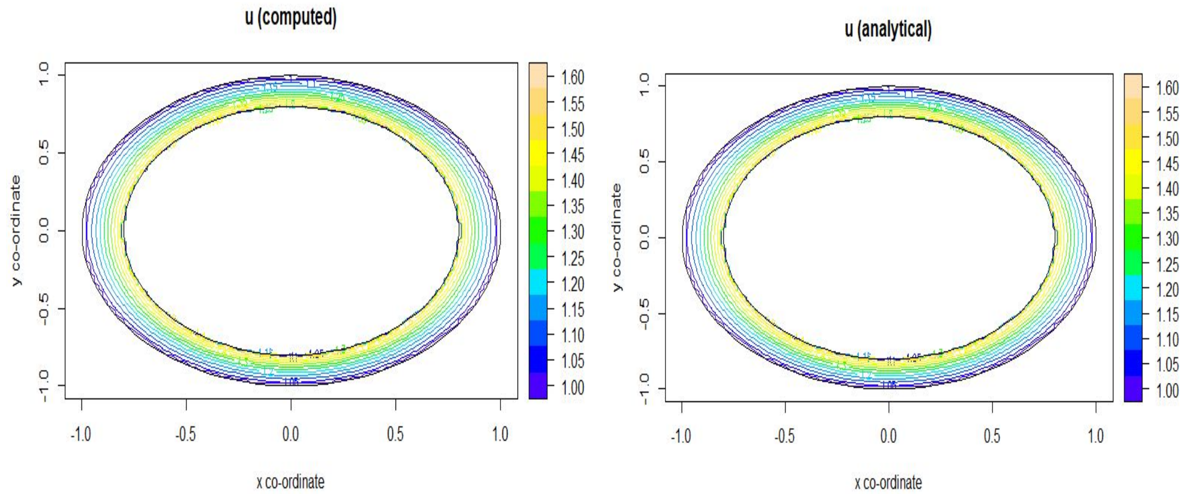
Substituting function for $u(x, y)$ in Eq.13 and simplifying

$$\begin{aligned}\beta_{ij}(x,y) = {} & x^i y^j\big(4(x^2+y^2-0.64) + 4(x^2+y^2-1) + 8x^2 + 8y^2\big) \\ & + ix^{i-1}y^j\left(4x(x^2+y^2-1) + 4x(x^2+y^2-0.64)\right) \\ & + i(i-1)x^{i-2}y^j(x^2+y^2-1)(x^2+y^2-0.64) \\ & + jx^i y^{j-1}\left(4y(x^2+y^2-1) + 4y(x^2+y^2-0.64)\right) \\ & + j(j-1)x^i y^{j-2}(x^2+y^2-1)(x^2+y^2-0.64)\end{aligned}$$

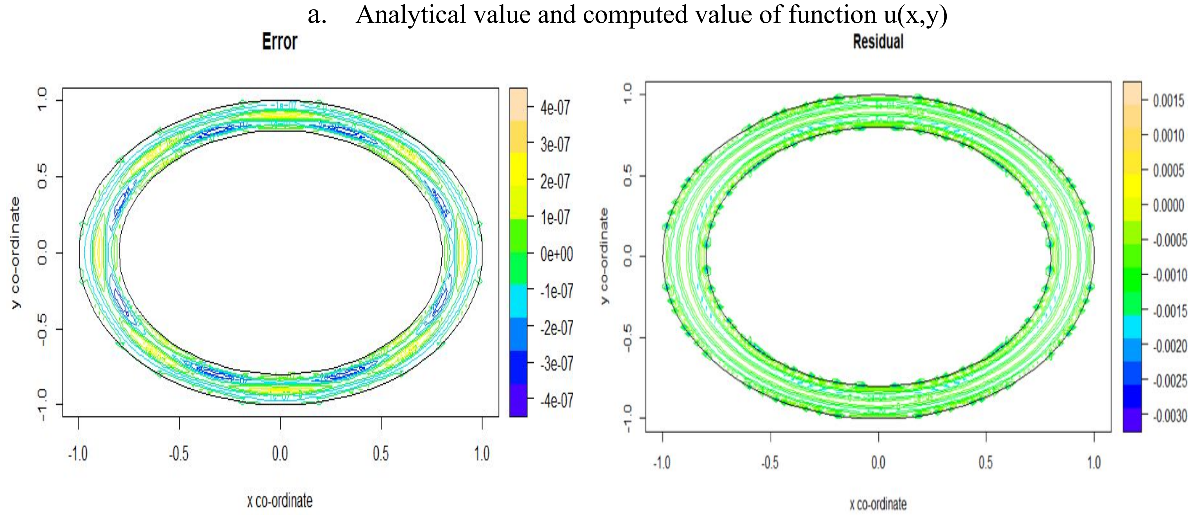
$for\ any\ i\ and\ j\ and\ for\ i \le 1\ and\ j \le 1$ negative power terms are replaced with zero

$$\gamma(x,y) = \frac{4}{(x^2+y^2)^2} + (4 * 1.5625)$$

Collocation grid of 200x200 was used and maximum Taylor polynomial was taken as 10. [A] and [B] matrices were formed and linear regression problem was solved to compute function $u(x, y)$. It can be observed from results in Fig.8 that the present methodology can solve problems with singularity with a single domain. Moreover the log10(maximal error) value of -6.4257obtained from the present methodology is better than accuracy obtained from TMM method with 10 sub domains and 10 degree Taylor polynomial where the log10(maximal error) is in between -5 and -6 [28].



a. Analytical value and computed value of function u(x,y)



a. Residual of PDE and error between u (computed) and u (analytical)

Figure.8 Results of Poisson's equation on circular disk

3.3 Poisson's equation for square with circular hole:

In this section, the proposed methodology is used to solve Poisson's equation for a square domain with a circular hole. The governing PDE along with associated boundary conditions can be given as

$$\Delta u = -\frac{8\pi^2}{9}\sin\left(\frac{2\pi x}{3}\right)\sin\left(\frac{2\pi y}{3}\right) \qquad (14)$$

On the edges of the squares the value of function $u(x,y)$ is 0 i.e. $u(1.5,y) = u(-1.5,y) = u(x,1.5) = u(x,-1.5) = 0$

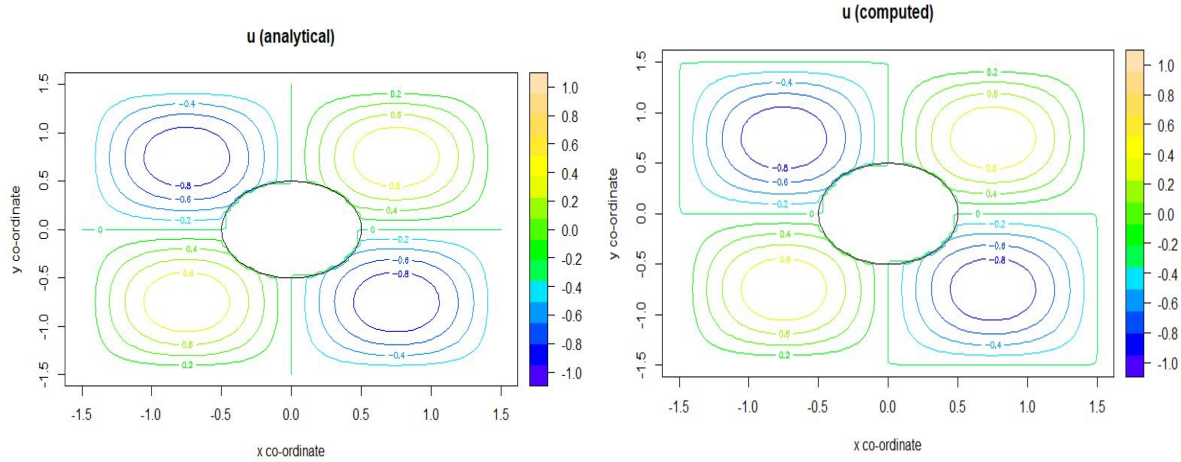
The function on the interior circle ($x^2 + y^2 - 0.25$=0) is given as $u(x,y) = \sin\left(\frac{2\pi x}{3}\right)\sin\left(\frac{2\pi y}{3}\right)$

The analytical solution for Eq.14 is $u(x,y) = \sin\left(\frac{2\pi x}{3}\right)\sin\left(\frac{2\pi y}{3}\right)$

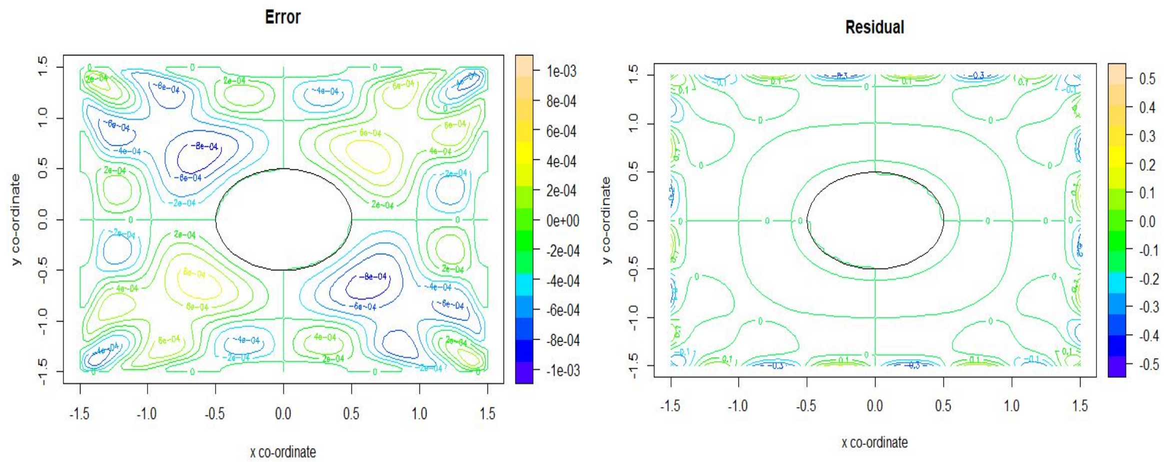
The generalized formulation of u which satisfies the boundary conditions can be given as

$$u(x,y) = \sin\left(\frac{2\pi x}{3}\right)\sin\left(\frac{2\pi y}{3}\right)\frac{(x^2-2.25)(y^2-2.25)}{x^2+y^2-0.25+(x^2-2.25)(y^2-2.25)} + (x^2+y^2-0.25)(x^2-2.25)(y^2-2.25)\sum_{i=0}^{i=d}\sum_{j=0}^{j=d} C_{ij}x^i y^j \qquad i+j \le d$$

[A] and [B] matrices were formed from formulated β and γ function but the formulated β and γ function are not presented here for sake of brevity. Same domain collation procedure was used as used for circular domain problems. Results were computed with 100x100 domain collocation grid and with Taylor polynomial of degree 10 is shown in Fig.9. It can be seen from Fig.9 that the $u(x,y)$ computed by the present methodology is close to analytical solution with error in the order of 10^{-3}. But it can also be observed that the residual of Eq.14 with computed function is very high when close to edges of the square. This may be due to difficulty in approximating the function near edges of the square which is of very small values. However, if the value of degree of Taylor polynomial is increased to 25 then the residual drops to a value of $1.5\text{x}10^{-3}$ near the edges of the square. Also the error between analytical and computed function drops to order of 10^{-6} as shown in Fig.10.

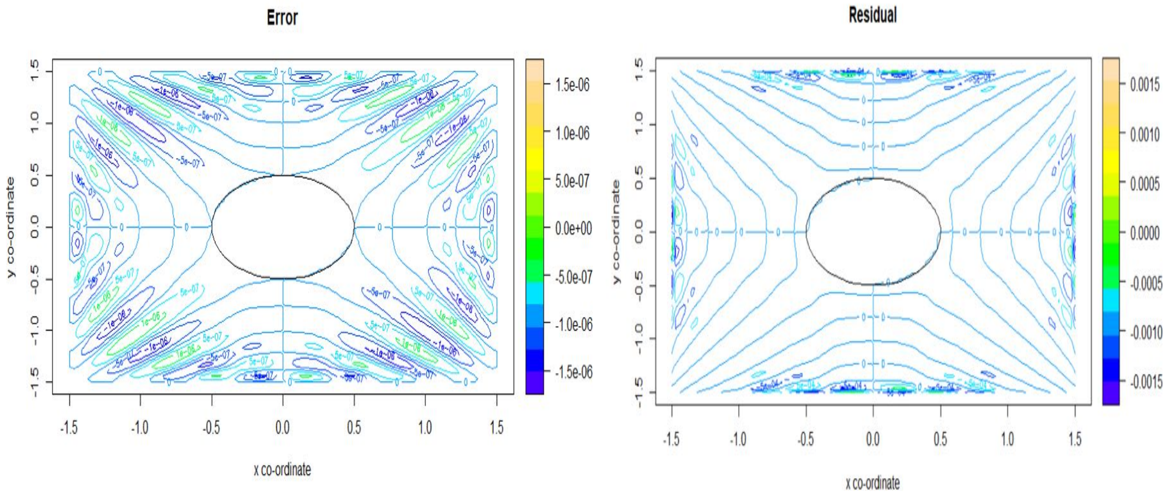


a. Analytical value and computed value of function u(x,y)



b. Residual of PDE and error between u (computed) and u (analytical)

Figure.9 Results of Poisson's equation on square with circular hole for Taylor polynomial degree 10



a. Residual of PDE and error between u (computed) and u (analytical)

Figure.10 Results of Poisson's equation on square with circular hole for Taylor polynomial degree 25

3.4 Non-homogenous Helmholtz equation in amoeba shaped domain:

In earlier sections, the proposed methodology has been used to solve PDEs where the boundary was defined using an analytic function. However, in many real world problems, the boundary would be defined only by using a set of points. Taking one such example problem of amoeba shaped domain with circular hole, the governing PDE and boundary conditions are given as

$$\Delta u - u = (4x^2 + 4y^2 + 1)\sin(x^2 + y^2 - 0.0625) - 4\cos(x^2 + y^2 - 0.0625) \quad (15)$$

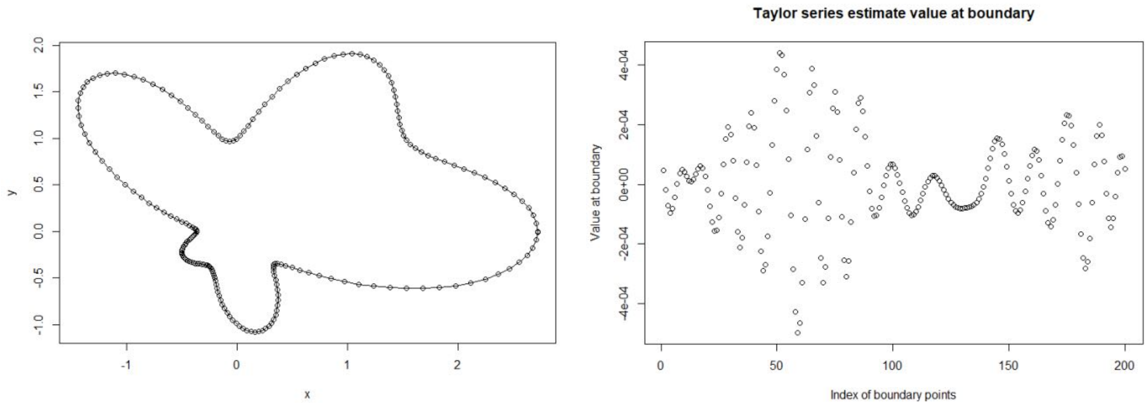
On the exterior amoeba shaped boundary defined in parametric form $(x, y) = R(s)\big(cos(s), sin(s)\big)$, $R(s) = e^{\sin(s)}sin^2(2s) + e^{\cos(s)}cos^2(2s)$, $0 \le s \le 2\pi$ the function takes value of $u(x, y) = sin(x^2 + y^2 - 0.0625)$

On the interior boundary defined by circle $(x^2 + y^2 - 0.0625 = 0)$, the function value $u(x, y) = 0$

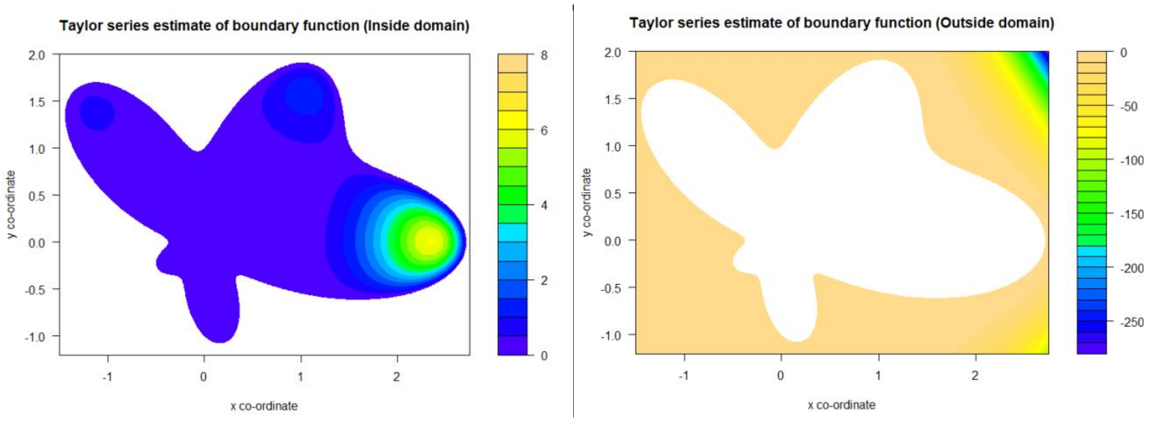
The analytical solution for Eq.15 is $u(x, y) = sin(x^2 + y^2 - 0.0625)$

Before the formulation $u(x, y)$ can be derived it is required that the exterior boundary be represented using an analytical function. Here, a generalized procedure is detailed which can be used to generate an analytic function in the form of Taylor series to define any boundary given a set on point lying on the boundary. Sufficient number of points to represent the amoeba shaped boundary was generated using the parametric equation which is 200 in this particular case as shown in Fig. 11(a). Using these set of points, the objective is to develop a function $f(x, y)$=0 which represent the boundary appropriately. Three conditions must met by the $f(x, y)$ to accurately represent the boundary viz; (a) on the boundary, function $f(x, y)$ should be close to zero (b) inside the domain created by the boundary, the function $f(x, y)$ should be of the same sign either positive or negative for all the points (c) outside the domain created by the boundary the function $f(x, y)$ should be same sign for all the points and opposite to sign of the points inside the boundary. The function $f(x, y)$ can be approximated using the Taylor series i.e. $f(x, y) = \sum_{i=0}^{i=d}\sum_{j=0}^{j=d} C_{ij}x^i y^j$, $i + j \le d$. However, equating the Taylor series $\sum_{i=0}^{i=d}\sum_{j=0}^{j=d} C_{ij}x^i y^j$ to zero for boundary points would result in a trivial problem with trivial solution of all Taylor coefficients being equal to zero. Furthermore, it difficult to know beforehand which Taylor coefficient (C_{ij}) is most significant. Hence a two-step process is used to compute the Taylor series representation of the function $f(x, y)$. In the first step, the equation

$\sum_{i=0}^{i=d}\sum_{j=0}^{j=d} C_{ij}x^i y^j - 1 = 0,\ i+j \le d\ ,\ (i,j) \ne (0,0)$ is solved for the boundary points using linear regression and the Taylor coefficient C_{ij} with highest t-statistic is considered as the most significant term. Let the most significant Taylor coefficient be represented by C_{kl}. In the second step, the Taylor series excluding the most significant term is equated to most significant term i.e. $\sum_{i=0}^{i=d}\sum_{j=0}^{j=d} C_{ij}x^i y^j - x^k y^l = 0,\ i+j \le d\ ,\ (i,j) \ne (k,l)$ and this equation is solved for the boundary points to find out the values of Taylor coefficients. The function $f(x,y)$ can be now represented using Taylor series as $f(x,y) = \sum_{i=0}^{i=d}\sum_{j=0}^{j=d} C_{ij}x^i y^j - x^k y^l = 0\ ,\ i+j \le d$. Using the procedure, with maximum degree of Taylor polynomial as 6 (d=6) the Taylor series function estimate for amoeba shaped domain is developed. The developed function values on the boundary are shown in Fig.11 (b) and it can be seen that the function values are very close to zero on the boundary. The function values inside and outside the domain created by the boundary are shown in Fig.11(c) and Fig. 11(d). It can be seen from Fig.11(c) and 11(d) that the function value is of opposite sign for points inside and outside the domain.



a. Selected points on the boundary b. Function value on the boundary



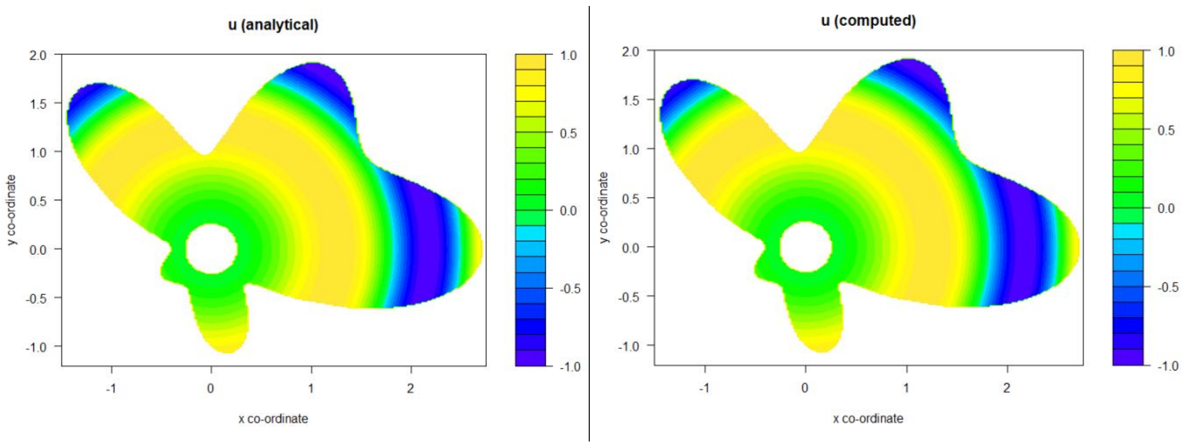
c. Function value inside the domain b. Function value outside the domain

Figure.11 Taylor series function estimate of a boundary represented by a set of points

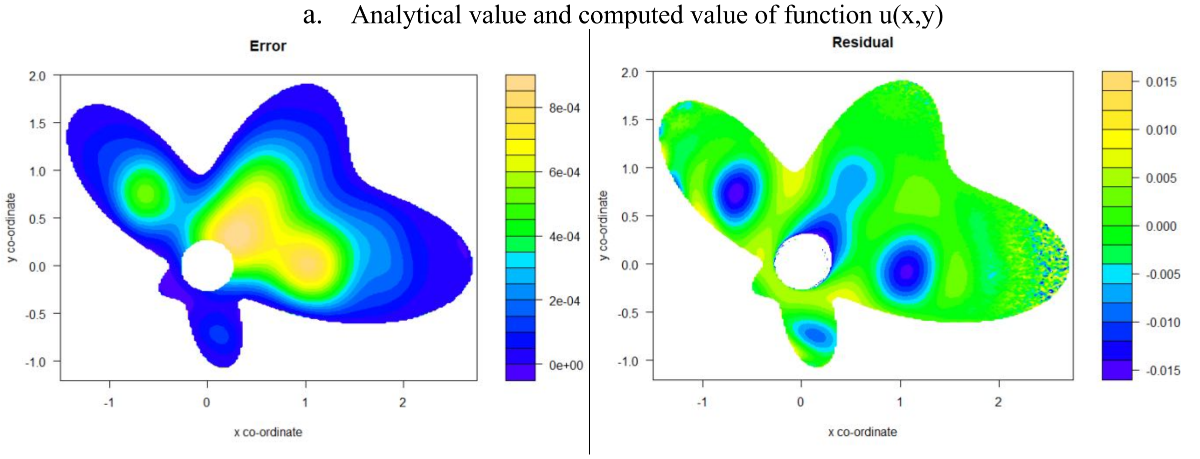
Denoting the analytical function representing the amoeba shaped domain as $poly\_val(x,y)$ the generalized form of u(x,y) which satisfies the boundary conditions can be given as

$$u(x,y) = \sin(x^2 + y^2 - 0.0625)\ \frac{x^2 + y^2 - 0.0625}{x^2 + y^2 - 0.0625 + poly\_val(x,y)} + (x^2 + y^2 - 0.0625)$$
$$* \, poly\_val(x,y) \sum_{i=0}^{i=d} \sum_{j=0}^{j=d} C_{ij} x^i y^j \qquad i + j \le d$$

The β and γ matrices to form the [A] and [B] matrix are derived using finite difference based numerical differentiation similar to section 3.1. The unknown function $u(x,y)$ is computed for collocation grid of 200x200 and maximum degree of Taylor polynomial is taken as10. The results from the computation are shown in Fig.12. It can be observed from Fig.12 that using the developed analytical function for exterior boundary, the governing PDE can be solved to compute the unknown function $u(x,y)$ with error in the order of 10^{-4}.



a. Analytical value and computed value of function u(x,y)



a. Residual of PDE and error between u (computed) and u (analytical)

Figure.12 Results of Helmholtz equation on amoeba shaped domain with circular hole

4. Conclusions

A meshless method has been proposed which can solve PDE having multiple boundary conditions including point boundary conditions. The main crux of the method is to develop a function which satisfies all the boundary conditions then the function is generalized into a family of functions which satisfies the boundary conditions by using Taylor series. For a linear governing PDE, substituting the family of functions in the PDE, the solution of the PDE transforms into a linear regression problem to solve for unknown Taylor coefficients over domain collocation points. A simple modification to method by using multi-point Taylor series makes the method capable of accounting for multiple point boundary conditions. The ability to account for point boundary conditions makes the method stand out as only few out of the available meshless methods can account for multiple point boundary conditions accurately. The method has been applied on wide range of problems in the paper and it seen that the method is very robust.

At a first glance, it might look like the proposed method requires more DOFs to solve a PDE as compared to TMM but as shown in the paper for problems with singularities, the TMM has to applied over several sub-domains which significantly increases the DOF whereas the proposed method solve those problems over a single domain. In the proposed method care should be taken that the function satisfying the boundary should not become singular at any point over the domain. This might be perceived as a drawback of the method but alternate formulations of the function have been provided which guarantees that the function would never be singular at any point over the real valued domain. However, these functions are more complicated hence direct differentiation might be tedious and cumbersome. Therefore, numerical differentiation method has been studied to solve the PDE using the proposed method. A simple finite difference based numerical differentiation was used in the paper which shows similar results as compared to direct differentiation.

Furthermore, the proposed method has been shown to work on problems where the boundary is defined as a set of points instead of an analytical function. A generalized procedure to develop an analytical function of boundary from a collection of points is detailed in the paper. The initial results from the proposed methodology are very encouraging and the method promises to become standard PDE solver in future. However, still further investigations are required into the method in regards to fidelity in handling non-linear PDE and large scale PDE problems.

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