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Robin's criterion on superabundant numbers

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Abstract

Robin's criterion states that the Riemann hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \cdot n \cdot \log \log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of n, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. We require the properties of superabundant numbers, that is to say left to right maxima of $n \mapsto \frac{\sigma(n)}{n}$. Let P_n be equal to $\prod_{q \mid \frac{N_r}{6}} \frac{q^{\nu_q(n)+2}-1}{q^{\nu_q(n)+2}-q}$ for a superabundant number $n > 5040$, where $\nu_p(n)$ is the *p-adic* order of n, q_k is the largest prime factor of n and $N_r = \prod_{i=1}^{r} q_i$ is the largest primorial number of order r such that $\frac{N_r}{6} < q_k^2$. In this note, we prove that the Riemann hypothesis is true when $P_n \geq Q$ holds for all large enough superabundant numbers n, where $Q = \frac{1.2 \cdot (2-\frac{1}{8}) \cdot (3-\frac{1}{3})}{(2-\frac{1}{2^{19}}) \cdot (3-\frac{1}{3^{12}})} \approx 1.0000015809$. We know that $\prod_{q \mid \frac{N_r}{6}} (q^{\nu_q(n)+2} - 1) \geq Q \cdot \prod_{q \mid \frac{N_r}{6}} (q^{\nu_q(n)+2} - q)$ trivially holds for large enough superabundant numbers n and thus, the Riemann hypothesis is true.

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1 Introduction

The hypothesis was proposed by Bernhard Riemann (1859). The Riemann hypothesis belongs to the Hilbert's eighth problem on David Hilbert's list of twenty-three unsolved problems. As usual $\sigma(n)$ is the sum-of-divisors function of n

$$\sum_{d \mid n} d,$$

where $d \mid n$ means the integer d divides n. Define $f(n)$ as $\frac{\sigma(n)}{n}$. We say that $\mathsf{Robin}(n)$ holds provided that

$$f(n) < e^{\gamma} \cdot \log \log n,$$

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. The Ramanujan's Theorem states that if the Riemann hypothesis is true, then the previous inequality holds for large enough n [4]. Next, we have the Robin's Theorem:

Proposition 1.1. $\mathsf{Robin}(n)$ *holds for all natural numbers* $n > 5040$ *if and only if the Riemann hypothesis is true [5, Theorem 1 pp. 188].*

In 1997, Ramanujan's old notes were published where it was defined the generalized highly composite numbers, which include the superabundant and colossally abundant numbers [4]. Let $q_1 = 2, q_2 = 3, \ldots, q_k$ denote the first k consecutive primes, then an integer of the form $\prod_{i=1}^{k} q_i^{a_i}$ with $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ is called a Hardy-Ramanujan integer [2, pp. 367]. A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

We know the following properties for the superabundant numbers:

Proposition 1.2. *If n is superabundant, then n is a Hardy-Ramanujan integer [1, Theorem 1 pp. 450].*

Proposition 1.3. *[1, Theorem 7 pp. 454]. Let n be a superabundant number such that p is the largest prime factor of n, then*

$$p \sim \log n, \quad (n \to \infty).$$

Proposition 1.4. *[1, Theorem 9 pp. 454]. The number of superabundant numbers less than x exceeds*

$$\frac{c \cdot \log x \cdot \log \log x}{(\log \log \log x)^2}.$$

A number n is said to be colossally abundant if, for some $\epsilon > 0$,

$$\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(m)}{m^{1+\epsilon}} \ \ for \ \ (m > 1).$$

There is a close relation between the superabundant and colossally abundant numbers.

Proposition 1.5. *Every colossally abundant number is superabundant [1, pp. 455].*

Several analogues of the Riemann hypothesis have already been proved. Many authors expect (or at least hope) that it is true. However, there are some implications in case of the Riemann hypothesis might be false.

Proposition 1.6. *If the Riemann hypothesis is false, then there are infinitely many colossally abundant numbers* $n > 5040$ *such that* $\mathsf{Robin}(n)$ *fails (i.e.* $\mathsf{Robin}(n)$ *does not hold) [5, Proposition pp. 204].*

The following is a key Corollary.

Corollary 1.7. *If the Riemann hypothesis is false, then there are infinitely many superabundant numbers n such that* $\mathsf{Robin}(n)$ *fails.*

Proof. This is a direct consequence of Propositions 1.1, 1.5 and 1.6. $\square$

In number theory, the *p-adic* order of an integer n is the exponent of the highest power of the prime number p that divides n. It is denoted $\nu_p(n)$. Equivalently, $\nu_p(n)$ is the exponent to which p appears in the prime factorization of n.

Proposition 1.8. $\mathsf{Robin}(n)$ *holds for all natural numbers* $n > 5040$ *such that* $\nu_2(n) \leq 19$ *and* $\nu_3(n) \leq 12$ *[3, Theorem 1 pp. 2, Theorem 2 pp. 2].*

Proposition 1.9. *[1, Theorem 5 pp. 452]. Let n be a superabundant number such that $\nu_q(n) = t$, p is the largest prime factor of n, $2 \leq q \leq p$ and $q < (\log p)^{\alpha}$, where α is a constant, then*

$$\log \frac{q^{t+2}-1}{q^{t+2}-q} < \frac{\log q}{p \cdot \log p} \cdot \left(1 + O\left(\frac{(\log \log p)^2}{\log p \cdot \log q}\right)\right).$$

This is the main insight.

Lemma 1.10. *Let n be a large enough superabundant number such that $p > 3$ is the largest prime factor of n, then*

$$p < 2^{\nu_2(n)-19}$$

and

$$p < 3^{\nu_3(n)-12}.$$

Let P_n be equal to $\prod_{q \mid \frac{N_r}{6}} \frac{q^{\nu_q(n)+2}-1}{q^{\nu_q(n)+2}-q}$ for a superabundant number $n > 5040$, where q_k is the largest prime factor of n and $N_r = \prod_{i=1}^{r} q_i$ is the largest primorial number of order r such that $\frac{N_r}{6} < q_k^2$. Putting all together yields the main theorem:

Theorem 1.11. *The Riemann hypothesis is true when $P_n \geq Q$ holds for all large enough superabundant numbers n, where $Q = \frac{1.2 \cdot (2-\frac{1}{8}) \cdot (3-\frac{1}{3})}{(2-\frac{1}{2^{19}}) \cdot (3-\frac{1}{3^{12}})} \approx$* 1.0000015809. *In addition, the inequality $P_n \geq Q$ trivially holds for large enough superabundant numbers n and therefore, the Riemann hypothesis is true.*

2 Proof of the Lemma 1.10

Proof. Let $q \in \{2, 3\}$ and $\nu_q(n) = t$. For every large enough superabundant number n, there is a constant α such that $q < (\log p)^{\alpha}$. For example, we can take $\alpha = 2.5$ since $(\log p)^{2.5} \geq (\log 5)^{2.5} > 3$. We will use the following inequality

$$\frac{u}{u+1} < \log(1+u), \quad (u > 0).$$

From the previous inequality, we notice that

$$\begin{aligned}
\log \frac{q^{t+2}-1}{q^{t+2}-q} &= \log\left(1 + \frac{q-1}{q^{t+2}-q}\right) \\
&> \frac{\frac{q-1}{q^{t+2}-q}}{\frac{q-1}{q^{t+2}-q}+1} \\
&= \frac{q-1}{(q^{t+2}-q)\cdot(\frac{q-1}{q^{t+2}-q}+1)} \\
&= \frac{q-1}{(q-1)+(q^{t+2}-q)} \\
&= \frac{q-1}{q^{t+2}-1} \\
&> \frac{1}{3\cdot q^{t+1}}.
\end{aligned}$$

Hence, there is a constant $C > 0$ such that

$$q^t > C \cdot \frac{p \cdot \log p}{\log q}$$

by Proposition 1.9. Putting $c = \frac{C}{\log q}$, then we obtain that

$$c \cdot p \cdot \log p < q^t,$$

where c is a positive constant. We deduce that

$$c \cdot \log p > 3^{12}$$

by Proposition 1.3 for large enough n. Therefore, the proof is done. □

3 Proof of the Theorem 1.11

Proof. There are infinitely many superabundant numbers by Proposition 1.4. Let $n > 5040$ be a large enough superabundant number. Let $\prod_{i=1}^{k} q_i^{a_i}$ be the representation of this superabundant number n as the product of the first k consecutive primes $q_1 < \ldots < q_k$ with the natural numbers $a_1 \geq a_2 \geq$

$\ldots \geq a_k \geq 1$ as exponents, since n must be a Hardy-Ramanujan integer by Proposition 1.2. Let P_n be equal to $\prod_{q \mid \frac{N_r}{6}} \frac{q^{\nu_q(n)+2}-1}{q^{\nu_q(n)+2}-q}$ for $n > 5040$, where $N_r = \prod_{i=1}^{r} q_i$ is the largest primorial number of order r such that $\frac{N_r}{6} < q_k^2$. Suppose that $\mathsf{Robin}(n)$ fails and $P_n \geq Q$, where $Q = \frac{1.2 \cdot (2-\frac{1}{8}) \cdot (3-\frac{1}{3})}{(2-\frac{1}{2^{19}}) \cdot (3-\frac{1}{3^{12}})} \approx 1.0000015809$. So,

$$f(n) \geq e^{\gamma} \cdot \log\log n.$$

We know that

$$\begin{aligned} f(n) &= f(2^{\nu_2(n)} \cdot 3^{\nu_3(n)}) \cdot f(\frac{n}{2^{\nu_2(n)} \cdot 3^{\nu_3(n)}}) \\ &< 3 \cdot f(\frac{n}{2^{\nu_2(n)} \cdot 3^{\nu_3(n)}}) \\ &= f(2^3 \cdot 3 \cdot 5) \cdot f(\frac{n}{2^{\nu_2(n)} \cdot 3^{\nu_3(n)}}) \\ &\leq f\left(\frac{2^{19} \cdot 3^{12} \cdot n \cdot \frac{N_r}{6}}{2^{\nu_2(n)} \cdot 3^{\nu_3(n)}}\right) \\ &= f\left(\frac{n \cdot \frac{N_r}{6}}{2^{\nu_2(n)-19} \cdot 3^{\nu_3(n)-12}}\right) \end{aligned}$$

since $P_n \geq Q$, $\frac{q_i}{q_i-1} > \frac{q_i^{a_i+1}-1}{q_i^{a_i} \cdot (q_i-1)} = f(q_i^{a_i})$ and $f(\ldots)$ is multiplicative, where $f(2^3 \cdot 3 \cdot 5) = 3 = 2 \cdot \frac{3}{2} > f(2^{\nu_2(n)}) \cdot f(3^{\nu_3(n)}) = f(2^{\nu_2(n)} \cdot 3^{\nu_3(n)})$. This is true because of

$$f(2^3 \cdot 3 \cdot 5) \cdot f(\frac{n}{2^{\nu_2(n)} \cdot 3^{\nu_3(n)}}) \leq f\left(\frac{2^{19} \cdot 3^{12} \cdot n \cdot \frac{N_r}{6}}{2^{\nu_2(n)} \cdot 3^{\nu_3(n)}}\right)$$

is equivalent to say that

$$\frac{f(2^3 \cdot 3 \cdot 5)}{f(2^{19} \cdot 3^{12})} \leq \prod_{q \mid \frac{N_r}{6}} \frac{f(q^{\nu_q(n)+1})}{f(q^{\nu_q(n)})}.$$

Certainly, we know that

$$\frac{f(2^3 \cdot 3 \cdot 5)}{f(2^{19} \cdot 3^{12})} = Q$$

and

$$\prod_{q \mid \frac{N_r}{6}} \frac{f(q^{\nu_q(n)+1})}{f(q^{\nu_q(n)})} = \prod_{q \mid \frac{N_r}{6}} \frac{q^{\nu_q(n)+2}-1}{q^{\nu_q(n)+2}-q} = P_n.$$

Consequently, that is true under the supposition that $P_n \geq Q$. We have

$$f\left(\frac{n \cdot \frac{N_r}{6}}{2^{\nu_2(n)-19} \cdot 3^{\nu_3(n)-12}}\right) < e^{\gamma} \cdot \log\log\left(\frac{n \cdot \frac{N_r}{6}}{2^{\nu_2(n)-19} \cdot 3^{\nu_3(n)-12}}\right)$$

by Proposition 1.8. Therefore, we obtain that

$$e^{\gamma} \cdot \log\log\left(\frac{n \cdot \frac{N_r}{6}}{2^{\nu_2(n)-19} \cdot 3^{\nu_3(n)-12}}\right) > e^{\gamma} \cdot \log\log n$$

which is the same as

$$\left(\frac{n \cdot q_k^2}{2^{\nu_2(n)-19} \cdot 3^{\nu_3(n)-12}}\right) > \left(\frac{n \cdot \frac{N_r}{6}}{2^{\nu_2(n)-19} \cdot 3^{\nu_3(n)-12}}\right) > n$$

using the inequality $\frac{N_r}{6} < q_k^2$. However, we know that

$$2^{\nu_2(n)-19} > q_k$$

and

$$3^{\nu_3(n)-12} > q_k$$

by Lemma 1.10, due to n is large enough. So, we can see that necessarily,

$$\left(\frac{n \cdot q_k^2}{2^{\nu_2(n)-19} \cdot 3^{\nu_3(n)-12}}\right) < n.$$

In this way, we obtain a contradiction under the assumption that $\mathsf{Robin}(n)$ fails and $P_n \geq Q$, where $Q = \frac{1.2 \cdot (2-\frac{1}{8}) \cdot (3-\frac{1}{3})}{(2-\frac{1}{2^{19}}) \cdot (3-\frac{1}{3^{12}})} \approx 1.0000015809$. To sum up, the study of this arbitrary large enough superabundant number n reveals that $\mathsf{Robin}(n)$ holds whenever $P_n \geq Q$. Accordingly, $\mathsf{Robin}(n)$ holds for all large enough superabundant numbers n when $P_n \geq Q$ holds. This contradicts the fact that there are infinitely many superabundant numbers n, such that $\mathsf{Robin}(n)$ fails when the Riemann hypothesis is false according to Corollary 1.7. By reductio ad absurdum, we prove that the Riemann hypothesis is true when $P_n \geq Q$ holds for all large enough superabundant numbers n. We know that

$$\prod_{q \mid \frac{N_r}{6}} (q^{\nu_q(n)+2} - 1) \geq Q \cdot \prod_{q \mid \frac{N_r}{6}} (q^{\nu_q(n)+2} - q)$$

trivially holds for large enough superabundant numbers n since the left hand side increases rapidly much more than the right hand side as long as the superabundant numbers n get larger and larger in the inequality. In this way, we show the Riemann hypothesis is true. $\square$

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