Research Article

Asymptotically Periodic and Bifurcation Points in Fractional Difference Maps

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The first step in investigating fractional difference maps, which do not have periodic points except fixed points, is to find asymptotically periodic points and bifurcation points and draw asymptotic bifurcation diagrams. Recently derived equations that allow calculations of asymptotically periodic and bifurcation points contain coefficients defined as slowly converging infinite sums. In this paper we derive analytic expressions for coefficients of the equations that allow calculations of asymptotically periodic and bifurcation points in fractional difference maps.

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1. Introduction

Fractional difference maps are maps with power-law-like memory. They are used to model biological (see, e.g. [1][2]) and socio-economic (see, e.g., [3][4]) systems, memristors (see, e.g., [5]), in image and signal encryption (see, e.g. [6][7]), to control systems (see, e.g., [8][9]), etc.

It is known that continuous and discrete fractional systems may not have periodic solutions except fixed points (see, for example, [10][11]). All bifurcation diagrams based on the finite time calculations on single trajectories are only approximations depending on the initial conditions and the number of iterations. But the asymptotically periodic solutions of fractional difference equations do exist, and the equations for finding these points in generalized fractional maps were derived in [12][13][14]. These equations contain coefficients which are slowly converging series. The numerical evaluation of these series, in the case fractional and fractional difference maps, requires calculations of finite sums of tens of thousands of terms and calculations of the Riemann ζ-function. It is also known, from the stability analysis of the discrete fractional systems (see [15]), that, in the case of fractional difference

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maps, the corresponding series may be summable (see, e.g., [16][17][18][19]). The equations that define bifurcation points of fractional difference maps [20] also depend on the same coefficients (sums).

In the following sections, after preliminaries in Section 2, we derive the analytic expressions for the coefficients (sums) of the equations defining periodic points in the case of fractional difference maps in Section 3. The concluding remarks are presented in Section 4.

2. Preliminaries

For $0 < \alpha < 1$, the generalized universal α-family of maps is defined as (see [12][13]):

$$x_n = x_0 - \sum_{k=0}^{n-1} G^0(x_k) U_\alpha(n-k), \tag{1}$$

where $G^0(x) = h^\alpha G_K(x)/\Gamma(\alpha)$, x_0 is the initial condition, h is the time step of the map, α is the order of the map, $G_K(x)$ is a nonlinear function depending on the parameter K, $U_\alpha(n) = 0$ for $n \le 0$, and $U_\alpha(n) \in \mathbb{D}^0(\mathbb{N}_1)$. The space $\mathbb{D}^i(\mathbb{N}_1)$ is defined as (see [13])

$$\mathbb{D}^i(\mathbb{N}_1) = \left\{ f: \ \sum_{k=1}^{\infty} \Delta^i f(k) > N, \ \forall N, \ \sum_{k=1}^{\infty} |\Delta^{i+1} f(k)| = C, \ C \in \mathbb{R}_+ \right\}, \tag{2}$$

where Δ is a forward difference operator defined as

$$\Delta f(n) = f(n+1) - f(n). \tag{3}$$

In the case Caputo fractional difference maps, which are defined as solutions of the Caputo h-difference equation[21][22][23]

$${}_0\Delta^{\alpha}_{h,*} x(t) = -G_K(x(t + (\alpha - 1)h)), \tag{4}$$

where $t \in (h\mathbb{N})_m$, with the initial conditions

$$({}_0\Delta^k_h x)(0) = c_k, \ k = 0, 1, \ldots, m-1, \ m = \lceil \alpha \rceil, \tag{5}$$

the kernel $U_\alpha(n)$ is the falling factorial function:

$$U_\alpha(n) = (n + \alpha - 2)^{(\alpha-1)}, \ U_\alpha(1) = (\alpha - 1)^{(\alpha-1)} = \Gamma(\alpha). \tag{6}$$

The definition of the falling factorial $t^{(\alpha)}$ is

$$t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}, \ t \ne -1, -2, -3.... \tag{7}$$

The falling factorial is asymptotically a power function:

$$\lim_{t\to\infty}\frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)t^{\alpha}}=1,\ \alpha\in\mathbb{R}. \tag{8}$$

The h-falling factorial $t_h^{(\alpha)}$ is defined as

$$t_h^{(\alpha)}=h^{\alpha}\frac{\Gamma(\frac{t}{h}+1)}{\Gamma(\frac{t}{h}+1-\alpha)}=h^{\alpha}\left(\frac{t}{h}\right)^{(\alpha)},\ \frac{t}{h}\neq-1,-2,-3,\ldots \tag{9}$$

Majority of the introduced, investigated, and used in applications maps are Caputo fractional difference maps.

The following equations define period-l points in generalized fractional maps of the orders $0<\alpha<1$ [12]:

$$x_{lim,m+1}-x_{lim,m}=S_{1,l}G^0(x_{lim,m})+\sum_{j=1}^{m-1}S_{j+1,l}G^0(x_{lim,m-j})$$

$$+\sum_{j=m}^{l-1}S_{j+1,l}G^0(x_{lim,m-j+l}),\ 0<m<l, \tag{10}$$

$$\sum_{j=1}^{l}G^0(x_{lim,j})=0, \tag{11}$$

where

$$S_{j+1,l}=\sum_{k=0}^{\infty}\left[U_\alpha(lk+j)-U_\alpha(lk+j+1)\right],\ 0\leq j<l. \tag{12}$$

It is easy to see that

$$\sum_{j=1}^{l}S_{j,l}=0. \tag{13}$$

In the case of p-dimensional maps ($1\leq i\leq p$) (see [14])

$$x_{i,n}=x_{i,0}-\sum_{k=0}^{n-1}G_i^0(x_{1,k},x_{2,k},\ldots,x_{p,k})U_{\alpha_i}(n-k), \tag{14}$$

the periodic points are defined as solutions of the system of $(l-1)\times p$ equations:

$$x_{i,l,m+1}-x_{i,l,m}=\sum_{j=0}^{m-1}S_{i,j+1,l}G_i^0(x_{1,l,m-j},x_{2,l,m-j},\ldots,x_{p,l,m-j})$$

$$+\sum_{j=m}^{l-1}S_{i,j+1,l}G_i^0(x_{1,l,m-j+l},x_{2,l,m-j+l},\ldots,x_{p,l,m-j+l}),$$

$$0<m<l,\ 0<i\leq p \tag{15}$$

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and additional p equations

$$\sum_{j=1}^{l} G_i^0(x_{1,l,j}, x_{2,l,j}, \dots, x_{p,l,j}) = 0, \ 0 < i \le p. \tag{16}$$

Bifurcation points in the maps of the order $0 < \alpha < 1$ are defined by the Theorem 1 from [20]:

Theorem 2.1. *The $T = 2^{n-1}$ – $T = 2^n$ bifurcation points, 2^{n-1} values of $x_{2^{n-1}bif,i}$ with $0 < i \le 2^{n-1}$ and the value of the nonlinear parameter $K_{2^{n-1}bif}$, of a fractional generalization of a nonlinear one-dimensional map $x_{n+1} = F_K(x_n)$ written as the Volterra difference equations of convolution type*

$$x_n = x_0 - \sum_{k=0}^{n-1} G^0(x_k) U_\alpha(n-k), \tag{17}$$

where $G^0(x) = h^\alpha G_K(x)/\Gamma(\alpha)$, x_0 is the initial condition, h is the time step of the map, α is the order of the map, $G_K(x) = x - F_K(x)$, $U_\alpha(n) = 0$ for $n \le 0$, $U_\alpha(n) \in \mathbb{D}^0(\mathbb{N}_1)$, and

$$\begin{aligned} \mathbb{D}^i(\mathbb{N}_1) \quad = \quad & \left\{ f : \left| \sum_{k=1}^{\infty} \Delta^i f(k) \right| > N, \right. \\ & \left. \forall N, N \in \mathbb{N}, \sum_{k=1}^{\infty} \left| \Delta^{i+1} f(k) \right| = C, C \in \mathbb{R}_+ \right\}, \end{aligned} \tag{18}$$

are defined by the system of $2^{n-1} + 1$ equations

$$\begin{aligned} x_{2^{n-1}bif,m+1} - x_{2^{n-1}bif,m} &= S_{1,2^{n-1}} G^0(x_{2^{n-1}bif,m}) \\ &+ \sum_{j=1}^{m-1} S_{j+1,2^{n-1}} G^0(x_{2^{n-1}bif,m-j}) \\ &+ \sum_{j=m}^{2^{n-1}-1} S_{j+1,2^{n-1}} G^0(x_{2^{n-1}bif,m-j+2^{n-1}}), \\ & 0 < m < 2^{n-1}, \end{aligned} \tag{19}$$

$$\sum_{j=1}^{2^{n-1}} G^0(x_{2^{n-1}bif,j}) = 0, \tag{20}$$

$$\det(A) = 0, \tag{21}$$

where

$$\begin{aligned} S_{j+1,l} = \sum_{k=0}^{\infty} \left[U_\alpha(lk+j) - U_\alpha(lk+j+1) \right], \\ 0 \le j < l, \quad S_{i,l} = S_{i+l,l}, \quad i \in \mathbb{Z}, \end{aligned} \tag{22}$$

and the elements of the 2^{n-1}-dimensional matrix A are

$$A_{i,j} = \frac{dG^0(x)}{dx}\Big|_{x_{2^{n-1}bif,j}} \sum_{m=i}^{i+2^{n-1}-1} S_{m-j+1,2^n} + \delta_{i,j}, \qquad (23)$$

$0 < i, j \leq 2^{n-1}$.

3. Sums $S_{p,l}$ for l-cycles of fractional difference maps

The definition of $S_{p,l}$ from [12] for fractional difference maps may be rewritten using the following chain of transformations:

$$S_{p,l} = \sum_{k=0}^{\infty} \left[(lk+p+\alpha-3)^{(\alpha-1)} - (lk+p+\alpha-2)^{(\alpha-1)} \right] = \sum_{k=0}^{\infty} \left[\frac{\Gamma(lk+p+\alpha-2)}{\Gamma(lk+p-1)} \right.$$

$$\left. - \frac{\Gamma(lk+p+\alpha-1)}{\Gamma(lk+p)} \right] = (1-\alpha) \sum_{k=0}^{\infty} \frac{\Gamma(lk+p+\alpha-2)}{\Gamma(lk+p)} = -\Gamma(\alpha) \sum_{k=0}^{\infty} \frac{\Gamma(lk+p+\alpha-2)}{\Gamma(\alpha-1)\Gamma(lk+p)}$$

$$= -\Gamma(\alpha) \sum_{k=0}^{\infty} \binom{lk+p+\alpha-3}{lk+p-1} = \Gamma(\alpha)(-1)^p \sum_{k=0}^{\infty} (-1)^{lk} \binom{1-\alpha}{lk+p-1}. \qquad (24)$$

Using absolute convergence of series and the following identity (see [24])

$$\sum_{k=0}^{\infty} \binom{\gamma}{t+ks} = \frac{1}{s} \sum_{j=0}^{s-1} \omega^{-jt} (1+\omega^j)^{\gamma}, \qquad (25)$$

where $\omega = e^{i2\pi/s}$, for the even and odd periods we obtain

$$S_{p,2n} = \Gamma(\alpha)(-1)^p \sum_{j=0}^{\infty} \binom{1-\alpha}{2nj+p-1} = \frac{\Gamma(\alpha)}{2n} (-1)^p \sum_{j=0}^{2n-1} e^{-i\pi j(p-1)/n} (1+e^{i\pi j/n})^{1-\alpha}$$

$$= \frac{\Gamma(\alpha)}{2n} (-1)^p \left[2^{1-\alpha} + \sum_{j=1}^{n-1} \left(e^{-i\pi j(p-1)/n} e^{i\pi j(1-\alpha)/(2n)} (2\cos(\pi j/(2n)))^{1-\alpha} \right. \right.$$

$$\left. \left. + e^{i\pi j(p-1)/n} e^{-i\pi j(1-\alpha)/(2n)} (2\cos(\pi j/(2n)))^{1-\alpha} \right) \right]$$

$$= \frac{\Gamma(\alpha) 2^{-\alpha}}{n} (-1)^p \left[1 + 2 \sum_{j=1}^{n-1} (\cos(\pi j/(2n)))^{1-\alpha} \cos(\pi j(2p+\alpha-3)/(2n)) \right], \qquad (26)$$

$$S_{p,2n+1} = \Gamma(\alpha)(-1)^p \sum_{j=0}^{\infty} (-1)^j \binom{1-\alpha}{(2n+1)j+p-1}$$

$$= \Gamma(\alpha)(-1)^p \left\{ \sum_{j=0}^{\infty} \binom{1-\alpha}{2(2n+1)j+p-1} - \sum_{j=0}^{\infty} \binom{1-\alpha}{2(2n+1)j+p+2n} \right\}$$

$$= \frac{\Gamma(\alpha)}{2(2n+1)} (-1)^p \left\{ \sum_{j=0}^{4n+1} e^{-\frac{i\pi j(p-1)}{2n+1}} (1+e^{\frac{i\pi j}{2n+1}})^{1-\alpha} - \sum_{j=0}^{4n+1} e^{-\frac{i\pi j(p+2n)}{2n+1}} (1 \right.$$

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$$+e^{\frac{i\pi j}{2n+1}})^{1-\alpha}\Big\} = \frac{\Gamma(\alpha)}{2(2n+1)}(-1)^p \sum_{j=1}^{2n}\Big\{\Big(e^{-\frac{i\pi j(p-1)}{2n+1}} - e^{-\frac{i\pi j(p+2n)}{2n+1}}\Big)(1$$

$$+e^{\frac{i\pi j}{2n+1}})^{1-\alpha} + \Big(e^{\frac{i\pi j(p-1)}{2n+1}} - e^{\frac{i\pi j(p+2n)}{2n+1}}\Big)(1+e^{-\frac{i\pi j}{2n+1}})^{1-\alpha}\Big\}$$

$$= \frac{\Gamma(\alpha)}{2n+1}(-1)^p i \sum_{j=1}^{2n}\left(2\cos\frac{\pi j}{2(2n+1)}\right)^{1-\alpha} \sin\frac{j\pi}{2}\Big(e^{-\frac{i\pi j(2p+2n-2+\alpha)}{2(2n+1)}}$$

$$-e^{\frac{i\pi j(2p+2n-2+\alpha)}{2(2n+1)}}\Big) = \frac{2^{2-\alpha}\Gamma(\alpha)}{2n+1}(-1)^p \sum_{j=1}^{2n}\left(\cos\frac{\pi j}{2(2n+1)}\right)^{1-\alpha} \sin\frac{j\pi}{2}$$

$$\times \sin\frac{\pi j(2p+2n-2+\alpha)}{2(2n+1)}$$

$$= \frac{2^{2-\alpha}\Gamma(\alpha)}{2n+1}(-1)^p \sum_{j=0}^{n-1}\left(\cos\frac{\pi(2j+1)}{2(2n+1)}\right)^{1-\alpha}(-1)^j$$

$$\times \sin\frac{\pi(2j+1)(2p+2n-2+\alpha)}{2(2n+1)} \tag{27}$$

Although in[24], Eq. (25) is proven for integer values of $\gamma \in \mathbb{N}_0$, it is valid for any real values $0 \le \gamma \le 1$. The calculated values of $S_{2,4}$ obtained when we used Eq. (26) for $\alpha = 0.5$, $\alpha = 0.99$, and $\alpha = 0.999$ are the same as the corresponding values obtained using the expression which allows a fast calculation of the series using tens of thousands of operations (see Eq. (35) in[12]): 1.029970, 0.2571808, and 0.2507111.

4. Conclusion

In this paper we derived the analytic expressions for the coefficients of the equations that define periodic points in fractional difference maps of the orders $0 < \alpha < 1$ (Eqs. (26) and (27)). To draw asymptotic bifurcation diagrams of fractional difference maps, researchers should solve Eqs. (10) and (11) in the case of maps of the orders $0 < \alpha < 1$, or Eqs. (15) and (16) in the case of multidimensional fractional maps. Calculations of coefficients (sums) $S_{p,l}$ of these equations should be a part of the corresponding numerical algorithms. Using analytic expressions derived in this paper instead of adding tens of thousands of terms based on Eq. (35) from[12] will make calculations of the coefficients thousands of times faster but will not change the time of the calculations of the periodic and bifurcation points. It took a couple of months to calculate data for the bifurcation diagram Fig. 1 and Table 1 from[20]. The most time-consuming task was to solve systems of large numbers of algebraic equations and analyze their solutions. Using Eqs. (26) and (27) will not significantly change the situation.

Based on the results of numerical simulations, in[20], the authors made a conjecture that the Feigenbaum number δ [25] exists in fractional difference maps and has the same value as in regular maps. Theorem 1 from[20] defining bifurcation points with analytic values of $S_{p,l}$ obtained in this paper could be used to prove this conjecture.

Statements and Declarations

Data availability

No data was used for the research described in the article.

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