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Characterizations that help explain particle and cosmic data

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Abstract

This paper suggests explanations for otherwise seemingly unexplained data about elementary particles and cosmology. The explanations have bases in coordinate-based modeling and in integer-based characterizations for some catalogs. One catalog features properties - including charge, mass, and spin - of objects. Another catalog features all known and some possible elementary particles. Assumptions include that multipole-expansion mathematics have uses regarding long-range interactions - such as gravity - and that nature includes six isomers of all elementary particles other than long-range-interaction bosons. One isomer associates with ordinary matter. Five isomers associate with dark matter. Multipole notions help explain large-scale aspects such as the rate of expansion of the universe.

Keywords: elementary particles, dark matter, rate of expansion of the universe, galaxy formation, neutrino masses, vacuum energy, dark energy, quantum gravity

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1. Introduction

This unit discusses the notion of characterizations. This unit discusses the notion of developing characterizations for catalogs of attributes that associate with physics models. This unit describes data that - currently - models seem not to explain. This unit suggests goals - regarding characterizations for attributes and regarding explaining data - for work that this paper discusses.

1.1. Characterizations

This unit discusses the notion of characterizations.

This paper suggests and uses integer-based characterizations for items in catalogs of attributes. One such catalog is a catalog of properties of objects, for which the attributes include properties such as charge, magnetic moment, mass, and spin. Another such catalog is a catalog of elementary particles, for which the attributes are families of elementary particles, such as the three-particle family of charged leptons, the three-particle family of neutrinos (or, zero-charge leptons), and the one-particle family that includes just the photon.

Such integer-based characterizations associate with mathematical notions of characterization. (Ref. [1] discusses characterization.) A characterization of an object is a set of conditions that, while different from the definition of the object, is logically equivalent to it.

In this paper, characterization schemes feature solutions to integer-arithmetic equations.

For example, for a catalog of families of elementary particles, the following notions pertain. If the integer 4 appears in the equation that is relevant to a family, each one of the particles in the family has zero charge. If the integer 4 does not appear in the equation that is relevant to a family, each one of the particles in the family has nonzero charge. If the integer 6 appears in the equation that is relevant to a family, each one of the particles in the family is a fermion. If the integer 6 does not appear in the equation that is relevant to a family, each one of the particles in the family is a boson.

1.2. Characterizations regarding physics catalogs and models

This unit discusses the notion of developing characterizations for catalogs of attributes that associate with physics models.

This paper assumes that society (or, people collectively) is an observer of nature (or, the universe) and that society's inferences about nature evolve.

This paper suggests that notions of inferred pertain broadly. For example, society does not directly experience (astrophysical) stars, but society (and equipment and other nearby objects, such as animals and the Earth) experiences effects that people infer as associating with inferred contributions - to, for example, an inferred overall electromagnetic field and to an inferred overall gravitational field - that associate with inferred stars.

Examples of themes for catalogs of inferences include the following.

- Inferred types of entities (such as objects or fields) and specific entities.
- Properties (such as energy, charge, or spin) that people infer as associating with objects or fields.
- Interactions (such as gravitational interactions or electromagnetic interactions) that people infer as occurring between objects or fields.
- Models (such as models that associate with Newtonian dynamics, electrodynamics, or special relativity) that people infer as interrelating entities, properties, interactions, and changes to entities and to values of properties of entities.
- Data (such as collected by people or equipment) that people interpret in contexts that can associate with themes (including data) in this list of themes.

This paper uses the two-word phrase inferred attributes to refer to inferred entities, inferred properties, and inferred interactions.

Below, this paper tends to de-emphasize using the word inferred (and synonyms of the word inferred), even though the notion of inferred applies ubiquitously in this paper.

Regarding the themes above, catalogs of attributes tend to feature discrete items. Models tend to associate with at least one of (a) use of a continuous temporal coordinate and (b) use of at least one continuous spatial coordinate. Equations pertaining to rates of decay of objects can exhibit (a) directly. Equations of motion can exhibit (a) and (b) directly. The following sentences reprise some themes regarding some such modeling. Objects exist and exhibit properties. Objects interact. Objects change. (For example, values of properties of objects change.) In many cases, a model associates with discrete lists of attributes (such as a list of objects or of types of objects, a list of properties, and a list of interactions) and with a set of discrete outputs (such as energy states, decay rates, or trajectories.)

The following notions can pertain regarding a catalog of attributes.

- The catalog associates with a collection that seems not to exhibit adequately meaningful patterns.
- The catalog associates with a collection that seems to exhibit meaningful patterns.
- The catalog associates with a collection for which at least one seemingly meaningful characterization scheme pertains.

Catalogs and models can co-evolve. A once seemingly unpatterned catalog of chemical elements and a now characterized catalog of isotopes exhibits such an evolution. Some pattern-centric evolutionary steps associate with cataloging chemical elements via similarities regarding chemical interactions and via atomic weights. (Ref. [2] proposes such cataloging.) Eventually, people developed the topic of nuclear physics and, within nuclear physics, modeling that suggests isotopes that nature might include. Some characterization-centric steps associate with cataloging isotopes via atomic number (or, number of protons), which is an integer, and via number of neutrons, which is an integer.

Regarding the evolution of physics, the following steps can pertain.

- People notice data that might suggest unexplained patterns.
- People propose patterns that the data might exhibit.
- People propose characterizations that seem to associate with the patterns or propose models that seem to output the patterns.

Table 1: Relationships between physics aspects, MOD, and CAT+. The leftmost column lists physics aspects. The symbol † denotes the two-word phrase list of. The symbol ‡ denotes the four-word phrase list of types of. In the middle column, the word assumes associates with the notion of inputs to MOD models and with notions of assumptions that are not necessarily based on outputs from MOD models. In the rightmost two columns, the word suggest associates with the notion of outputs and notions that use of respective (MOD or CAT+) techniques does suggest or might suggest relevant outputs. In the rightmost column, the three-word phrase suggest notions regarding suggests that CAT+ might, for example, suggest relevant numbers of spatial coordinates.

Aspects	MOD ... the aspects.	CAT+ might ... the aspects.
† Properties of objects	Assumes	Suggest
† Fields	Assumes	Suggest
‡ Interactions between objects and fields	Assumes	Suggest
† Families of elementary particles	Assumes	Suggest
‡ Objects	Assumes or suggests	Suggest
Values of properties of objects or of fields	Assumes or suggests	Suggest
MOD coordinate systems	Assumes	Suggest notions regarding
MOD models	Uses	Suggest bases for
Motions of objects	Suggests	-
Rates of changes of properties of objects	Suggests	-

- People propose principles that might underlie the characterizations or models.
- People propose reuses for the principles, characterizations, or models.

A goal for physics can be increased consistency within collections of catalogs and models. Other goals can include increased breadth, increased accuracy, and increased usefulness.

This paper uses the following acronyms.

- MOD - Modeling that associates with space-time coordinates.
- CAT - Cataloging that associates with - or might eventually associate with - patterns or characterizations.
- CAT+ - Cataloging that associates with characterizations.
- PAT - Pattern matching, including pattern matching that develops CAT+.

Table 1 suggests relationships between physics aspects, MOD, and CAT+.

The rightmost column in Table 1 provides motivation for work that this paper discusses. The work suggests integer-based characterizations that pertain regarding catalogs that associate with the first six rows of Table 1.

In CAT+, a set of integers can characterize an item in a catalog.

1.3. Unexplained data

This unit describes data that - currently - models seem not to explain.

1.3.1. Elementary particles

This unit discusses some particle data that MOD seems not to explain.

MOD seems not to explain the current catalog of elementary particles.

1.3.2. Ratios of not-ordinary-matter effects to ordinary-matter effects

This unit discusses some cosmology data that physics seems not to explain.

Physics suggests ratios of effects that seem not to link to ordinary matter to effects that seem to link to ordinary matter.

Table 2 lists observed ratios of NOM (as in not-ordinary-matter) effects to OM (as in ordinary-matter) effects. (The following three sentences pertain regarding Table 2a. Refs. [3, 4, 5, 6] provide data and discussion regarding the amount of cosmic optical background. Refs. [7, 8, 9] provide data and discussion regarding absorption of CMB. The acronym CMB abbreviates the four-word phrase cosmic microwave background radiation. The following sentences pertain regarding Table 2b. Refs. [10, 11] provide data and discussion. Ref. [10] influenced the choice - that this paper reflects - of a time range to associate with the word early. Regarding the combination of 0^{+}:1 and later, Refs. [12, 13, 14, 15, 16, 17, 18]

Table 2: Observed ratios of not-ordinary-matter effects to ordinary-matter effects. The notation NOM:OM abbreviates the seven-element phrase ratio of not-ordinary-matter effects to ordinary-matter effects.

(a) Ratios - that pertain to light that dates to about 380,000 years after a so-called Big Bang - of observed effects. The three-word phrase cosmic optical background links to radiation that - recently - measures as optical radiation or measures as close (with respect to wavelengths) to optical radiation. The acronym CMB links to radiation that - recently - measures as cosmic microwave background radiation.

Aspect	NOM:OM
Amount of cosmic optical background	$1:1$
Some absorption of CMB	$1:1$

(b) Ratios - that pertain to some galaxies - of observed effects. Regarding galaxies, the notion of early links to observations that pertain to galaxies that link to high redshifts. High might link to $z > 7$ and possibly to smaller values of z. Here, z denotes redshift. The word later links to the notion that observations pertain to objects later in the history of the universe. Possibly, early NOM galaxies (as in NON:OM $1:0^+$ galaxies) existed and people have yet to detect any such galaxies.

Objects	NOM:OM
Some early galaxies	$0^+:1$
Some later galaxies	$0^+:1$
Some later galaxies	$1:0^+$
Some later galaxies	$\sim 4:1$
Many later galaxies	$5^+:1$

(c) Ratios - that pertain to larger-than-galaxies-scale phenomena - of observed effects. For a galaxy cluster that has collided with another galaxy cluster, a ratio of other than $5^+:1$ might pertain.

Aspect	NOM:OM
Densities of the universe	$5^+:1$
Some galaxy clusters	$5^+:1$

provide data and discussion. Ref. [19] discusses a galaxy that might have started as containing mostly OM. Ref. [20] discusses a NOM-deficient galaxy. Regarding observed NOM:OM $1:0^+$ galaxies, Refs. [21, 22, 23, 24] provide data and discussion. Current techniques might not be capable of observing early NOM:OM $1:0^+$ galaxies. Refs. [25, 26] suggest, regarding galaxy clusters, the existence of clumps of NOM-stuff that might be individual galaxies. Extrapolating from results that Refs. [21, 27] discuss regarding ultrafaint dwarf galaxies that orbit the Milky Way galaxy might suggest that the universe contains many NOM:OM $1:0^+$ later galaxies. Ref. [28] discusses a trail of galaxies for which at least two galaxies have little NOM-stuff. Ref. [28] suggests that the little-NOM galaxies might have resulted from a collision that would have some similarities to the Bullet Cluster collision. Regarding galaxies for which NOM:OM ratios of $\sim$4:1 pertain, Refs. [29, 30] provide data and discussion. Regarding later galaxies for which NOM:OM ratios of 5^+:1 pertain, Ref. [21] provides data and discussion. Refs. [31, 32] provide data about collisions of galaxies. The following two sentences pertain regarding Table 2c. Ref. [33] provides data and discussion regarding densities of the universe. Refs. [34, 35, 36, 37] provide data and discussion regarding galaxy clusters.)

Possibly, the notion that ratios that Table 2 lists are ratios of near-integers associates with a useful pattern.

1.3.3. Large-scale gravitational phenomena

This unit discusses a possible cosmology pattern that physics seems not to explain.

MOD suggests three eras regarding the so-called rate of expansion of the universe. (The notion of a rate of expansion of the universe associates with notions of typical speeds of moving away from each other regarding neighboring large objects, such as galaxy clusters.) The first era (about which there may be no data) would feature a typical speed of moving away that rapidly increases. The second era (about which there is data) features a typical speed of moving away that - while remaining positive - decreases. The third (and current) era (about which there is data) features a typical speed of moving away that increases.

Possibly, notions of such eras point to a possibly useful pattern of swings between repulsion between objects and attraction between (similar) objects.

MOD uses the two-word term dark energy to denote some possible mechanisms that could lead to increases in the typical speed of moving away from each other of large clumps.

MOD underestimates the third-era rate of expansion of the universe. (Refs. [38, 39, 40, 41, 42, 43, 44, 45] provide further information. Ref. [46] suggests that the notion that dark matter is like ordinary matter might help resolve the relevant tension. Ref. [47] discusses various possible resolutions. Ref. [48] provides data about the Hubble constant.)

MOD overestimates so-called large-scale clumping of matter. (Refs. [41, 49, 50, 51] provide data and discussion. Ref. [52] provides further information.)

Possibly, notions of such miscalculations associate with aspects of repulsion and attraction between large objects.

1.4. Goals for work that this paper discusses

This unit suggests goals - regarding characterizations for attributes and regarding explaining data - for work that this paper discusses.

This paper discusses work for which the following goals pertain.

- Produce characterizations that link types of objects and fields, properties that associate with objects and fields, interactions between objects and fields, and aspects of modeling.
- Use the characterizations to help explain data that - as of now - modeling alone seems not to explain.

2. Methods

This unit develops CAT+ techniques, interrelates CAT+ with familiar aspects of MOD, discusses interpretations of some sets of CAT+ so-called solution-pairs, and discusses the notion that dark matter might associate with five additional (with respect to the one isomer that associates with ordinary matter) isomers of most elementary particles.

This unit features the notion that CAT+ solution-pairs seem to echo - and to characterize - familiar aspects of MOD. PAT suggests methods that lead to associating CAT+ solution-pairs with aspects of MOD.

2.1. Integer-arithmetic equations that associate with CAT+

This unit discusses integer-arithmetic equations that associate with CAT+ characterizations.

The next paragraph discusses mathematics and not necessarily physics. (Notation from set theory that this paper uses includes the following. $\{a, b, \cdots\}$ denotes the set that has members a, b, and so forth. $a \in b$ denotes that a is a member of set b. $a \notin b$ denotes that a is not a member of set b. $a \subset b$ denotes that set a is a subset of set b. $a \cap b$ denotes the largest set for which each member is a member of each one of set a and set b. $\emptyset$ denotes the set that includes no - as in zero - members. Other mathematics notation that this paper uses includes the following. $a \Rightarrow b$ denotes that, if a pertains, b pertains. $a \Leftrightarrow b$ denotes that a pertains if and only if b pertains.)

The following rules pertain.

- Each term in a sum (of terms) has the form ks_k. k is a positive integer. One of $s_k = +1$ and $s_k = -1$ pertains. k appears no more than once in the sum.
- Eq. (1) depicts a sum. K denotes the subset of Z_+ (as in the set of all positive integers) for which one of $s_k = +1$ and $s_k = -1$ pertains.

$$s = \sum_{k \in K} ks_k \tag{1}$$

- Eq. (2) defines the symbol n_k.

$$n_k = \text{ the number of } k \text{ in the set } K \tag{2}$$

- CAT+ links the word solution (as in solution to an integer-arithmetic equation) to a set of integers s, k, and s_k that satisfy Eq. (1).

- For each solution, there is exactly one second solution for which s is the negative of the s for the first solution and each s_k is the negative of the respective s_k for the first solution. CAT+ links the one-element term solution-pair to such a pair of a first solution and the second solution. For a solution-pair, one nonnegative value of $|s|$ pertains. (Regarding showing arithmetic pertaining to a solution-pair for which - for each solution - $k_{max \in K}$ denotes the maximum k in K, this paper adopts a convention of showing the solution for which $s_{k_{max \in K}} = +1$ and not showing the solution for which $s_{k_{max \in K}} = -1$. For example, for $K = \{1\}$, this paper might show the expression $1 = |s| = |+1|$ and would not show the expression $1 = |s| = |-1|$.)

- CAT+ links the word cascade (and the phrase one-step cascade) to forming a new (as in second) K by adding one (new) positive integer to an original (as in first) K. For a one-step cascade that starts with a K for which $n_k = l_1$ pertains, $n_k = l_1 + 1$ pertains for the second K. CAT+ links the two-element phrase multi-step cascade to multiple uses of the notion of a one-step cascade. At each step n_k increases by one.

- This paper extends the notion of cascade to refer to solution-pairs. With respect to one solution-pair, the notions of one $|s|$ and one K pertain. A one-step cascade regarding that K results in a new K for which - across the one-step cascade solution-pairs - more than one $|s|$ pertains. (For example, the solution-pair that associates with $1 = |s| = |+1|$ cascades in one step to the two solution-pairs that associate with, respectively, $1 = |s| = |-1+2|$ and $3 = |s| = |+1+2|$.) Except where this paper makes a specific statement such as non-same-$|s|$, the notion of one-step cascade solution-pair links to the notions of the original $|s|$ and the new K. (For the example, $3 = |s| = |+1+2|$ does not equal the original $|s|$ - as in $|s| = 1$ - and - absent a reference to non-same-$|s|$ - would not pertain.)

2.2. MOD notions that associate with long-range interactions

This unit discusses aspects regarding so-called LRI (as in long-range-interaction) fields and regarding properties - of objects - with which LRI fields interact.

MOD currently considers two LRI fields - an electromagnetic field and a gravitational field.

The following notions pertain regarding MOD.

- MOD links the electromagnetic field to two modes - left-circular polarization and right-circular polarization - and to a spin of one (in units of $\hbar$).

- MOD suggests that the electromagnetic field conveys to an object-O (as in observer object) information about an object-I (as in inferred object).

- MOD includes the notion that an object-O can characterize contributions (that link to object-I) to the electromagnetic field in terms of contributions (that link to object-I) to an inferred (by object-O) electric field and contributions (that link to object-I) to an inferred (by object-O) magnetic field.

- From the point of view of an object-O and regarding an object-I, the following notions pertain. (Here, for MOD in which object-I models as point-like, the word position links to a specific position. For MOD in which object-I models as point-like, the word time links to a specific time. For MOD in which object-I does not model as point-like, position can link to a region and time can link to a time range.)
  - The inferred electric field that object-O links to object-I links to a property of inferred charge q (of object-I), an inferred position (of object-I), and an inferred time that links the charge and position.
  - The inferred magnetic field that links to object-I links to the following properties.
    * An inferred magnetic moment μ (of object-I), an inferred position (of object-I), and an inferred time that links the magnetic moment and position. For some modeling (for example, regarding an object-I that models as having a rigid distribution of charge), the inferred magnetic moment links to the inferred charge (of object-I) and an inferred angular velocity (of object-I).
    * An inferred charge current I (of object-I), an inferred position (of object-I), and an inferred time that links the charge current and position. The inferred charge current links to the inferred charge (of object-I) and an inferred velocity (of object-I).

- MOD regarding a point-like object links the inferred electric field to the two-word phrase monopole potential.

- Similar notions might pertain regarding the gravitational field. (Refs. [53, 54] discuss gravitoelectromagnetism, which suggests parallels between gravitation and electromagnetism.) For gravitation, the spin is two and, for example, the monopole potential associates with the property of energy (or, for Newtonian dynamics, rest-energy).

- ND (as in Newtonian dynamics) aspects of MOD can link a total potential to a sum of a monopole (contribution to the total potential) potential, a dipole (contribution to the total potential) potential, a quadrupole (contribution to the total potential) potential, and so forth.
  - The monopole potential can link to a radial (as in distance r away from an object-I) r^{-1} spatial dependence. The dipole potential can link to a radial (as in distance r away from an object-I) r^{-2} spatial dependence. The quadrupole potential can link to a radial (as in distance r away from an object-I) r^{-3} spatial dependence. And so forth. (For a dipole potential, a quadrupole potential, and so forth, angular dependence can also - along with radial dependence - pertain.)
  - Regarding a scalar property (such as charge, which links to electromagnetism), the following can pertain.
    * For a system (of objects) that models as having exactly one point-like object, a monopole potential can pertain. The point links to a MOD notion of zero dimensions.
    * For a system (of objects) that models as having exactly two identical point-like objects, a dipole potential can pertain. The line that links the two objects links to a MOD notion of one dimension.
    * For a system (of objects) that models as having exactly four identical objects arrayed as the corners of a square, a quadrupole potential can pertain. The square links to a MOD notion of two dimensions.
    * For a system (of objects) that models as having exactly eight identical objects arrayed as the corners of a cube, an octupole potential can pertain. The cube links to a MOD notion of three dimensions.

2.3. Some associations between CAT+ and MOD

This unit posits that some $|s| > 0$ CAT+ solution-pairs associate with properties (of objects) with which LRI fields interact and that some $|s| = 0$ CAT+ solution-pairs associate with specific objects.

PAT suggests the following MOD notions regarding an observer object-O and an inferable (by object-O) object-I.

- The electromagnetic field (that object-O links to object-I) links to two modes, each one of which links to a magnitude (as in $|s| = 1$) of spin s (in units of $\hbar$). CAT+ (arbitrarily, but without unintended consequences) links $s = +1$ to the left-circular-polarization mode and links $s = -1$ to the right-circular-polarization mode.

- Regarding ND and MOD that considers object-I to be point-like, the electric field (that object-O links to object-I) links to a position (that object-O infers regarding object-I), to a monopole (as in r^{-1}) potential and to one (as in q) property that does not link directly to properties (such as position) that link directly to space-time-coordinates.

PAT extends the above notions as follows.

- Notions of monopole potentials, dipole potentials, and so forth are not directly relevant within CAT+. (CAT+ does not necessarily need to directly discuss the dynamics of objects. CAT+ should be invariant with respect to MOD choices between, for example, ND and SR - as in special relativity. CAT+ does not necessarily need to directly discuss the excitation of fields. CAT+ should not disturb MOD modeling regarding excitations of fields and regarding de-excitations of fields.)

- Fields with integer spins of $|s| \geq 2$ can pertain. For example, for gravity, CAT+ links $s = +2$ to the left-circular-polarization mode and links $s = -2$ to the right-circular-polarization mode. (Refs. [55, 56, 57] discuss notions of gravitational circular polarization.)

- Some parallels between electromagnetic properties of objects and gravitational properties of objects pertain. For example, PAT suggests that Eq. (3) illustrates a relevant pattern regarding three MOD electromagnetic properties of objects and three MOD gravitational properties of objects. The following notation pertains. E denotes energy. P denotes momentum. s_{IAM} denotes spin (as in intrinsic angular momentum). (This paper uses the symbol s_{IAM} to denote spin and to not associate spin with the symbol s.) MOD associates each one of q and E with the word scalar. MOD associates each one of I, μ, E, and s_{IAM} with the word vector. (In MOD general relativity, so-called frame-dragging can associate with intrinsic angular momentum.) The symbol : links to the two-word phrase is to and pertains to the two symbols that bracket the symbol :. The symbol :: links to the word as and pertains to the two trios that bracket the symbol ::. (One difference between electromagnetism and gravity is that q for an object can be nonpositive but E for an object is positive.)

$$q : I : \mu :: E : P : s_{IAM} \tag{3}$$

- The CAT+ symbols 1x, 2x, and so forth associate with $|s| \geq 1$ and with MOD notions of position (and time), velocity (and time), and so forth.
  - The symbol 1x links to inferred properties - such as q (as in charge), μ (as in magnetic moment), E (as in energy) and s_{IAM} (as in spin) - that link (in the sense of discussion above) to inferred position but that do not necessarily link to inferred linear velocity.
  - The symbol 2x links to inferred properties - such as I (as in charge current) and P (as in momentum) - that link (in the sense of discussion above) to inferred position and to inferred (linear) velocity.
  - Each one of 1x and 2x does not necessarily link to inferred (linear) acceleration.
  - The symbol 3x links to the hypothetical possibility of inferred properties that link to inferred position, inferred (linear) velocity, and inferred (linear) acceleration. MOD suggests that the notion of an unchanging object and the notion of a nonzero 3x property are not compatible with each other. For example, a linearly accelerating charged object-I changes at least one of inferred energy and inferred (linear) momentum. A related notion is that inferred acceleration links to the notion that an inferred object-I - itself - would not link to at least one of conservation of energy and conservation of momentum. A related notion is that an inferred object-I models as being part of a system - of inferred objects and/or inferred fields - that includes more than just object-I and inferred fields that link to object-I.
  - The symbol 1x links to the MOD notion of zero derivatives (with respect to time) of position. The symbol 2x links to the MOD notion of one derivative (with respect to time) of position. The symbol 3x links to the MOD notion of two derivatives (with respect to time) of position.

The CAT+ symbols 1f and 2f associate with $|s| = 0$. The symbols 1f and 2f can associate with MOD notions of transitions (such as decays) in which one set of objects becomes another set of objects. (For example, a Z boson can decay into two fermion elementary particles, with each fermion being the antiparticle to the other fermion.) Regarding such a transition, parallels to a MOD three-vertex interaction can pertain. (For example, regarding a decay of a Z boson, 1f can associate with the Z-boson field - which de-excites - and 2f can associate with the two fermion fields - which excite.)

The following notions pertain regarding 1f and 2f and regarding 1x, 2x, and so forth.

- The symbol f in the symbols 1f and 2f might seem to link to MOD notions of temporal. The symbol x in the symbols 1x, 2x, and 3x links to (at least) MOD notions of spatial.
- Whereas 2x (as in velocity) associates with MOD notions of rates of changes regarding 1x (as in position), 2f (as in a set of objects) does not necessarily associate directly with MOD notions of rates of changes from (or to) 1f (as in another set of objects).

2.4. CAT+ expressions that associate with some LRI properties of objects

This unit associates specific uses of specific CAT+ solution-pairs with specific properties that MOD associates with objects and fields.

PAT suggests the following notions.

- For some $|s| \geq 1$, the following notions pertain.

- $|s|$ associates with the spin (in units of $\hbar$) that MOD associates with an LRI (as in long-range interaction) field. $|s| = 1$ associates with electromagnetism. $|s| = 2$ associates with gravitation.
- n_k echoes some aspects of MOD notions of multipole expansions. $n_k = 1$ can associate with monopole. $n_k = 2$ can associate with dipole.

- For some $|s| \geq 1$, some solution-pairs can associate with properties of objects.
- For $|s| = 1$, the following notions pertain regarding some electromagnetic properties of objects.
  - 1x use of $1 = |s| = |+1|$ associates with $n_k = 1$ and with the object property of charge (which MOD associates with electromagnetic interactions).
  - 1x use of $1 = |s| = |-1+2|$ associates with $n_k = 2$ and with the object property of magnetic (dipole) moment (which MOD associates with electromagnetic interactions).
  - 2x use of $1 = |s| = |-1+2|$ associates with $n_k = 2$ and with the object property of charge current (which MOD associates with electromagnetic interactions).
- For $|s| = 2$, the following notions pertain regarding some gravitational properties of objects. (Here, each solution-pair associates with doubling each value of k that associates with the respective - just above - electromagnetic solution-pair.)
  - 1x use of $2 = |s| = |+2|$ associates with $n_k = 1$ and with the object property of energy (which MOD associates with gravitational interactions).
  - 1x use of $2 = |s| = |-2+4|$ associates with $n_k = 2$ and with the object property of spin (which MOD can associate with gravitational interactions).
  - 2x use of $2 = |s| = |-2+4|$ associates with $n_k = 2$ and with the object property of momentum (which MOD associates with gravitational interactions).
- For $|s| = 2$, the following notions pertain regarding some gravitational properties of objects. (Here, each solution-pair does not associate with doubling each value of k that associates with a respective electromagnetic solution-pair.)
  - 1x use of $2 = |s| = |+1-2+3|$ associates with two distinct (axes and associated) moments of inertia and with some object-internal stress-energy.
  - 1x use of $2 = |s| = |-1+2-3+4|$ links to rotation relative to one of the two distinct axes of moment of inertia. 1x $2 = |s| = |-1-2-3+4|$ links to rotation relative to the other one of the two distinct axes of moment of inertia.

Discussion above links to two uses of the terms monopole, dipole, and so forth.

- MOD use: For an object-I that an object-O models as point-like, the following statements pertain. $n_k = 1$ links to monopole potential. $n_k = 2$ links to dipole potential. $n_k = 3$ links to quadrupole potential. And so forth.
- CAT+ use: $n_k = 1$ links to one solution-pair (and to one value of $|s|$). $n_k = 2$ links to two solution-pairs (for which the $|s|$ for one solution-pair does not equal the $|s|$ for the other solution-pair). $n_k = 3$ links to four solution-pairs (and to up to four values of $|s|$). $n_k = 4$ links to eight solution-pairs (and, at least for the case of $K = \{1, 2, 3, 4\}$, to six values of $|s|$). And so forth.

The following notions pertain regarding MOD descriptions of excitations of LRI fields.

- Mathematics of a one-dimensional harmonic oscillator links (in MOD) to aspects regarding the $s = +|s|$ (as in left-circular polarization) mode.
- Mathematics of a one-dimensional harmonic oscillator links (in MOD) to aspects regarding the $s = -|s|$ (as in right-circular polarization) mode.
- CAT+ suggests that individual terms in a sum of the form $1 \leq |s| = |\cdots|$ do not link directly to MOD notions of excitations of modes. Below, this paper discusses notions that some individual terms (that are in or that are lacking from sums of the form $1 \leq |s| = |\cdots|$) link to MOD notions of ground-state symmetries.

Discussion - other than regarding Table 1 - above does not necessarily cover the following.

- Some properties - of objects - that people infer (or might infer) via electromagnetism. One example features (energy-level) states of atoms. Another example features surface temperature or other properties that might associate with electromagnetic thermal radiation.
- Some properties - of objects - that people infer (or might infer) via gravitation. Some examples might associate with MOD notions of possible quantum gravity.
- Properties - of objects - that might associate with LRI fields for which $|s| \geq 3$.
- Properties - of objects - that might associate with non-LRI fields (such as a strong-interaction field or a weak-interaction field) that MOD associates with intermediation by some boson (not-necessarily elementary) particles.
- Types of objects.
- Specific objects that associate with some types - such as elementary particles - of objects.
- MOD modeling that addresses internal aspects of objects.

For an object such as an atom, some MOD models model the object as having one component and some MOD models model the object as having two components. One-component modeling might feature the overall (regarding the atom) properties of mass, charge, and spin. Two-component modeling might feature - as the two components - the electron cloud and the atomic nucleus. Featured properties might include the principal and fine-structure properties of the electron cloud, the charge and spin of the nucleus, and the hyperfine state (that links the spin of the electron cloud and the spin of the nucleus).

CAT+ associates the notion of a component (of a two-component object) with notions of an object or field that associates with nonzero mass. Fields (such as the electromagnetic field) can, in effect, bind two components to each other. CAT+ does not consider a field that binds to each other the two components to be a component of such a two-component object.

CAT+ associates the symbol 1CO with MOD one-component modeling. CAT+ associates the symbol 2CO with MOD two-component modeling.

2.5. CAT+ characterization-centric interpretations of specific sets of integers

This unit lists matches between CAT+ uses of integers and aspects of MOD.

Table 3 discusses relationships between CAT+ expressions and MOD modeling. The relationships stem from pattern matching.

Additionally, PAT suggests the following notions.

- For a set K for which more than one $|s| = 0$ solution-pair exists, the solution-pairs do not have 1f uses.
- For a 2f use of a set K for which more than one $|s| = 0$ solution-pair exists, 1f use of another set K (from which the 2f set cascades) for which one $|s| = 0$ solution-pair exists associates with an even number (either zero or two) of elementary particles.

One might expect Table 3 to evolve, based on further pattern-matching exploration of uses of CAT+. An evolutionary step might feature adding rows or refining the content of existing rows.

2.6. Spin states of two-component systems

This unit provides details regarding the suggestion that CAT+ can provide a basis for insight regarding MOD modeling regarding two-component systems.

Table 3 lists CAT+ expressions that associate with 2CO, 2CO5, and 2CO7.

The following $0 = |s|$ solution-pairs would link to $S = 0$.

- For 2CO, $|+1-3-5+7|$ could link to nonzero charge; $|+2-4-5+7|$ could link to zero charge.
- For 2CO5, $|-1-2-5+8|$ could link to nonzero charge; $|+1-2-4+5|$, $|+1-4-5+8|$, and $|+2-3-4+5|$ could link to zero charge.
- For 2CO7, $|+2-3-7+8|$ could link to nonzero charge; $|-1-2-4+7|$ and $|+3-4-7+8|$ could link to zero charge.

Table 3: Relationships between CAT+ expressions and MOD modeling. The table lists MOD notions that associate with CAT+ expressions. Regarding MOD notions of objects and fields, use of the word field in place of the word object can be appropriate. For cases in which $6 \in K$, use of the word field can be preferred, because MOD suggests that a net change in the number of fermions associates with an even-number change in the number of fermions. (For example, the creation of a matter fermion elementary particle can associate with the creation of an antimatter fermion elementary particle.) The symbol S denotes the magnitude of total spin, as in the MOD expression $S(S+1)\hbar^2$. In MOD, $2S$ is a nonnegative integer.

CAT+ expression	MOD interpretation
$K \subset (\{k\|1 \leq k \leq 8\} \cup \{16, 32, 64, \cdots\})$	-
$1 \leq n_k \leq 2 \Rightarrow \|s\| \geq 1$	-
For $K \neq \emptyset$, $\|s\| = 0$ implies $n_k \geq 3$	-
$\|s\| \geq 1$	Known or possible LRI field
$\|s\| = 0$	State(s) of a type of object
$4 \notin K$	The charge is nonzero
$4 \in K$	The charge is zero or the notion of charge is not relevant
$6 \notin K$	Classical physics or boson physics can pertain
$6 \in K$	Fermion physics pertains
1f, $6 \notin K$	Magnitude of spin (in units of $\hbar$): $S = \|n_k - 4\|$
1f, $6 \in K$	Magnitude of spin (in units of $\hbar$): $S = \|n_k - 4.5\|$
1f, $8 \notin K, \{5, 6, 7\} \cap K \neq \emptyset$	Nonzero-mass object
1f, $\{1, 3, 8\} \subset K, \{5, 6, 7\} \cap K = \emptyset$	Zero-mass boson elementary particle
1f, $\{1, 3, 8, 16\} \subset K, \{5, 6, 7\} \cap K = \emptyset$	Zero-mass LRI boson elementary particle
1f, $\{1, 3\} \subset K, \{5, 7\} \cap K = \emptyset$	1CO (an object models as having only one component)
1f, $\{5, 7\} \subset K$	2CO (an object models as having exactly two components)
1f, $5 \in K, 7 \notin K$	2CO5 - one of the two 2CO components
1f, $7 \in K, 5 \notin K$	2CO7 - the other one of the two 2CO components

For each $S = 0$ solution-pair, one series of one-step cascades starts by adding 6 to K (to provide a solution-pair that links to $S = 0.5$) and continues to fill out the sequence $S = 0.5$, $S = 1.5$, and so forth. For each $S = 0$ solution-pair, one series of one-step cascades fills out the sequence $S = 1$, $S = 2$, and so forth.

Discussion above assumes that two components (for each of which $S \geq 0$) exist. CAT+ suggests the possibility that - for purposes of modeling - some solution-pairs might link to the notion of uninferred (as in not inferred or as in not relevant within the relevant modeling; for example, some MOD classical mechanics modeling does not necessarily include the notions of boson and fermion). For 2CO, $|-2-5+7|$ might link to the notion of uninferred. For 2CO5, $|-2-3+5|$, $|-1-4+5|$, or $|-1-5+6|$ might link to the notion of uninferred. For 2CO7, $|-3-4+7|$ or $|-1-6+7|$ might link to the notion of uninferred.

2.7. Relationships between 1f and 2f uses of some solution-pairs

This unit discusses solution-pairs that might prove useful regarding cataloging elementary particles.

For a positive integer l, CAT+ uses the notation $+l\hat{}$ to denote the series to which Eq. (4) alludes. Each item in the series totals to l.

$$+l\hat{}\ \text{denotes the series}\ +l,\ -l+2l,\ -l-2l+4l,\ -l-2l-4l+8l,\ \cdots \tag{4}$$

CAT+ uses the notation $+l_n\hat{}$ to denote the item in Eq. (4) that includes exactly n terms. For example, $+l_2\hat{}$ denotes , $-l+2l$.

Table 4 discusses some solution-pairs that might have relevance regarding the topic of elementary particles.

The following notions associate with items - in Table 4 - for which the notes (A), (B), and (C) pertain.

One such notion associates with 1f $|-1+2-3-6+8|$, (A), and (B).

Table 5 discusses the two component fields that associate with an S=0.5 solution-pair that does not necessarily directly associate with exactly one elementary particle.

Regarding the other two rows - in Table 4 - that associate with S =0.5 and with (A), the following notions pertain.

CAT+ does not (yet) suggest usefulness for a parallel, that would start with 1f $|-1+3-4-6+8|$, to Table 5, which starts with 1f $|-1+2-3-6+8|$. The solution-pair $|-4-5-6+7+8|$ would associate with 2CO. The two 1f solution-pairs $|-1+4-5-6+8|$ and $|-3-4+5-6+8|$ would associate with

Table 4: Some solution-pairs that might have relevance regarding the topic of elementary particles. The leftmost column lists solution-pairs. The next four columns provide interpretations that CAT+ links to the 1f solution-pairs. Q denotes the magnitude of the charge (q) in units of $|q_e|$, for which q_e is the charge of an electron. In this table, Q is an integer. (The 1f column does not list solution-pairs that would associate with quarks.) m denotes mass. S denotes spin (in units of $\hbar$). Each 2f solution-pair cascades - in one step - from the corresponding 1f solution-pair or solution-pairs. The symbol ; separates cascades from each other. Eq. (4) and related remarks define notation of the form $+16_n\hat{}$. The 2f notation $|\pm(-1-3+4)-2-6+8|$ links to 2f use of the two solution-pairs $|-1-2-3+4-6+8|$ and $|+1-2+3-4-6+8|$. (Here, $-1-3+4=+1+3-4=0$.) In this table, each non-2CO 1f solution-pair for which $n_k \geq 4$ cascades (in n_k-3 steps) from (at least) one of the solution-pair $|-1-2+3|$ and the solution-pair $|-1-3+4|$. The following notes pertain. (A) Two 2f solution-pairs pertain and (per discussion related to Table 3) the 1f solution-pair does not associate with exactly one elementary particle. (B) The previous row points to the possibility for two elementary particles. Each one of the two components that associate with the present row might associate with the field for an elementary particle. (C) The notion of two 2f solution-pairs can associate (per discussion related to Table 3) with the notion of fields that associate with two elementary particles.

1f	Q	m	S	Notes	2f
$\|-1-2+3\|$	>0	>0	1	1CO	$\|-1-2-3+6\|$; $\|+1-2-3+4\|$.
$\|-1-3+4\|$	=0	>0	1	1CO	$\|+1-3-4+6\|$; $\|+1-2-3+4\|$; $\|-1-3-4+8\|$.
$\|-1-2-3+6\|$	>0	>0	0.5	1CO	$\|-1+2-3-4+6\|$; $\|-1+2-3-6+8\|$.
$\|+1-3-4+6\|$	=0	>0	0.5	1CO	$\|-1+2-3-4+6\|$; $\|-1+3-4-6+8\|$.
$\|+1-2-3+4\|$	=0	>0	0	1CO	$\|-1+2-3-4+6\|$; $\|+1-2-3-4+8\|$.
$\|-1-3-4+8\|$	=0	=0	0	1CO	$\|-1+3-4-6+8\|$; $\|+1-2-3-4+8\|$; $\|-1-3-4-8+16\|$.
$\|-1+2-3-6+8\|$	>0	>0	0.5	1CO (A)	$\|\pm(-1-3+4)-2-6+8\|$.
$\|-2-5+6-7+8\|$	>0	>0	0.5	2CO (B)	
$\|-1+3-4-6+8\|$	=0	>0	0.5	1CO (A)	$\|\pm(-1-3+4)-2-6+8\|$.
$\|-4-5-6+7+8\|$	=0	>0	0.5	2CO (B)	
$\|-1+2-3-4+6\|$	=0	>0	0.5	1CO (A)	$\|\pm(-1-3+4)-2-6+8\|$.
$\|-2-4+5-6+7\|$	=0	>0	0.5	2CO (B)	
$\|+1-2-3-4+8\|$	=0	=0	1	1CO (A) (C)	$\|\pm(-1-3+4)-2-6+8\|$.
$\|-1-3-4-8+16_S\hat{}\|$	=0	=0	Integer S, with $S \geq 1$		$\|-1-3-4-8+16_{(S+1)}\hat{}\|$; $\|+1-2-3-4-8+16_S\hat{}\|$; $\|-1+3-4-6-8+16_S\hat{}\|$.

Table 5: The two component fields that associate with an S=0.5 solution-pair that does not necessarily directly associate with exactly one elementary particle. The first row associates with treating an original solution-pair (that would associate - for some MOD modeling - with being a single object) with a next 1f solution-pair, which would associate - for some MOD modeling - with the object having two components. Each one of the next two rows shows solution-pairs that would associate with a field for one of the two components. The letter-string interp abbreviates the word interpretation. An interpretation pertains to the nearest 1f solution-pair to the left of the interpretation.

Original 1f solution-pair	Interp.	Next 1f solution-pair	2f solution-pair	Interp.
$\|-1+2-3-6+8\|$	1CO	$\|-2-5+6-7+8\|$	$\|-2-5+6-7-8+16\|$	2CO
$\|-2-5+6-7+8\|$	2CO	$\|+1+2-5-6+8\|$	$\|+1+2-5-6-8+16\|$	2CO5
$\|-2-5+6-7+8\|$	2CO	$\|+2+3-6-7+8\|$	$\|+2+3-6-7-8+16\|$	2CO7

2CO5. The two 1f solution-pairs $|+1-4-6-7+8|$ and $|-3-4+6-7+8|$ would associate with 2CO7. CAT+ does not (yet) include any examples of 1f associating with more than one solution-pair.

CAT+ does not (yet) suggest usefulness for a parallel, that would start with 1f $|-1+3-4-6+8|$, to Table 5, which starts with 1f $|-1+2-3-4+6|$. The solution-pair $|-2-4+5-6+7|$ would associate with 2CO. The two 1f solution-pairs $|+1+2-4-5+6|$ and $|-2-3+4-5+6|$ would associate with 2CO5. The two 1f solution-pairs $|+1+2-4-6+7|$ and $|-2-3+4-6+7|$ would associate with 2CO7. CAT+ does not (yet) include any examples of 1f associating with more than one solution-pair.

Regarding 1f $|+1-2-3-4+8|$, (A), and (C), CAT+ suggests that the existence of the 2f solution-pairs associates with the notion that 1f $|+1-2-3-4+8|$ associates with two 1CO elementary particles.

Regarding the possibility of 2CO 1f $|-2-4+5-7+8|$, the following notions pertain.

- A direct parallel to Table 5 seems not to be relevant to physics. 2CO5 would associate with $|-1+2-4-5+8|$ and $|-2+3-4-5+8|$. 2CO7 would associate with $|+1+2-4-7+8|$ and $|-2-3+4-7+8|$.
- If one replaces, in the four would-be 2CO5 and 2CO7 solution-pairs, the $4 \in K$ by $6 \in K$, the following notions pertain. 2CO5 associates with 1f $|+1+2-5-6+8|$ and 2f $|+1+2-5-6-8+16|$. 2CO7 associates with 1f $|+2+3-6-7+8|$ and 2f $|+2+3-6-7-8+16|$.
- 2CO 1f $|-2-4+5-7+8|$ associates with $Q=0$, $m>0$, and even-S particles. Each particle would associate with excitements of two distinct odd-S fields.

2.8. An assumption regarding a specification for dark matter

This unit states an assumption - regarding a specification for dark matter - that underlies work in this paper.

The two-word term dark matter arose in conjunction with data that link to the galaxy cluster row in Table 2c and data that link to the many later galaxies row in Table 2b. The term dark-matter galaxy links to the NON:OM $1:0^{+}$ row in Table 2b.

Physics proposes and debates notions that link to the two-word term dark matter. Some MOD modeling might suggest that gravitational phenomena might explain (without needing to consider matter other than ordinary matter) some of the ratios that Table 2 lists. (Ref. [58] provides an example.) Some MOD assumes that NOM links to dark matter being stuff (as in matter that interacts via gravity). MOD proposes (various) general characteristics (beyond the characteristic of interacting with gravity) of dark matter. MOD suggests ranges of properties of dark matter. Possibly, people consider that no data links directly to suggested general characteristics (beyond the characteristic of interacting with gravity) or specific properties.

Possibly, patterns link to the notion that ratios that Table 2 lists are ratios of near-integers.

This paper makes the following assumption, which seems compatible with data that Table 2 presents.

- Nature includes six isomers of most elementary particles. Stuff that has bases in five isomers underlies dark-matter effects (as in not-ordinary-matter effects).

3. Results

This unit suggests explanations for data that MOD alone seems not to explain.

3.1. Elementary particles

This unit discusses an integer-characterized catalog of all elementary particles of which people know or that CAT+ suggests.

CAT+ uses the following notions to catalog elementary particles. A symbol of the form SΦ links to a so-called family of elementary particles. Each elementary particle links to one family. Each family links to one of one, three, or eight elementary particles. For a family, the value S denotes the spin (in units of $\hbar$) for each elementary particle in the family. S links to the MOD expression $\mathrm{S}(\mathrm{S}+1)\hbar^2$ that links to spin. Regarding MOD and known elementary particles, values of S are 0, 0.5, and 1. The symbol Φ links to a symbol of the form X_Q, in which X is a capital letter and Q is the magnitude of the charge (in units of $|q_e|$, in which q_e denotes the charge of an electron) for each particle in the family. For cases for which $Q=0$, CAT+ omits - from the symbols for families - the symbol Q. Regarding quarks, CAT+ uses the symbol $\mathrm{Q}_{>0}$ to link to cases for which either one of $\mathrm{Q}_{1/3}$ or $\mathrm{Q}_{2/3}$ pertains.

Regarding elementary particles, the following notions pertain regarding 1f uses of solution-pairs and regarding 2f uses of solution-pairs.

Table 6: All known elementary particles and some elementary particles that CAT+ suggests nature might include. Each 1f solution-pair appears as a 1f solution-pair in one of Table 4 and Table 5. Each 2f solution-pair cascades - in one step - from the corresponding 1f solution-pair. The leftmost column lists 1f solution-pairs. The next five columns provide interpretations that CAT+ links to the 1f solution-pairs. Q denotes a magnitude of charge (in units of $|q_e|$, in which q_e denotes the charge of an electron). m denotes mass. n_{EP} denotes the number of elementary particles in a family. The symbol † denotes that the elementary particles are as-yet unfound. The three charged leptons are the electron, the muon, and the tau. The word inflaton links to MOD notions of a possible inflaton elementary particle. For one of the two rows that uses the symbol l_Q, $l_Q = 1$. For the other one of the two rows that uses the symbol l_Q, $l_Q = 2$. 2L cascades from 1L, 3L cascades from 2L, and so forth. The acronym TBD abbreviates the three-word phrase to be determined. Eq. (4) and related remarks define notation of the form $+16_n\hat{}$. The 2f notation $|\pm(-1-3+4)-2-6+8|$ links to 2f use of the two solution-pairs $|-1-2-3+4-6+8|$ and $|+1-2+3-4-6+8|$. (Here, $-1-3+4 = +1+3-4 = 0$.) Discussion related to Eq. (46) suggests that 2f $|\pm(-1-3+4)-2-6+8|$ associates with nine elementary particles. In this table, except regarding quarks, each 1f solution-pair for which $n_k \geq 4$ cascades (in $n_k - 3$ steps) from (at least) one of solution-pair $|-1-3+4|$ and solution-pair $|-1-2+3|$.

1f	Q	m	Names	Families	n_{EP}	2f
$\vert-1-2+3\vert$	1	>0	W	$1W_1$	1	$\vert-1-2-3+6\vert$; $\vert+1-2-3+4\vert$.
$\vert-1-3+4\vert$	0	>0	Z	1Z	1	$\vert+1-3-4+6\vert$; $\vert+1-2-3+4\vert$; $\vert-1-3-4+8\vert$.
$\vert-1-2-3+6\vert$	1	>0	Charged leptons	$0.5C_1$	3	$\vert-1+2-3-4+6\vert$; $\vert-1+2-3-6+8\vert$.
$\vert+1-3-4+6\vert$	0	>0	Neutrinos	0.5N	3	$\vert-1+2-3-4+6\vert$; $\vert-1+3-4-6+8\vert$.
$\vert+1-2-3+4\vert$	0	>0	Higgs boson	0H	1	$\vert-1+2-3-4+6\vert$; $\vert+1-2-3-4+8\vert$.
$\vert-1-3-4+8\vert$	0	=0	Inflaton	0I	1 †	$\vert-1+3-4-6+8\vert$; $\vert+1-2-3-4+8\vert$; $\vert-1-3-4-8+16\vert$.
$\vert+1+2-5-6+8\vert$	$l_Q/3$	>0	Quarks	$0.5Q_{l_Q/3}$	3	$\vert+1+2-5-6-8+16\vert$
$\vert+2+3-6-7+8\vert$	$l_Q/3$	>0	Quarks	$0.5Q_{l_Q/3}$	3	$\vert+2+3-6-7-8+16\vert$
$\vert+1-2-3-4+8\vert$	0	=0	Gluons	1G	8	$\vert\pm(-1-3+4)-2-6+8\vert$
$\vert+1-2-3-4+8\vert$	0	=0	Jay	1J	1 †	$\vert\pm(-1-3+4)-2-6+8\vert$
$\vert-1-3-4-8+16_1\hat{}\vert$	0	=0	Photon	1L	1	$\vert-1-3-4-8+16_2\hat{}\vert$, $\vert+1-2-3-4-8+16_1\hat{}\vert$, $\vert-1+3-4-6-8+16_1\hat{}\vert$.
$\vert-1-3-4-8+16_S\hat{}\vert$, with S being the S in SL	0, 0, ...	=0, =0, ...	Graviton, TBD, ...	2L, 3L, ...	1 †, 1 †, ...	$\vert-1-3-4-8+16_{(S+1)}\hat{}\vert$, $\vert+1-2-3-4-8+16_S\hat{}\vert$, $\vert-1+3-4-6-8+16_S\hat{}\vert$

- For elementary particles, 1f uses of solution-pairs link to sets of specific elementary particles.
- For boson elementary particles, 2f uses of solution-pairs link to types of interactions in which the counterpart 1f elementary bosons partake. Here, a 2f solution-pair is a one-step cascade from a counterpart 1f solution-pair.
- For elementary particles, 1f uses of a 2f solution-pair (that is a one-step cascade from a counterpart elementary-particle 1f use of a solution-pair) can link to specific elementary particles.

Table 6 catalogs all known elementary particles and some elementary particles that CAT+ suggests nature might include. A primary organizing principle for the rows is that lesser n_k precedes greater n_k. A secondary organizing principle for the rows is fermions precede bosons. A tertiary organizing principle is that nonzero charge precedes zero charge. Table 6 stems from Table 4 and Table 5.

Table 6 de-emphasizes (but CAT+ does not necessarily rule out) the possibilities that - for each one of some S≥ 1 - 1f use of the solution-pair $|+1-2-3-4-8+16_S\hat{}|$ links to an elementary boson that has spin S+1. Such elementary bosons might link to notions of (S + 1)G families or (S + 1)J families.

Table 6 de-emphasizes (but CAT+ does not necessarily rule out) the possibilities that - for each one of some S ≥ 1 - 1f uses of the solution-pair $|+1+2-5-6-8+16_S\hat{}|$ and 1f uses of the solution-pair $|+2+3-6-7-8+16_S\hat{}|$ link to elementary fermions that have spins of S + 0.5. Such elementary fermions might link to notions of $(S+0.5)Q_{>0}$ families

The following notions associate with CAT+. CAT+ regarding elementary particles ...

- Points to all known elementary particles.
  - The eight gluons link to 1f use of one solution-pair.
  - Each one of the other known boson elementary particles links to 1f use of a unique (to that boson elementary particle) solution-pair.
  - Each trio of three equally charged lepton fermion elementary particles links to 1f use of a unique (to those three fermion elementary particles) solution-pair.
- Suggests the following notions.

$$\text{1f use and } 6 \notin K \Rightarrow \text{one boson or eight bosons} \tag{5}$$

$$\text{1f use and } 6 \in K \Rightarrow \text{three flavours of fermions} \tag{6}$$

- Suggests (at least) the following new elementary particles.
  - An inflaton: MOD concordance cosmology suggests that nature might include an inflaton elementary particle.
  - A so-called jay boson: MOD does not necessarily link the repulsive component of the strong interaction or the Pauli exclusion principle to a boson elementary particle.
  - A graviton: MOD suggests that nature might include a graviton elementary particle.
- Suggests that, for boson elementary particles, 2f uses of solution-pairs link to types of interactions or decays that nature exhibits (or might exhibit).

Table 6 provides a possible analog (for elementary particles) to the original periodic table for chemical elements. The original periodic table organizes chemical elements based on two notions - the atomic weight that links to an element and the types of interactions in which an element participates. In the sense of Table 6, one might link atomic weight to 1f and the notion of property. Types of chemical interactions might link to 2f.

Regarding Table 6, the following notions pertain.

- For a family of elementary particles, each 1f item links to one n_k and to the various k that appear in the sum that leads to $0 = |s| = |\cdots|$. For a boson family for which $n_{EP} = 1$, no further consideration pertains. For a fermion family (for which $n_{EP} = 3$), one further aspect - flavour - pertains. For a boson family for which $n_{EP} = 8$, one further aspect - choice of specific particle - pertains.
- Each 2f item links to one n_k and to the various k that appear in the sums that lead to $0 = |s| = |\cdots|$. For some families of boson elementary particles, the relevant set of k links to types of interactions that nature exhibits.

3.2. Dark matter

This unit discusses a candidate specification for dark matter.

Useful modeling might link with the following notions.

- CAT+ categorizes elementary particles into two categories.
  - LRI (as in long-range interaction) boson elementary particles include the photon, the (would-be) graviton, and any higher-spin zero-mass bosons that would extend the series that starts with the photon and the graviton.
  - Isomeric-set elementary particles include all elementary particles that are not LRI boson elementary particles.
- Nature includes six isomers (as in sets of isomeric-set elementary particles).
  - So-called isomer-zero links to one set of isomer-set elementary particles and to MOD notions of elementary particles, left-handedness, charge, charge current, and magnetic moment.
  - Each one of so-called isomer-one through so-called isomer-five links to its own instance of an isomer-set of elementary particles and to its own instance of each one of charge, charge current, and magnetic moment.

  - Each one of the pair isomer-zero and isomer-three, the pair isomer-one and isomer-four, and the pair isomer-two and isomer-five links to the one-element term isomer-pair.
  - Each one of isomer-two and isomer-four links to MOD notions of left-handedness. Each one of isomer-one, isomer-three, and isomer-five links to MOD notions of right-handedness.

- CAT+ includes notions of n_i (as in number of instances) and $R_{/i}$ (as in reach per instance). Regarding isomers, $n_i = 6$. Each isomer includes its own instance of boson elementary particles (as in the W boson and the Z boson) that intermediate the weak interaction and its own instance of boson elementary particles (as in the eight gluons) that intermediate the strong interaction. For each of the weak interaction and the strong interaction, $n_i = 6$ and the reach (as in number of isomers) of an instance of the interaction is one isomer (as in $R_{/i} = 1$).

- CAT+ includes Eq. (7).

$$n_i R_{/i} = 6 \tag{7}$$

- To a first approximation, each isomer does not detect (much in the way of) electromagnetic aspects of the other five isomers. (Data to which Table 2a alludes suggests that isomer-zero stuff detects some electromagnetic aspects of isomer-three stuff.) PAT suggests that $n_i = 6$ and $R_{/i} = 1$ pertain regarding a 1x $1 = |s| = |+1|$ so-called component of electromagnetism (as in the electromagnetic interaction) and regarding the property of charge.

- To a first approximation, each isomer interacts gravitationally equally with itself and with each one of the other five isomers. PAT suggests that $n_i = 1$ and $R_{/i} = 6$ pertain regarding the 1x $2 = |s| = |+2|$ component of gravity (as in the gravitational interaction) and regarding the property of energy.

- CAT+ categorizes LRI-linked object properties (as in object properties that link to LRI) into three categories. (PAT suggests a means for computing - given a set K - the category to which - for $|s| \geq 1$ - a 1x use of a solution-pair links. One considers the integers from 1 to the maximum k - in K - that is less than or equal to four. One counts the number of integers that are not in K but are in the range from one to the previously mentioned maximum k. If the count is zero or three, $R_{/i} = 1$. If the count is one, $R_{/i} = 6$. If the count is two, $R_{/i} = 2$. PAT suggests that discussion related to Eq. (45) might associate with these values of $R_{/i}$. Based on the notion of an object, PAT suggests that the reach that links to a 2x use of a solution-pair equals the reach for a 1x use of a solution-pair from which the 2x-solution-pair cascades. The result regarding 2x-use is independent of the choice of 1x-used solution-pair.) The following notions pertain regarding properties.
  - Each so-called $R_{/i} = 1$ (as in reach-one or in reach of one per instance of property) LRI-linked property associates with the notion of six instances (as in one instance per isomer) of the property. Each one of charge (a 1x property), charge current (a 2x property for which the 1x counterpart is charge), and magnetic moment (a 1x property) is a reach-one property.
    * PAT suggests that ordinary matter does not interact directly with dark-matter charge, dark-matter charge current, or with dark-matter magnetic moment.
  - Each so-called $R_{/i} = 2$ (as in reach-two or in reach of two per instance of property) LRI-linked property associates with the notion of three instances (as in one instance per isomer-pair) of the property.
  - Each so-called $R_{/i} = 6$ (as in reach-six or in reach of six per instance of property) LRI-linked property associates with the notion of one instance (as in one instance that pertains to all six isomers) of the property. Each one of energy (a 1x property) and momentum (a 2x property for which the 1x counterpart is energy) is a reach-six property.
    * PAT suggests that ordinary matter interacts directly with dark-matter energy and with dark-matter momentum.

- PAT suggests that counterpart (across isomers) elementary particles are identical with respect to spin S and with respect to rest energy (as in mass).

- PAT suggests that counterpart (across isomers) nonzero-mass elementary particles are identical with respect to ratios of magnitudes of the respective charges to the respective masses.

Table 7: Matches between masses and flavours, for isomers of charged elementary fermions. The symbol l_i denotes the isomer number. The symbol f_{l-r} denotes - for fermion elementary particles - whether a matter particle (in the context of matter particle and antimatter particle) elementary fermion is left-handed ($f_{l-r} = +1$), does not associate with handedness ($f_{l-r} = 0$, in which case the elementary fermion is its own antiparticle), or is right-handed ($f_{l-r} = -1$). The symbol $0.5Q_{>0}$ denotes the pair $0.5Q_{1/3}$ and $0.5Q_{2/3}$. For each row, the $0.5Q_{>0}$ column assigns the three flavour numbers in order of increasing geometric-mean mass, with the geometric mean associating with the masses for the two quarks that are relevant to the flavour. For each row, the $0.5C_1$ column assigns the three flavor numbers in order of increasing mass for the one relevant charged lepton.

l_i	f_{l-r}	Respective flavour ($0.5Q_{>0}$)	Respective flavour ($0.5C_1$)
0	$+1$	Flavour-1, flavour-2, flavour-3	Flavour-1, flavour-2, flavour-3
1	-1	Flavour-1, flavour-2, flavour-3	Flavour-3, flavour-1, flavour-2
2	$+1$	Flavour-1, flavour-2, flavour-3	Flavour-2, flavour-3, flavour-1
3	-1	Flavour-1, flavour-2, flavour-3	Flavour-1, flavour-2, flavour-3
4	$+1$	Flavour-1, flavour-2, flavour-3	Flavour-3, flavour-1, flavour-2
5	-1	Flavour-1, flavour-2, flavour-3	Flavour-2, flavour-3, flavour-1

- PAT suggests that each isomer-pair links to a different pairing between charged lepton flavours and rest-energies. Table 7 provides more information. PAT suggests that 0.5N flavours follow the patterns for the respective charged leptons.

- PAT suggests that the notion of counterpart links to mass but not to lepton flavour. MOD suggests that - at least for neutrinos - mass eigenstates do not necessarily equal flavour eigenstates.

- PAT suggests that the fermion flavour-and-mass pairings for isomer-one, isomer-two, isomer-four, and isomer-five lead to those isomers forming stable counterparts to isomer-zero neutrons and to those isomers not forming significant numbers of counterparts to isomer-zero atoms.
  - Here, PAT uses the one-element term alt-isomer to designate an isomer other than isomer-zero and isomer-three.
  - For each one of the six isomers, a charged baryon that includes exactly three flavour-3 quarks is more massive than the counterpart (within the same isomer) zero-charge baryon that includes exactly three flavour-3 quarks. (For example, the hadron that includes just two tops and one bottom has a larger total mass than does the hadron that includes just one top and two bottoms.)
  - Per Table 7, alt-isomer flavour-3 charged leptons are less massive than isomer-zero flavour-3 charged leptons. When flavour-3 quark states are much populated (and based on interactions mediated by W bosons), the stuff that associates with an alt-isomer converts more charged baryons to zero-charge baryons than does the stuff that associates with isomer-zero. Eventually, regarding the stuff that associates with the alt-isomer, interactions that entangle multiple W bosons result in the stuff that associates with the alt-isomer having more counterparts to isomer-zero neutrons and fewer counterparts to isomer-zero protons than does the stuff that associates with isomer-zero. The sum of the mass of a counterpart-to-isomer-zero proton and the mass of an alt-isomer flavour-1 charged lepton exceeds the mass of a counterpart-to-isomer-zero neutron. Compared to isomer-zero neutrons, alt-isomer neutrons scarcely decay.
  - The IGM (or, intergalactic medium) that associates with the alt-isomer scarcely interacts with itself via electromagnetism.
  - Stuff that links to the four alt-isomers might not form large enough clumps for detection (by current means) - as clumps - in isomer-zero solar systems or in isomer-three solar systems.

3.3. Gravitational phenomena

This unit discusses phenomena that MOD associates with gravitational and dark-energy phenomena. This unit suggests insights regarding the evolution of the universe. This unit suggests insights regarding the formation and evolution of galaxies.

Useful modeling might link with the following notions.

- Regarding the gravity that an object-O links to an object-I, the following notions pertain.

- Some gravitational properties (including energy) link to attracting and some gravitational properties (including momentum) link to repelling. An example that links with MOD regarding special relativity illustrates the attracting or repelling notion. For a non-rotating object-I, a rest-energy mc^2 pertains. For any object-O, inferences regarding the E (as in energy) of object-I and the P (as in momentum) of object-I link via Eq. (8). The greater the inferred value of $|P|$, the greater the inferred value of E and the greater the gap - between E and mc^2 - that object-O infers. The inferred gravitational effect that links to momentum subtracts from the inferred gravitational effect that links to energy. (This notion of subtraction associates with an analogy to the electromagnetic invariance of $(E')^2 - (B')^2c^2$, in which E' denotes the inferred magnitude of the electric field - that links to object-I - and B' denotes the inferred magnitude of the magnetic field - that links to object-I. E' scales with inferred charge. B' scales with inferred charge current plus inferred magnetic moment.)

$$E^2 - P^2c^2 = (mc^2)^2 \tag{8}$$

- PAT divides gravitational properties into the following two sets.
  * 1x gravitational properties that link to $|s| = 2$ include the following properties. Here, the notion of an object-I implies that each 1x gravitational property links to one common inferred position.
    1. Energy, which links to $2 = |s| = |+2|$, links to $n_k = 1$. $R_{/i} = 6$ pertains.
    2. Spin, which links to $2 = |s| = |-2+4|$, links to $n_k = 2$. $R_{/i} = 2$ pertains.
    3. Two moments of inertia, which link to $2 = |s| = |+1-2+3|$, link to $n_k = 3$. $R_{/i} = 1$ pertains.
    4. Two so-called rotations-that-link-to-moments-of-inertia which link to, respectively, $2 = |s| = |-1+2-3+4|$ and $2 = |s| = |-1-2-3+4|$, link to $n_k = 4$. $R_{/i} = 1$ pertains.
  * 2x gravitational properties that link to $|s| = 2$ include the following properties. Here, the notion of an object-I implies that each 2x gravitational property links to the one common position and to one common inferred velocity. Here, the notion of an object-I implies that the reaches match the reaches of the counterpart 1x gravitational properties. For example, the $R_{/i}$ for momentum equals the $R_{/i}$ for energy.
    1. Momentum, which links to $2 = |s| = |-2+4|$, links to $n_k = 2$. $R_{/i} = 6$ pertains.
    2. So-called angular-momentum momentum, which links to $2 = |s| = |-2-4+8|$, links to $n_k = 3$. $R_{/i} = 2$ pertains.
    3. Two so-called moments-of-inertia momenta, which link to $2 = |s| = |-1+2-3+4|$ and $2 = |s| = |-1-2-3+4|$, link to $n_k = 4$. $R_{/i} = 1$ pertains.
    4. Two so-called rotations-that-link-to-moments-of-inertia momenta, which link to $2 = |s| = |-1+2-3-4+8|$ and $2 = |s| = |-1-2-3-4+8|$, link to $n_k = 5$. $R_{/i} = 1$ pertains.
- $n_k = 1$, $n_k = 3$, and so forth $|s| = 2$ properties link to notions of attracting of object-O toward object-I.
  * The property energy links to $2 = |s| = |+2|$, $n_k = 1$, and attracting.
  * The properties moments of inertia link to $2 = |s| = |+1-2+3|$, $n_k = 3$, and attracting.
- $n_k = 2$, $n_k = 4$, and so forth $|s| = 2$ properties link to notions of repelling of object-O away from object-I.
  * The property spin links to $2 = |s| = |-2+4|$, $n_k = 2$, and repelling.

• PAT suggests that, regarding two objects that move (generally radially) away from each other, the gravitational interaction transits some portion of the sequence dominance by $n_k = 4$-property-based repelling, dominance by $n_k = 3$-property-based attracting, dominance by $n_k = 2$-property-based repelling, and dominance by $n_k = 1$-property-based attracting. (Here, one might consider the notions that $n_k = 4$ links to Newtonian dynamics octupole potentials, $n_k = 3$ links to Newtonian dynamics quadrupole potentials, $n_k = 2$ links to Newtonian dynamics dipole potentials, and $n_k = 1$ links to Newtonian dynamics monopole potentials.)

• PAT suggests that eras in the rate of expansion of the universe link to transitions regarding dominance.

Table 8: Six possible eras regarding a typical speed of moving away from each other of large clumps. The rightmost three columns suggest eras. The leftmost three columns describe phenomena that PAT suggests as noteworthy causes for the eras. Generally, a noteworthy cause links to dominant forces and to notions of accelerations. Generally, an era links to notions of speeds. The symbol $\rightarrow$ links to the notion that a noteworthy cause may gain prominence before an era starts. Subsequent rows associate with later eras. MOD suggests notions of a Big Bang (or - at least - of a time that MOD links to the two-word term Big Bang). The symbol † denotes a possible association between the relevant era and some MOD notions of a Big Bang. CAT+ points to the possibility for the first two eras that the table discusses. 1f use of the solution-pair $0 = |-1-2-3-4+8|$ links to the Pauli exclusion principle (and to the might-be jay boson). The 1x solution-pairs to which the table alludes associate with gravitation. The notation $2 = |-1 \pm 2-3+4|$ links to the two solution-pairs $2 = |-1+2-3+4|$ and $2 = |-1-2-3+4|$. MOD uses the word inflation (or, the two-word term inflationary epoch) to name the era that links to the third row in the table. MOD suggests that the inflationary epoch started about 10^{-36} seconds after the Big Bang. MOD suggests that the inflationary epoch ended between 10^{-33} seconds after the Big Bang and 10^{-32} seconds after the Big Bang. Possibly, no direct evidence exists for the inflationary epoch. The following notions pertain regarding the column with the one-word label notes. The symbol 1 denotes the notion that MOD interpretations of data support the notions of each one of the two billions-of-years eras. The symbol 2 denotes the notion that MOD suggests the era. The symbol 3 denotes the notion that PAT suggestions regarding resolving MOD tensions (between data and modeling) that associate with the fifth row do not necessarily depend on the existence of the era.

Force	1x or 1f solution-pairs	$R_{/i}$	$\rightarrow$	Typical speed	MOD duration	Notes
Attractive	$2 = \|+1-3-4-8+16\|$	6	$\rightarrow$	Is negative	-	3
Repulsive	$0 = \|-1-2-3-4+8\|$	1	$\rightarrow$	Turns positive †	-	3
Repulsive	$2 = \|-1 \pm 2-3+4\|$	1	$\rightarrow$	Increases rapidly	Less than a second	2, 3
Attractive	$2 = \|-1-2+3\|$	1	$\rightarrow$	Decreases	Billions of years	1
Repulsive	$2 = \|-2+4\|$	2	$\rightarrow$	Increases	Billions of years	1
Attractive	$2 = \|+2\|$	6	$\rightarrow$	Would decrease	-	3

  - Inflation might link to dominance by ($|s| = 2$) $n_k = 4$-property-based repelling.
  - The start of the first one of the two multi-billion-years eras links to dominance by ($|s| = 2$) $n_k = 3$-property-based attracting.
  - The start of the second (and current) multi-billion-years era links to dominance by ($|s| = 2$) $n_k = 2$-property-based repelling.

- PAT suggests that MOD based on general relativity and an equation of state that suits a period early in the first multi-billion-years era would link to $R_{/i} = 1$ and extrapolate to underestimate later repelling effects - which would link to $R_{/i} = 2$ - that link to the equation of state.

- Discussion above suggests that MOD overestimates large-scale clumping of stuff (as in stuff that has bases in at least one of DM - as in dark matter - and OM). PAT suggests the following notions.

  - MOD links to notions that each one of 1x use of $2 = |s| = |-2+4|$ repulsion and 2x use of $2 = |s| = |-2+4|$ repulsion links to $R_{/i} = 1$.
  - More appropriate application would link to $R_{/i} = 2$ regarding 1x use of $2 = |s| = |-2+4|$ repulsion and would link to $R_{/i} = 6$ regarding 2x use of $2 = |s| = |-2+4|$ repulsion
  - The MOD underestimates (based on $R_{/i} = 1$) link to (at least some part of) the MOD overestimates regarding clumping.

- Table 8 discusses six possible eras regarding a typical speed of moving away from each other of large clumps. (Refs. [59, 60, 61, 62] discuss the possible inflationary epoch. Refs. [63, 64, 65, 66] provide data and discussion about the two multi-billion-years eras. Ref. [67] discusses attempts to explain the rate of expansion of the universe.)

PAT suggests that some CAT+ notions regarding eras that follow the inflationary epoch might not necessarily depend significantly on CAT+ notions regarding the inflationary epoch or on CAT+ notions regarding eras that might precede the inflationary epoch.

This paper does not try to explore the possibility that (or to estimate a time at which) a transition - for the largest observable objects - from repelling based on $2 = |-2+4|$ to attracting based on $2 = |+2|$ might occur.

PAT suggests that smaller astrophysical objects generally transit segments of the series that includes $n_k = 3$ attraction, $n_k = 2$ repulsion, and $n_k = 1$ attraction more rapidly than do larger astrophysical objects.

A proto-solar system that features ordinary matter (as in isomer-zero stuff) forms based on $n_k = 3$, $R_{/i} = 1$ attraction. That proto-solar system expels isomer-three stuff (and some ordinary-matter stuff)

Table 9: Possible stages and eras regarding galaxy formation and evolution. The rightmost three columns suggest stages and eras. The leftmost three columns describe phenomena that PAT suggests as noteworthy causes for the stages and eras. Generally, a noteworthy cause links to dominant forces and to notions of accelerations. The symbol † denotes the notion the 2x $2 = |-2+4|$ - with $R_{/i} = 6$ - also pertains. The symbol → links to the notion that a noteworthy cause may gain prominence before a stage starts. Subsequent rows associate with later stages. This table associates with a scenario in which a galaxy forms based on one original one-isomer clump and initially does not significantly collide with other galaxies. Currently, some galaxies associate with the first era, some galaxies associate with the second era, and some galaxies associate with the third era. The following notes pertain. (S1) A one-isomer original clump forms. (S2) The original clump repels (some) stuff that associates with the isomer that associates with the original clump and (most) stuff that associates with one other isomer. (S3) The original clump attracts stuff that associates with the four not-repelled isomers and stuff that associates with the isomer that associates with the original clump. (S4) Another galaxy subsumes the original clump and might subsequently merge with yet other galaxies.

Force	1x solution-pair	$R_{/i}$	→	Stage	Era	Note
Attractive	$2 = |-1-2+3|$	1	→	1	First	(S1)
Repulsive	$2 = |-2+4|$ †	2	→	2	First	(S2)
Attractive	$2 = |+2|$	6	→	3	Second	(S3)
Attractive	$2 = |+2|$	6	→	4	Third	(S4)

based on $n_k = 2$, $R_{/i} = 2$ repulsion. The proto-solar system continues to evolve based on the presence of ordinary matter and on $n_k = 1$ attraction. (Discussion above suggests that isomer-three stuff evolves similarly to isomer-zero stuff. Isomer-three solar systems would evolve similarly to isomer-zero solar systems. (Discussion above suggests that isomer-one, isomer-two, isomer-four, and isomer-five might not form protons or atoms. If so, stuff that links to those four isomers might not form large enough clumps for detection (by current means) - as clumps - in isomer-zero solar systems or in isomer-three solar systems.)

Table 9 discusses possible stages and eras regarding galaxy formation and evolution.

PAT suggests that galaxy clusters generally evolve based on somewhat equal presences of all isomers and on $n_k = 1$, $R_{/i} = 6$ attraction.

3.4. Ratios of dark-matter effects to ordinary-matter effects

This unit discusses the notion that PAT and CAT+ specifications for dark matter and gravity suffice to explain a variety of inferred ratios of dark-matter effects to ordinary-matter effects.

PAT suggests, based on the specification for dark matter and on discussion above regarding gravity and dark-energy phenomena, the following notions (which, in effect, extend Table 2).

- DM (as in dark-matter effects) replaces NOM (as in not-ordinary-matter effects) throughout Table 2.
- MOD suggests that each one of cosmic optical background and CMB formed based on atomic transitions. PAT suggests the following notions.
  - The amount of observed CMB links to a DM:OM ratio of 1 : 1.
  - PAT suggests the following possibilities.
    * 1x uses of solution-pairs $1 = |s| = |-2-3+5-7+8|$ and $1 = |s| = |+2+3-5-7+8|$ associate with the charge of the electron cloud and with the charge of the atomic nucleus. The two solution-pairs associate with interactions via which the atom transits to a new overall charge state.
    * 1x use of solution-pair $1 = |s| = |-1-4+5-7+8|$ associates with electron-cloud principal energy levels and with electromagnetic interactions via which the atom transits to new principal energy-level states.
    * 1x uses of solution-pairs $1 = |s| = |-1-4-5-7+8|$ and $1 = |s| = |+1-4-5-7+8|$ associate with fine-structure energy levels and hyperfine energy levels and with electromagnetic interactions via which the atom transits to new fine-structure and hyperfine energy-level states.
    * For each one of the above five solution-pairs ($1 = |s| = |-2-3+5-7+8|$, $1 = |s| = |+2+3-5-7+8|$, $1 = |s| = |-1-4+5-7+8|$, $1 = |s| = |-1-4-5-7+8|$, and $1 = |s| = |+1-4-5-7+8|$), $R_{/i} = 6$ pertains.
  - Per discussion above in this paper, isomer-zero and isomer-three form atoms. Isomer-one, isomer-two, isomer-four, and isomer-five do not form atoms. PAT suggests a notion of an effective reach $(R_{/i})_{\rm ef}$. PAT suggests Eq. (9).

$$\text{Regarding electromagnetism: for atomic states and atomic transitions, } (R_{/i})_{\text{ef}} = 2 \quad (9)$$

- Table 9 suggests galaxy formation and evolution scenarios that might link to the ratios that Table 2b lists.
  - Some early galaxies link to a DM:OM ratio of $1 : 0^+$, but possibly observations do not (yet) point directly to such DM galaxies.
- PAT suggests that $R_{/i} = 6$ links with the fives in the $5^+ : 1$ ratios that Table 2 lists.
- PAT suggests three mechanisms that might link to the pluses in the $5^+ : 1$ ratios that Table 2 lists (or equivalently, that might contribute toward the notion that measurements of large-scale presences of DM might exceed five times measurements of large-scale presences of OM.)
  - Dissimilar evolution regarding isomeric stuff. For each one of some properties that associate with $|s| = 1$, $R_{/i} = 6$ can pertain. The evolution of alt-isomer stuff might deviate - compared to the evolution of isomer-zero stuff - early enough that (nominally) isomer-zero high-energy excitations of the electromagnetic field produce alt-isomer stuff significantly more copiously than (nominally) alt-isomer excitations of the electromagnetic field produce isomer-zero stuff.
  - Isomer-zero dark matter. Some MOD suggests notions - such as notions of primordial black holes or yet-to-be-found elementary particles - of stuff that might measure as DM and (in the context of PAT and CAT+) associate mainly with isomer-zero stuff. (PAT does not necessarily suggest isomer-zero elementary particles that would associate with notions of DM.)
  - Misinterpreted measurements. Interpretations of measurements might - based on notions that, for example, the $R_{/i} = 2$ for 1x uses of $2 = |-2+4|$ differs from the $R_{/i} = 6$ for 1x uses of $2 = |+2|$ and for 2x uses of $2 = |-2+4|$ - might lead to inferred ratios of DM effects to OM effects that do not associate exactly with actual ratios of DM stuff to OM stuff.
- PAT suggests that most stars form from one-isomer clumps. DM (as in dark-matter) stuff can form one-isomer stars. OM (as in ordinary-matter) stuff does not interact directly with thermal radiation from dark-matter stars. (This notion of dark-matter stars differs from notions of dark stars that Ref. [68] discusses.)
  - PAT suggests that thermal states of objects might link to 1x use of $1 = |s| = |-4+5|$, which links to $R_{/i} = 1$. (Unlike for 1f 2CO5 uses of a solution-pair, PAT does not require - for 1x uses of a solution-pair - the existence of a matching 2CO7 solution-pair or of a matching 2CO solution-pair. Additionally, 1x uses of the following $R_{/i} = 1$ solution-pairs might pertain: $1 = |s| = |-1-2+3-4+5|$, $1 = |s| = |+1+2-3-4+5|$, $1 = |s| = |-1+2-3-4+5|$, and $1 = |s| = |+1+2-3-4+5|$.)
- Based on the notion that atoms link to 1x and $(R_{/i})_{\text{ef}} = 2$, it might (someday) be possible to detect light that isomer-three DM atoms emit recently (compared to when cosmic optical background radiation formed). The following sentences point to possible bases for such light. Some isomer-three DM atoms would exist near isomer-three DM stars. Some isomer-three DM atoms might exist in laboratory settings (and, OM light might - occasionally - excite such isomer-three DM atoms).
- PAT suggests that a combination of MOD and CAT+ might comport with observations regarding the aftermath of the so-called Bullet Cluster collision of two galaxy clusters. The following notions pertain.
  - Ref. [69] discusses the Bullet Cluster collision of two galaxy clusters.
  - MOD suggests two general types of trajectories for stuff. Most DM - from either one of the clusters - exits the collision with trajectories that are consistent with having interacted just gravitationally with the other cluster. Also, OM stars - from either cluster - exit the collision with trajectories that are consistent with having interacted just gravitationally with the other cluster. However, OM IGM - from either cluster - lags the cluster's OM stars and DM. MOD suggests that the OM IGM interacted electromagnetically with the other cluster's OM IGM, as well as gravitationally with the other cluster.

- PAT might comport (regarding each cluster) with the interpretations of data, with one possible exception. The possible exception associates with the notion that PAT suggests that isomer-three IGM interacts electromagnetically and might follow trajectories that are consistent with OM IGM trajectories.
- Regarding the possible exception, at least three possibilities arise.
  * For one possibility, per discussion related to Eq. (9), the light that MOD associates with OM IGM might include light that CAT+ associates with OM IGM and light that CAT+ associates with isomer-three IGM.
  * For one possibility, isomer-three IGM measures as DM and MOD does not adequately report (or otherwise account for) lagging isomer-three IGM.
  * For one possibility, isomer-three IGM follows trajectories that are consistent with other DM trajectories.
- PAT suggests that interpretations of data may not be sufficient to rule out each one of the first two possibilities or to rule out a combination of the first two possibilities.
- PAT suggests that CAT+ notions of DM are not necessarily incompatible with constraints - that have bases in observations of collisions of galaxy clusters - regarding DM.

• PAT suggests that people might want to interpret observations regarding light from IGM in the context of isomers and reaches.

4. Discussion

This unit discusses aspects that - while not necessarily germane to discussion above - tend to support or extend work above regarding CAT+.

4.1. Additional unexplained data

This unit describes data that - currently - models seem not to explain.

4.1.1. Elementary-particle properties, interactions, and physics constants

This unit discusses data that might interrelate physics constants in ways that MOD seems not to discuss.

MOD seems not to explain some possible relationships between properties of specific elementary particles. The next two paragraphs provide examples.

Eq. (10) might link the masses (as in rest-energies divided by c^2) of the known nonzero-mass elementary bosons. (Refs. [33, 70] provide relevant data.) m denotes mass. The symbols W, Z, and Higgs link to, respectively, the W boson, the Z boson, and the Higgs boson. (MOD links the ratio $(m_{\mathrm{W}})^2 : (m_{\mathrm{Z}})^2$ to the three-word term weak mixing angle. MOD does not necessarily discuss the ratio $7 : 9$.)

$$(m_{\mathrm{W}})^2 : (m_{\mathrm{Z}})^2 : (m_{\mathrm{Higgs}})^2 :: 7 : 9 : 17 \tag{10}$$

Eqs. (11), (12), and (13) might link aspects of electromagnetic interactions with aspects of gravitational interactions. (Ref. [33] provides relevant data.) Eq. (11) defines β' as the ratio of the mass of the tau (a charged lepton elementary particle) to the mass of the electron (another charged lepton elementary particle). The right-hand side of Eq. (12) is - for two electrons - a ratio of electromagnetic repelling to gravitational attracting. Eq. (12) defines β. Eq. (13) might suggest a relationship between electromagnetism and gravity.

$$\beta' \equiv m_\tau / m_e \tag{11}$$

$$(4/3) \cdot (\beta^2)^6 = ((q_e)^2/(4\pi\varepsilon_0))/(G_N (m_e)^2) \tag{12}$$

$$\beta' = \beta \tag{13}$$

Eqs. (10) and (13) might point to new links between physics constants and to possibilities for reducing in number the elements of a set that MOD might associate with the notion of independent fundamental constants.

4.1.2. Phenomena that might involve the PAT-suggested jay boson elementary particle

This unit discusses phenomena that might associate with the PAT-suggested jay (or, 1J) boson (that Table 6 lists).

4.1.2.1. Pauli repulsion. MOD includes the notion that two identical fermions cannot occupy the same state. Regarding quantum mechanics, one notion is that repelling between identical fermions associates with overlaps of wave functions. Another quantum mechanics notion features wave functions that are antisymmetric with respect to the exchange of two identical fermions.

PAT might be compatible with such aspects of MOD and, yet, not necessitate - regarding MOD dynamics modeling - the use of wave functions. Quantum mechanics based on jay bosons might suffice. Classical mechanics based on potentials that would associate with effects of jay bosons might suffice.

PAT suggests that quantum mechanics based on jay bosons or classical mechanics based on jay bosons might suggest that the prevention of two identical fermions from occupying the same state might associate with, in effect, interactions - mediated by jay bosons - that try to change aspects related to the fermions. Notions of changing a spin orientation might pertain. For elementary fermions, notions of changing a flavour might pertain.

4.1.2.2. Energy levels in positronium. Ref. [71] discusses the transition - between two states of positronium - characterized by the expression that Eq. (14) shows.

$$2^3S_1 \rightarrow 2^3P_0 \tag{14}$$

Four standard deviations below the nominal observed value of the energy that associates with the transition approximately equals four standard deviations above the nominal value of the energy that MOD suggests.

PAT notions regarding jay bosons might explain the might-be discrepancy regarding positronium. Compared to MOD quantum field theory, a new notion of virtual charge exchange or a new notion of virtual flavour change might pertain.

To the extent that MOD quantum field theory does not suffice to explain positronium energy levels, PAT notions related to the jay boson might help to close the gap between observations and modeling.

4.1.2.3. Pauli crystals. Ref. [72] reports detection of Pauli crystals. PAT suggests that modeling based on the notion of jay bosons might help explain relevant phenomena.

4.2. Additional methods

This unit discusses mathematics PAT suggests has use regarding PAT, MOD, and CAT+.

4.2.1. Isotropic harmonic oscillator math - solutions to partial differential equations

This unit discusses mathematics - and shows solutions - related to the three-word term isotropic harmonic oscillators.

Modeling for a j-dimensional isotropic harmonic oscillator can feature j linear coordinates x_{l_x} - each with a domain $-\infty < x_{l_x} < \infty$ - and an operator that is the sum - over l_x - of j operators of the form that Eq. (15) shows. The number C is positive and is common to all j uses of Eq. (15). The word isotropic associates with the commonality - across all j uses of Eq. (15) - of the number C.

$$-\frac{\partial^2}{\partial(x_{l_x})^2} + C \cdot (x_{l_x})^2 \tag{15}$$

For $j \geq 2$, one can split the overall operator into pieces. Eq. (16) associates with a split into two pieces. Here, each of j_1 and j_2 is a positive integer.

$$j = j_1 + j_2 \tag{16}$$

In discussion below, the symbol D might be any one of j, j_1, and j_2.

For $D \geq 2$, mathematics related to isotropic harmonic oscillators can feature partial differential equations, a radial coordinate, and $D-1$ angular coordinates. Eq. (17) defines a radial coordinate.

$$x = (\sum_{l_x} (x_{l_x})^2)^{1/2} \tag{17}$$

PAT suggests replacing x via the expression that Eq. (18) shows. Here, r_{HO} denotes the radial coordinate and has dimensions of length. The parameter η has dimensions of length. The parameter η is a nonzero real number. The magnitude $|\eta|$ associates with a scale length. (Here, r_{HO} associates with mathematics for HO - or, harmonic oscillators - and does not necessarily associate with uses of r elsewhere - for example, regarding MOD Newtonian dynamics - in this paper.)

$$x = r_{HO}/\eta \tag{18}$$

In MOD applications, the following HO notions can pertain. Solutions - that can associate with wave functions - to the pair of Eqs. (19) and (20) can have the form $\Psi = \phi_R(r_{HO})Y$, in which Y is a function of $D-1$ angular coordinates and is not a function of r_{HO}. Ω associates with operators that associate with angular coordinates. (For $D = 3$, Ref. [73] shows a representation for Ω in terms of an operator that is a function of spherical coordinates.) D is a nonnegative integer. The domain for r_{HO} is $0 \leq r_{HO} < \infty$. Each one of ξ and ξ' is an as-yet unspecified constant. For $D = 1$, Eqs. (19) and (20) might not be appropriate.

$$\xi\Psi = (\xi'/2)(-\eta^2\nabla^2 + (\eta)^{-2}(r_{HO})^2)\Psi \tag{19}$$

$$\nabla^2 = (r_{HO})^{-(D-1)}(\partial/\partial r_{HO})(r_{HO})^{D-1}(\partial/\partial r_{HO}) - \Omega(r_{HO})^{-2} \tag{20}$$

This paper considers solutions that comport with Eqs. (21), (22), (23), (24), (25), (26), and (27). With respect to the domain $0 \leq r_{HO} < \infty$, ϕ_R associates with the mathematics notion of having a definition almost everywhere. In MOD, solutions that associate with Eq. (15) and with $D = 1$ have the form $H(x)\exp(-x^2)$, in which $H(x)$ is a Hermite polynomial. In this paper, for each relevant D, each solution that is relevant associates with - in effect - a one-term polynomial. In this paper, $D = 1$ is a relevant D. Eqs. (25), and (26) echo Eqs. (19) and (20). (Per Eq. (30), that the function $\phi_R(r_{HO})$ normalizes will be significant. Per the equal-sign symbol in Eq. (27), normalization to a value of one is not necessarily relevant in this paper.)

$$D \text{ is a real number} \tag{21}$$

$$\Omega \text{ is a constant} \tag{22}$$

$$\phi_R(r_{HO}) \text{ is a function of just } r_{HO},\ \eta, \text{ and a number } \nu \tag{23}$$

$$0 < r_{HO} < \infty \tag{24}$$

$$\xi\phi_R(r_{HO}) = (\xi'/2)(-\eta^2\nabla^2 + (\eta)^{-2}(r_{HO})^2)\phi_R(r_{HO}) \tag{25}$$

$$\nabla^2 = (r_{HO})^{-(D-1)}(\partial/\partial r_{HO})(r_{HO})^{D-1}(\partial/\partial r_{HO}) - \Omega(r_{HO})^{-2} \tag{26}$$

$$\phi_R(r_{HO}) \ = \ (r_{HO}/\eta)^{\nu}\exp(-(r_{HO})^2/(2\eta^2)), \text{ with } \eta^2 > 0 \tag{27}$$

Eqs. (28) and (29) characterize solutions of the form that Eq. (27) shows. The parameter η does not appear in Eqs. (28) and (29).

$$\xi = (D + 2\nu)(\xi'/2) \tag{28}$$

$$\Omega = \nu(\nu + D - 2) \tag{29}$$

$\phi_R(r_{HO})$ normalizes if and only if Eq. (30) pertains. The symbol $(\phi_R(r_{HO}))^*$ denotes the complex conjugate of $\phi_R(r_{HO})$.

$$\int_0^\infty (\phi_R(r_{HO}))^*\phi_R(r_{HO}) \cdot (r_{HO})^{D-1}dr_{HO} < \infty \tag{30}$$

Eq. (31) associates with the domains of D and ν for which normalization pertains for $\phi_R(r_{HO})$. For $D + 2\nu = 0$, normalization pertains in the limit $\eta^2 \rightarrow 0^+$. Regarding mathematics relevant to normalization for $D + 2\nu = 0$, the delta function that Eq. (32) shows pertains. Here, $(x')^2$ associates with $(r_{HO})^2$ and 4ϵ associates with η^2. (Ref. [74] provides Eq. (32).) The difference in domains, between $-\infty < x' < \infty$ and Eq. (24), is not material here.

$$D + 2\nu \geq 0 \tag{31}$$

$$\delta(x') = \lim{}_{\epsilon \to 0^+} (1/(2\sqrt{\pi\epsilon}))e^{-(x')^2/(4\epsilon)} \tag{32}$$

MOD uses Eq. (33) to describe the energy E of the ν-times excited state of a D-dimensional isotropic harmonic oscillator. (Compare with Eq. (28) and with Eq. (31).) In MOD, ν is a nonnegative integer. In MOD, $\nu = 0$ links to the two-word term ground state. ω is a frequency. MOD links a wavelength λ to the frequency via the equation $\lambda = c/\omega$. The symbol c denotes the speed of light.

$$E = 0.5(D + 2\nu)\hbar\omega \geq 0 \tag{33}$$

4.2.2. Isotropic harmonic oscillator math - ground-state symmetries

This unit discusses ground-state symmetries for isotropic D-dimensional harmonic oscillators for which $D \geq 2$.

Per Ref. [75], for $D \geq 2$, $SU(D)$ symmetry associates with the ground state of an isotropic D-dimensional harmonic oscillator. Eq. (34) pertains. Here, gen(GX) denotes the number of generators of the group GX.

$$\text{gen}(SU(D)) = D^2 - 1, \text{ for } D \geq 2 \tag{34}$$

4.2.3. Excitations for bosons and fermions

This unit discusses raising operators and lowering operators related to boson states and discusses raising operators and lowering operators related to fermion states.

Eqs. (35) and (36) echo MOD regarding, respectively, a boson raising (as in excitation) operator and a boson lowering (as in de-excitation) operator. Here, a_b^+ denotes raising operator. $|n>$ denotes a state that links to the notion of n excitations. a_b^- denotes lowering operator.

$$a_b^+|n>= (1 + n)^{1/2}|n + 1>, \text{ for } 0 \leq n < \infty \tag{35}$$

$$a_b^-|n>= (n)^{1/2}|n - 1>, \text{ for } 0 \leq n < \infty \tag{36}$$

PAT suggests that Eqs. (37) and (38) might pertain regarding, respectively, a fermion raising (as in excitation) operator and a fermion lowering (as in de-excitation) operator.

$$a_f^+|n>= (1 - n)^{1/2}|n + 1>, \text{ for } 0 \leq n \leq 1 \tag{37}$$

$$a_f^-|n>= (n)^{1/2}|n - 1>, \text{ for } 0 \leq n \leq 1 \tag{38}$$

Regarding Eqs. (37) and (38), harmonic oscillator mathematics does not pertain. Regarding interactions that involve fermion elementary particles, PAT suggests that Eqs. (39) and (40) can pertain. Here, n_l links to the excitation state of a left-handed fermion elementary particle. n_r links to the excitation state of a right-handed fermion elementary particle.

$$a_f^+|n_l, n_r>= (1 - n_l)^{1/2}(1 - n_r)^{1/2}|n_l + 1, n_r + 1>, \text{ for } 0 \leq n_l \leq 1 \text{ and } 0 \leq n_r \leq 1 \tag{39}$$

$$a_f^-|n_l, n_r>= (n_l)^{1/2}(n_r)^{1/2}|n_l - 1, n_r - 1>, \text{ for } 0 \leq n_l \leq 1 \text{ and } 0 \leq n_r \leq 1 \tag{40}$$

PAT suggests that Eqs. (39) and (40) link to the notion that excitation of a left-handed fermion elementary particle can link to excitation of a right-handed fermion elementary particle.

PAT suggests that Eq. (41) links to the notion that - for two fermion elementary particles that link to the same (either left or right) handedness, an interaction can de-excite one fermion (as in, in the equation, fermion one) and excite the other fermion (as in, in the equation, fermion two).

$$a_f^+|n_2> a_f^-|n_1>= (1 - n_2)^{1/2}(n_1)^{1/2}|n_2 + 1> |n_1 - 1>, \text{ for } 0 \leq n_2 \leq 1 \text{ and } 0 \leq n_1 \leq 1 \tag{41}$$

4.2.4. Some symmetries related to the factors s_k

This unit discusses symmetries that relate to factors s_k.

PAT suggests considering the following regarding a value of an integer k. Here, the pair $s_k = +1$ and $s_k = -1$ can associate with the notion of $k \in K$ and $s_k = 0$ can associate with $k \notin K$.

- The pair $s_k = +1$ and $s_k = -1$ links to modeling for the ground state of a $D = 2$ isotropic harmonic oscillator. (This notion echoes the notion that MOD links aspects of excitations of each one of the $s = +|s|$ mode and the $s = -|s|$ mode to mathematics for a one-dimensional harmonic oscillator.) Per discussion related to Eq. (34), the ground state of a $D = 2$ isotropic harmonic oscillator links to $SU(2)$ symmetry and to three generators. For PAT, the notion of three generators can link to three MOD (spatial, velocity, angular-velocity, or other) dimensions or to three CAT+ discrete values.

- $s_k = 0$ links to modeling for excitations of a $D = 1$ harmonic oscillator. Mathematics links excitations of a $D = 1$ harmonic oscillator to $U(1)$ symmetry. The notion of one generator links to the group $U(1)$. For PAT, the notion of one generator can link to one MOD (temporal or other) dimension or to one CAT+ discrete value.

4.2.5. Some notions related to values of reaches $R_{/i}$

This unit discusses notions that might associate with reaches $R_{/i}$ for 1x uses of solution-pairs for which $|s| > 0$.

Discussion related to Eq. (7) suggests reaches $R_{/i}$ that associate with 1x uses of solution-pairs for which $|s| > 0$.

PAT suggests that each relevant solution-pair comports with Eq. (42).

$$\{1, 2, 3, 4\} \cap K \neq \emptyset \tag{42}$$

For each solution-pair, Eq. (43) defines k_{n_0}. (That is, k_{n_0} denotes the largest value of k for which k is less than or equal to four and $k \in K$.)

$$k_{n_0} \equiv \max\{k|1 \leq k \leq 4 \text{ and } k \in K\} \tag{43}$$

For each solution-pair, Eq. (44) computes n_0.

$$n_0 = \text{ the number of } k \text{ for which } 1 \leq k < k_{n_0} \text{ and } k \notin K \tag{44}$$

Eq. (42) and Eq. (44) imply that the range $0 \leq n_0 \leq 3$ pertains regarding n_0.

PAT suggests that Eq. (45) pertains for each 1x use of a solution-pair for which $1 \leq n_0 \leq 3$.

$$R_{/i} = \text{gen}(SU(7))/\text{gen}(SU(2n_0 + 1)) \tag{45}$$

PAT suggests the possibility that the factor (in Eq. (45)) of two might associate with the notion that - for a k that is not a member of K - the two notions ($s_k = +1$ and $s_k = -1$) of nonzero s_k do not play a role in Eq. (1).

4.2.6. Some properties that do not necessarily associate directly with LRI

This unit discusses PAT notions regarding properties - of objects - that do not necessarily associate directly with long-range-interaction properties.

PAT suggests extrapolating from the notion of positive integers k (as in members of sets K) to a larger (than associates with the notion of K) set of integers that includes integers k' for which the values of k' are nonpositive. Paralleling notions above regarding various values of k, PAT suggests that a pair $s_{k'} = +1$ and $s_{k'} = -1$ links to modeling for the ground state of a $D = 2$ isotropic harmonic oscillator. PAT suggests that $s_{k'} = 0$ links to modeling for excitations of a $D = 1$ harmonic oscillator. PAT suggests that the following notions pertain.

- For $k = 1$ (and for $k = 2$) and $n_k = 1$, MOD Newtonian dynamics suggests - regarding an object that models as point-like - the spatial dependence r^{-n_k} for the related potential.

- MOD can embrace potentials that link to r^0. Here, a force would have no effect. MOD can embrace potentials that link to r^1. Here, notions of asymptotic freedom link to the strong interaction. MOD can embrace potentials that link to r^2. Here, modeling links to notions of three-dimensional isotropic harmonic oscillators.

- Related to the statements just above, PAT posits (respectively) the following.
  - $k' = 0$ has relevance. The pair $s_0 = +1$ and $s_0 = -1$ links to the notion of three isomer-pairs. Given such, the notion of two-solutions per solution-pair links to six (as in three times two) isomers. $s_0 = 0$ links to the notion that LRI boson elementary particles do not necessarily conceptually link to single isomers or to single isomer-pairs and can have components for which $R_{/i} = 6$.
  - $k' = -1$ has relevance. The pair $s_{-1} = +1$ and $s_{-1} = -1$ links to the notion of three color charges. $s_{-1} = 0$ links to the notion that elementary particles other than quarks and gluons do not link to the three color charges. MOD links all known objects and all known elementary particles (other than quarks and gluons) to notions of no color charge or to notions of a white (or clear) color charge.
  - $k' = -2$ might have relevance. The pair $s_{-2} = +1$ and $s_{-2} = -1$ links to the notion of three dimensions. In MOD, the notion of three dimensions can link to three spatial dimensions (which, in turn, in PAT link to K-related aspects of 1x), three velocity-centric dimensions (which, in turn, in PAT link to K-related aspects of 2x), three angular-velocity dimensions (which, in turn, in PAT link to K-related aspects of 1x), and so forth. $s_{-2} = 0$ links to the notion of one dimension. In MOD, the notion of one dimension can link to one temporal dimension (which, in turn, in PAT links to K-related aspects of 1f and to K-related aspects of 2f).

4.2.7. Notions regarding the PAT-suggested jay boson

This unit discusses notions that PAT suggests interrelate the PAT-suggested jay boson and the eight gluons.

Regarding Table 6, except for the case of gluons and the jay boson, adding a six to the K set (for a boson 1f solution-pair) to produce a basis for a 2f solution-pair links to a notion of the number three, as in three fermion flavours for the 1f use of the new solution-pair. Similarly regarding 1x use of a solution-pair, adding an integer to the K set for the 1x solution-pair to produce a basis for a 2x solution-pair links to a notion of a factor of three, as in three dimensions that link to velocity.

Regarding (gluons and the jay boson and) 1f use of the $0 = |+1-2-3-4+8|$ solution-pair, there are two 2f solution-pairs ($0 = |-1-2-3+4-6+8|$ and $0 = |+1-2+3-4-6+8|$) that share $K = \{1, 2, 3, 4, 6, 8\}$. PAT suggests that a factor of three links to each one of the 2f solution-pairs - in the sense that $n_{EP} = 9$, as in three times three. Out of the nine elementary particles, eight elementary particles link to the MOD notion of eight gluons. PAT suggests that the ninth elementary particle - the jay boson - links to repulsion between the two fermion objects (that might be fermion elementary particles but are not necessarily fermion elementary particles) in a pair of adequately similar fermion objects that (in the sense of MOD) are in adequately similar states. PAT suggests considering that the jay boson interacts with the MOD notion of white (or clear) color charge.

Discussion that includes Eq. (14) suggests MOD examples for which the notion of a jay boson might prove helpful.

Regarding groups and symmetries, PAT suggests the following notions.

Eq. (46) states a result from mathematics.

$$SU(3) \subset U(3) \tag{46}$$

MOD links a representation of the group $SU(3)$ to modeling regarding gluons. PAT suggests that the notion of gluons plus the jay boson might link (in a sense of an extension to MOD) to the group $U(3)$. Here, one can consider that the jay boson links to the identity matrix that a representation of $U(3)$ includes but that a similar representation for $SU(3)$ lacks.

4.2.8. A possible limit regarding the types of long-range interactions

This unit discusses a PAT-suggested limit on the series for which the first two members are electromagnetism and gravitation.

Regarding Eq. (12), PAT suggests linking a factor of four to the solution-pair $1 = |s| = |+1|$ and linking a factor of three to the solution-pair $2 = |s| = |+2|$. PAT suggests that the series four, three, and so forth might extrapolate to link a factor of zero to the solution-pair. $5 = |s| = |+5|$. PAT suggests that, for an integer $l \geq 5$, the solution-pair $l = |s| = |+l|$ might not link to a physics property. PAT suggests that such a limit regarding $|s|$ links to the notion that, possibly, no boson elementary particles link - for example, in the sense of Eqs. (49) and (50) - to $S \geq 5$.

Possibly, no data pertains regarding whether nature includes LRI bosons for which $S \geq 3$.

4.3. Additional results

This unit discusses relationships between data plus MOD and PAT plus CAT+.

4.3.1. Gauge symmetries and the Higgs mechanism

This unit discusses possible similarities between gauge symmetries that MOD features and some symmetries that PAT suggests. This unit discusses possible associations between the MOD notion of the Higgs mechanism and notions that PAT suggests.

MOD links the electromagnetic, electroweak, and strong interactions to the respective Gauge symmetries $U(1)$, $SU(2) \times U(1)$, and $SU(3)$.

For boson elementary particles, MOD includes the following notions. For each $S = 1$ zero-mass elementary particle, there are two excitation modes (for example, left-circular polarization and right-circular polarization). For each $S = 1$ nonzero-mass elementary particle, there are three excitation modes (for example, relative to an axis, the spin state can measure as plus one, zero, or minus one). For each $S = 0$ nonzero-mass elementary particle, there is one excitation mode. Across all these cases, for each excitation mode, $D = 1$ harmonic oscillator mathematics links to notions about excitations.

For each of three interaction types - electromagnetic, weak, and strong - PAT suggests excitation modeling that has bases in mathematics for a D^*-dimensional isotropic harmonic oscillator. For each case, D^* is the product of the number of excitation modes and the number (in Table 6) of 2f uses of each K. The respective (regarding interaction types) products are two (as in two times one), three (as in three times one), and four (as in two times two). For each case, PAT links the MOD notion of modeling excitations (via the mathematics of a one-dimensional harmonic oscillator) to the notion that a symmetry related to a $(D^* - 1)$-dimensional isotropic harmonic oscillator pertains.

The respective symmetries are $U(1)$ for the electromagnetic interaction, $SU(2)$ for the weak interaction, and $SU(3)$ for the strong interaction.

MOD includes the notion of a Higgs field and the notion that the ground state energy for Higgs bosons is less than the ground state energy of the Higgs field. MOD links such to the two-word term Higgs mechanism.

PAT suggests that MOD for the Higgs field links to $D = 3$ (as in three spatial dimensions) and to the MOD notion that the ground state links to $\nu = 0$. MOD links the ground state of the Higgs boson to a lower energy than the ground state energy of the Higgs field. PAT suggests that MOD links the ground state of the Higgs boson to PAT notions of $D = 3$ and $\nu = -1$. Relative to the ground state for the Higgs boson, MOD links modeling for excitations to a ground state that links to $D = 1$ and $\nu = 0$.

4.3.2. Interaction patterns - for bosons and fermions - beyond Gauge and Higgs notions

This unit notes possibilities for parallels to MOD Gauge symmetries and to MOD notions of a Higgs field.

PAT suggests - regarding boson elementary particles - the following parallels to discussion above about Gauge and Higgs notions. Each LRI boson elementary particle might link to a $U(1)$ symmetry. (PAT does not yet explore the notion that - across LRI boson elementary particles - the respective symmetries might point to more than one MOD symmetry.) Regarding the inflaton boson elementary particle, a parallel - the inflaton field is to the inflaton elementary particle as the Higgs field is to the Higgs boson - to the Higgs boson might pertain.

This paper does not discuss notions that - regarding aspects of the MOD elementary particle Standard Model - the Higgs mechanism might link to nonzero mass for one or both of nonzero-mass boson elementary particles and fermion elementary particles. This paper does not discuss notions that a would-be inflaton mechanism might link to nonzero properties for one or both of nonzero-property boson elementary particles and nonzero-property fermion elementary particles.

4.3.3. Interrelations for properties of boson elementary particles

This unit discusses interrelations regarding properties of boson elementary particles.

Regarding boson elementary particles, PAT defines $(N')^2$ via Eqs. (47) and (48). M' denotes $m/(m_Z/3)$. S' denotes S (as in the spin, in units of $\hbar$), Q' denotes the magnitude of the charge, in units of the magnitude of the charge of the W boson. (MOD equates the magnitude of the charge of the W boson to the magnitude of the charge of the electron.) μ' denotes the magnitude of the magnetic moment, in units of the magnitude of the magnetic moment of the W boson.

$$(N')^2 \equiv (M')^2 + (S')^2 + (Q')^2 + (\mu')^2 - (T')^2 \tag{47}$$

$$(T')^2 = 1 \ \Leftrightarrow \ M' > 0 \ ; \quad (T')^2 \ \Leftrightarrow \ M' = 0 \tag{48}$$

PAT suggests that Eqs. (49) and (50) might pertain regarding all known boson elementary particles and all boson elementary particles that PAT suggests. (PAT does not suggest any elementary bosons for which $M' > 0$ and $N' \leq 2$.)

$$N' \in \{0, 1, 2, 3, 4\} \tag{49}$$

$$N' = 4 - S' \geq 3 \ \Leftrightarrow \ M' > 0 \ ; \quad N' = S' \ \Leftrightarrow \ M' = 0 \tag{50}$$

Eqs. (47), (48), (49), and (50) might point to the possibility that the notion of N' links to modeling that lies beyond both MOD and the scope of this paper.

PAT suggests that Eqs. (47), (48), (49), and (50) might associate with the notion that inferred spin (as in 1x $2 = |-2+4|$) detracts from inferred energy (as in 1x $2 = |+2|$). PAT suggests that Eqs. (47), (48), (49), and (50) might associate with notions that inferred charge (as in 1x $1 = |+1|$) and inferred magnetic moment (as in 1x $1 = |-1+2|$) detract from inferred energy.

4.3.4. Interrelations for properties of elementary fermions, including neutrino masses

This unit discusses interrelations regarding properties of fermion elementary particles. This unit suggests neutrino masses.

Table 3.9.1 in Ref. [76] shows an orderly arraying (based on charges and flavours) of logarithms of the masses of the three charged leptons and the six quarks.

PAT interprets Table 3.9.1 in Ref. [76] as suggesting that - for fermion elementary particles - an analog to Eq. (47) pertains and that the analogous equation has bases in logarithms (and not in squares) of ratios of magnitudes of particle properties.

Ref. [77] includes a formula that approximately links the masses of the six quarks and the three charged leptons.

Ref. [78] suggests that data point to the notion that the sum of the three neutrino rest energies is at least approximately 0.06 eV and not more than approximately 0.12 eV. Ref. [79] discusses data and modeling regarding upper bounds for the sum of the rest energies of the three neutrinos. Ref. [80] discusses a lower bound of 0.06 eV, an upper bound of 0.15 eV, and a possible upper bound of 0.12 eV. Ref. [33] suggests that an upper bound might be approximately 0.10 eV.

Eq. (51) shows a result based on extrapolation based on the formula that links the masses of the nine nonzero charge fermion elementary particles.

$$mc^2 \approx 3.448 \times 10^{-2} \text{ eV} \tag{51}$$

PAT suggests the following two possibilities, either of which might comport with bounds regarding the sum of the three neutrino rest energies.

1. $mc^2 \approx 3.448 \times 10^{-2}$ eV pertains for each of the three neutrinos.
2. $mc^2 \approx 3.448 \times 10^{-2}$ eV pertains for each of two neutrinos. For one neutrino, one of $mc^2 \approx 4.2 \times 10^{-6}$ eV and $mc^2 \approx 4.4 \times 10^{-4}$ eV might pertain.

This paper does not try to explore the extent to which PAT notions - such as notions regarding anomalous spin and 1x use of the $|6| = |+2+4|$ solution-pair or such as notions regarding interactions that associate with 1x $|s| > 0$ properties for which $\{5, 7\} \cap K \neq \emptyset$ - might suffice to explain neutrino oscillations, including for the case in which just one rest energy pertains for all three neutrinos.

4.3.5. Baryon asymmetry

This unit suggests an explanation for baryon asymmetry.

MOD suggests that each interaction that creates a left-handed (as in matter) elementary fermion also produces a right-handed (as in antimatter) elementary fermion.

PAT uses the notation f_{l-r} to denote the handedness of an elementary fermion. PAT links $f_{l-r} = +1$ to left-handedness. PAT links $f_{l-r} = -1$ to right-handedness. PAT posits that conservation of fermion handedness - as in $0 = +1 - 1$ - pertains regarding each interaction that - in the sense of MOD - creates (or destroys) two fermion elementary particles.

PAT suggests that each one of the three isomer-pairs links to its own instance of conservation of fermion handedness. PAT suggests that an interaction can create (for example) a pair of fermion elementary particles for which the left-handed fermion elementary particle links to isomer-zero and the right-handed fermion elementary particle links to isomer-three.

PAT suggests the following scenario.

- At sometime early in the history of the universe, for each isomer, the number of left-handed fermion elementary particles equaled the number of right-handed fermion elementary particles.
- For the isomer-pair that links to isomer-zero and isomer-three, at some time, a run-away (as in lasing-like) burst of interactions led to the dominance (in what PAT happens to call isomer-zero stuff) of the MOD notion of left-handed fermion elementary particles. Here, dominance links to particle counts. PAT suggests that a dominance (in what PAT happens to call isomer-three stuff) of right-handed fermion elementary particles exists.

Examples of 1x uses of solution-pairs that might associate with such lasing include solution-pairs for which $|s| = 1$ and $K = \{3, 4\}$ or $K = \{3, 4, 5, 7\}$, solution-pairs for which $|s| = 2$ and $K = \{1, 4, 5\}$ or $K = \{1, 4, 7\}$, and solution-pairs for which $|s| = 3$ and $K = \{1, 4\}$ or $K = \{1, 4, 5, 7\}$. For each one of the six (as in three values of $|s|$ times two sets K for each value of $|s|$) cases, for the 1x use, $R_{/i} = 2$ pertains, $6 \notin K$ pertains, and a 2x use of a one-step cascade that associates with $6 \in K$ can pertain.

4.3.6. Vacuum energy and long-range-interaction-boson ground states

This unit discusses the notion that MOD might want to include the possibility that the total - regarding electromagnetism, regarding gravity, and regarding the combination of electromagnetism and gravity - vacuum energy can be zero.

MOD explores the notion of a so-called vacuum energy density that might link to the sum over all possible photon modes of the ground-state energies that Eq. (33) suggests. To keep the sum from being infinite, MOD explores notions of limiting the range of ω (or, equivalently, of limiting the range of λ).

MOD also explores the notion of a so-called vacuum energy density that might link to the term Λ (as in a so-called cosmological constant) in the general relativity Einstein field equations. MOD explores the notion that a positive value of Λ might link to dark-energy phenomena.

MOD indicates that, compared to values of Λ that might comport with data, vacuum energy densities that link to Eq. (33) are too large - perhaps by a factor of 10^{120}.

PAT suggests that MOD for the ground state of a photon could link to the ground state of a $D = 2$ (as in two-dimensional) isotropic harmonic oscillator. In effect, $D = 2$ links to the sum of $D = 1$ for the $s = +1$ (as in left-circular polarization) mode and $D = 1$ for the $s = -1$ (as in right-circular polarization) mode.

PAT suggests that the ground state for a photon (and for any other LRI boson) links to $D = 2$ and $\nu = -1$. Per Eq. (33), each ground state links to $E = 0$. The sum over all such ground states (even without invoking notions of limiting the ranges of frequencies) is $E = 0$.

Possibly, MOD might want to consider the possibility that photons (or electromagnetism) and gravitons (or gravity) might contribute nothing to a cosmological constant Λ.

4.3.7. General relativity and quantum gravity

This unit suggests limits regarding the applicability of general relativity, discusses interpretations of general relativity, and provides insight that might have uses regarding the notion of quantum gravity.

Table 10 discusses possibly useful notions of matches (that PAT suggests) between some CAT+ aspects and the MOD general relativity notion of a stress-energy tensor.

PAT suggests bounds on the appropriateness of general relativity. For example, MOD based on general relativity would not be appropriate to the extent that some relevant objects include stuff that links to more than one isomer-pair and at least one such relevant object models as having significant spin.

To date, possibly no precision test of general relativity has involved objects that link to more than one isomer-pair.

PAT might not disturb MOD notions of so-called equivalence principles - regarding mass or energy - that MOD links to general relativity. However, PAT points to the notion that - regarding spin (as in 1x $|-2+4|$) and other possibly relevant properties (such as moments of inertia, as in 1x $|+1-2+3|$) - no similar principle would pertain.

MOD discusses two interpretations of modeling based on general relativity. One interpretation links to notions of forces. The other interpretation links to notions of geodesic motions (of small-mass objects)

Table 10: Possible matches between some CAT+ aspects and the MOD general relativity notion of a stress-energy tensor. Regarding components of a stress-energy tensor, the equations $1 \leq a \leq 3$, $1 \leq b \leq 3$, and $a < b$ pertain. T^{00} associates with energy density. T^{0a} associates with momentum density. T^{a0} associates with energy flux. T^{aa} associates with pressure and with spin density. T^{ab} associates with shear stress. T^{ba} associates with momentum flux. The first four rows in the table associate with $1 \leq n_k \leq 3$. The last three rows in the table associate with $4 \leq n_k$. For each $|s| = |\cdots|$ to which the table alludes, $|s| = 2$. The symbol C-1 links to the notion of the use of the solution-pair $2 = |s| = |+2|$ and of uses of solution-pairs that cascade from $2 = |s| = |+2|$. The symbol C-2 links to the notion of the use of the solution-pair $2 = |s| = |+1-2+3|$ and of uses of solution-pairs that cascade from $2 = |s| = |+1-2+3|$.

Components	Force	C-1 solution-pair	C-1 $R_{/i}$	C-2 solution-pairs	C-2 $R_{/i}$
T^{00}	Attracting	1x $\|+2\|$	6	-	-
T^{0a}, T^{a0}	Repelling	2x $\|-2+4\|$	6	-	-
T^{aa}	Repelling	1x $\|-2+4\|$	2	-	-
T^{ab}, T^{ba}	Attracting	2x $\|-2-4-8\|$	2	1x $\|+1-2+3\|$	1
?	Repelling	-	-	2x $\|\pm 2-1-3+4\|$	1
?	Repelling	-	-	1x $\|\pm 2-1-3+4\|$	1
?	Attracting	-	-	2x $\|\pm 2-1-3-4+8\|$	1

and a curved space-time. PAT suggests that (even if modeling ignores electromagnetic and other non-gravitational interactions) geodesic motion does not pertain, (at least) because the reach $R_{/i}$ regarding spin is (two and thus) less than six.

Possibly, MOD inabilities to develop models for would-be quantum gravity link to the notion that general relativity is not a basis for adequately accurate modeling regarding gravitational phenomena.

MOD regarding electromagnetic interactions with properties such as charge might use classical physics notions. MOD regarding electromagnetic interactions with properties such as atomic states use quantum physics notions.

PAT might (regarding gravitational interactions) suggest the following notions.

- For gravitational aspects related to properties such as energy, momentum, and spin, classical modeling might suffice.
- For gravitational aspects related to properties that link to 1x uses of solution-pairs such as the two solution-pairs $2 = |s| = |\pm(-1-3+4)-5+7|$ and the two solution-pairs $2 = |s| = |\pm(-1-2+3)-5+7|$, modeling based on quantum gravity might be useful.

In this sense, PAT might suggest that future MOD regarding quantum gravity could be as easy as - and as hard as - MOD quantum electrodynamics regarding multi-component systems.

4.3.8. ΛCDM cosmology and modified gravity

This unit discusses the extents to which PAT results associate with MOD notions of ΛCDM cosmology and MOD notions of modified gravity.

Some MOD modeling associates with the two-element term ΛCDM cosmology.

In the term ΛCDM cosmology, the symbol Λ associates with a term in the Einstein field equations. PAT suggests (based on notions that associate with reaches) that the Einstein field equations do not adequately comport with nature. PAT suggests that - in and beyond uses of the Einstein field equations - notions paralleling $\Lambda \neq 0$ are not necessarily needed to explain data.

In the term ΛCDM cosmology, the term CDM associates with the three-word phrase cold dark matter. PAT suggests the following notions. Stuff based on isomer-one, isomer-two, isomer-four, and isomer-five associates with notions of cold dark matter. Stuff based on isomer-three associates with the two-word term dark matter and does not necessarily associate with notions of CDM.

Some MOD modeling associates with the two-word phrase modified gravity. Possibly, people would consider that some notions that associate with PAT notions of reaches associate with notions of modified gravity.

4.3.9. Some notions that might motivate further research

This unit discusses notions that might motivate further research.

To what extent might the six in Eq. (12) associate with the six in the notion of six isomers?

To what extent might CAT+ associate, with the notion of anomalous magnetic moments, notions that charged leptons might exhibit distributions of charge or mass that are spatially oval or spatially oblate, and that, therefore, physics might anticipate notions of beyond (in the sense of smallness) elementary

Table 11: Aspects of CAT+ and aspects of MOD. The leftmost column lists notions that have bases in CAT+ uses of integers and integer-arithmetic equations. The symbol ‡ links to the notion that 1x $1 = |\pm(-1-2+3)-4+5|$ might pertain (also or instead of 1x $1 = |-4+5|$). The symbol CBEPF abbreviates the six-word term candidate basis for elementary particle families. This table assumes that CBEPF does not include families that Table 6 does not list. EP abbreviates elementary particle. The next two columns feature notions - from PAT - of instances and reaches. The fourth column names properties. MOD includes each property, except isomer-pair and possibly precessing magnetic moment. (The Earth exhibits precessing magnetic moment. Precessing magnetic moment does not necessarily associate directly with the notion of charge or with the notion of Larmor precession.) The symbol EF abbreviates the two-word term elementary fermion. The symbol 2CO abbreviates the two-element phrase two-component object. LRI abbreviates long-range interaction. In the symbol column, each symbol (except for f_{l-r}) echoes MOD notation. Regarding the column with the label A, the following four sentences pertain. The symbol A denotes that LRI interactions do not transmit the related property and that the property adds across objects. The symbol A' denotes that LRI interactions transmit the related property and that the property adds across objects and LRI fields. PAT suggests that - for each A property - n_i instances of a conservation law pertain. PAT suggests that - for each A' property - one instance of a conservation law pertains. The following sentences pertain regarding the column with the one-word label note. Rows that note three rational numbers point to values that elementary particles exhibit. The symbol †1 links to the notion of $|Q| = |q|/|q_e|$ and to the triad - which pertains for nonzero-charge elementary particles of $|Q| = 1/3$, $|Q| = 2/3$, and $|Q| = 1$. Regarding the property of energy, MOD includes the notion of one temporal CV (as in complementary variable) - time. Regarding the property of momentum, MOD includes the notion of three spatial CV - three components of a position 3-vector. The symbol †2 links to the notion that one of uninferred, boson, or fermion might pertain. The symbol †3 links to the notion that one temporal dimension and three spatial dimensions might pertain.

CAT+ notion	n_i	$R_{/i}$	Property	Symbol	A	Note
1x $1 = \|+1\|$	6	1	Charge	q	A	†1
1x $2 = \|+2\|$	1	6	Energy	E	A'	1 temporal CV
2x $2 = \|-2+4\|$	1	6	Momentum	P	A'	3 spatial CV
1x $2 = \|-2+4\|$	3	2	Spin	$S\hbar$	A'	†2
1x $3 = \|+3\|$	3	2	EF handedness	f_{l-r}	A	+1, 0, −1
1x $4 = \|+4\|$	6	1	EF baryon number	B	A	+1, 0, −1
1x $4 = \|+4\|$	6	1	EF lepton number	L	A	+1/3, 0, −1/3
1x $4 = \|+4\|$	6	1	$B-L$	$B-L$	A	-
1x $6 \in K$	6	1	EF flavour	-	-	3 flavours
$k' = 0$	3	2	Isomer-pair	-	-	3 isomer-pairs
$k' = -1$	6	1	Quark color charge	-	-	3 colors
$k' = -2$	-	-	Modeling dimensions	-	-	†3
1x $1 = \|-1-2+4\|$	6	1	Precessing magnetic moment	-	-	-
1x $1 = \|-4+5\|$ ‡	6	1	Surface temperature	-	-	-
1x $1 = \|\cdots 5-7+8\|$	3	2	2CO atomic state	-	-	-
1f CBEPF $16 \notin K$	6	1	Non-LRI EP	-	-	-
1f CBEPF $16 \in K$, $S \leq 2$	1	6	LRI EP	-	-	-
1f $\{5,7\} \cap K \neq \emptyset$	-	-	Spins related to 2CO systems			†2
EP, even $2S$, 2f aspects	-	-	Gauge symmetries	-	-	-

particles? Here, for $|s = 2|$ and 1x, $n_k = 1$ might associate with spherical, $n_k = 2$ might associate with one axis and with oval or oblate, and $n_k = 3$ might associate with two axes and two moments. For example, regarding 1x and $n_k = 2$, if $1 = |-1+2|$ associates with a spherical distribution of charge, perhaps $3 = |+1+2|$ associates with oval or oblate.

4.3.10. A characterizations-centric catalog of attributes and conservation laws

This unit discusses a catalog of properties of objects, minimum magnitudes for nonzero values of some properties, interaction forces, and conservation laws.

Table 11 points to some links between aspects of CAT+ and aspects of MOD.

The following notions pertain regarding Table 11.

- PAT suggests linking elementary-fermion handedness to 1x $3 = |s| = |+3|$, based on the notion that - for 1x $3 = |s| = |+3|$ - $n_i = 3$.
- PAT suggests linking elementary-fermion baryon number and elementary-fermion lepton number to 1x $4 = |s| = |+4|$, based on the notion that - for 1x $4 = |s| = |+4|$ - $n_i = 6$.
- PAT suggests that each property that associates with A or A' associates with a conservation law.
  - For each item (except the $2 = |s| = |\cdots|$ items) for which one of A or A' pertains, the relevant LRI field does not transmit the property from one object to another object. n_i instances of the property pertain. n_i instances of conservation of the property pertain.

  - For each one of the $2 = |s| = |\cdots|$ items for which A' pertains, the gravitational field exhibits and can transmit the property. Independent of the notion that $n_i = 3$ pertains for 1x $2 = |-2+4|$, one instance of conservation of the property 1x $2 = |-2+4|$ pertains across the gravitational field and all six isomers.

- PAT suggests the following regarding properties that associate with A or A'.
  - For each property other than energy and momentum, a nonzero minimum absolute value exists for nonzero values. For example, for charge the minimum is $|Q| = |q|/|q_e| = 1/3$. The minimum for spin (divided by $\hbar$) is 0.5.
  - Energy (a 1x property) and momentum (the 2x property that associates with energy) are the only two properties for which MOD notions of complementary variables (as in, respectively, time and position) pertain and for which some MOD notions of uncertainty pertain.

- PAT suggests (but might not necessarily imply) that 1x $3 = |s| = |+3|$ links to an elementary particle (which would be a long-range-interaction $S = 3$ boson elementary particle).

- PAT suggests (but might not necessarily imply) that 1x $4 = |s| = |+4|$ links to an elementary particle (which would be a long-range-interaction $S = 4$ boson elementary particle).

4.3.11. Possible other uses of PAT and CAT+ techniques

This unit suggests that techniques like PAT techniques and CAT+ techniques might have uses regarding materials science.

A combination of MOD and CAT+ seems to associate with various notions that parallel notions regarding MOD for some solid-state materials. Such notions include the following.

- Band gaps between minimal observable nonzero magnitudes of properties (such as energy or charge) and zero.

- Fractional (compared to boson-physics) values of some integer properties (such as spin and charge).

- Boson aspects that associate with paired fermions.

- Aspects that MOD might associate with reduced numbers of degrees of freedom or with reduced numbers of dimensions. For example, modeling for the linear motion of a hadron might associate with three degrees of freedom or with three spatial dimensions, but individual quarks (and individual gluons) model as entangled and MOD modeling for individual particles might not necessarily associate with three linear degrees of freedom or with three spatial dimensions.

Also perhaps noteworthy is the notion that - for Eqs. (28), (29), (30), and (31) - D does not necessarily need to be an integer.

Possibly, PAT and CAT+ notions can prove useful regarding fermion-pairing in solid-state materials and regarding phenomena that associate with so-called topological materials. (An example of the notion of topological material is a material that conducts electric current near a surface of the material, but not significantly within the remainder of the material.)

5. Conclusion

This paper uses PAT (as in pattern matching) and suggests so-called CAT+ (as in cataloging that associates with characterizations) for some areas of physics.

PAT suggests possibly useful reuses (as in isomers of non-long-range-interaction elementary particles) of some MOD (as in modeling that associates with space-time coordinates).

PAT suggests that CAT+ branches from successful MOD.

This paper suggests that one can also consider that CAT+ might stand on its own and provide a basis from which MOD branches.

This paper suggests that co-use of and co-evolution of PAT, MOD, and CAT+ might beneficially impact modeling, cataloging, and explaining data and other aspects that associate with (at least) the following topics: properties of objects and fields, elementary particles and their properties, dark-matter phenomena, gravitational phenomena, and general physics.

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