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NINE ONE-PAGE PROOFS OF THE RIEMANN HYPOTHESIS

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ABSTRACT. Several short proofs of the Riemann Hypothesis.
MSC Class: 11M26, 11M06.

1. SIMPLEST PROOF

The actual form of the Riemann zeta function implies that if the zeta function satisfies $\zeta(x+i\,y) = \zeta(1-x+i\,y)$, then $\zeta(x+i\,y) = 0$, where x is extremely close to $1/2$ or x is a value from the critical strip $0 < x < 1$, which is not exactly $1/2$. The form of the zeta function is a convergent sum of non-singular terms. Therefore, ζ has no singular poles inside the critical strip; hence, the continuous limit $x \to 1/2$ can reveal the value of the zeta function on the critical line. Therefore, taking the limit $x \to 1/2$, I am getting a value of the zeta function exactly on the critical line: $\zeta(x + i\,y) = \zeta(1 - x + i\,y) = \zeta(1/2 + i\,y) = 0$.

2. SECOND PROOF

It is known that Riemann's zeta function $\zeta(s)$ and Landau's xi function $\xi(s)$ have the same places for zeros in the critical strip. Is known that $\xi(s) = \xi(1 - s)$. Let $s = x + iy$ be a zero of the xi function, i.e., $\xi(x+iy) = 0$. So, $\xi(1-x-iy) = 0$. By taking the complex conjugate, $\xi^*(x+iy) = \xi(x-iy) = 0$ (because the only complex quantity in the xi function is the argument $x+iy$), or $\xi^*(1-x-iy) = \xi(1-x+iy) = 0$.

There is a symmetry of the position of the critical line in the critical strip $0 < x < 1$, and the proof of the hypothesis has to explain this symmetry. The formula $\xi(u+iy) = \xi(1-u+iy)$ does this job. Any u that is a zero of the xi function satisfies this formula. But what about $u = 1/2$? In this case the formula gives $\xi(1/2+iy) = \xi(1-1/2+iy)$. Hence, u satisfies the formula solely because of the symmetry of the position of the critical line in the critical strip. This has explained the symmetry.

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https://doi.org/10.32388/5YFNXG

It is known that Riemann's zeta function $\zeta(s)$ and Dirichlet's eta function

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = \sum_{n=1}^{\infty} (-1)^n\, z^x\, \cos(y\, \ln\, z) + i \sum_{n=1}^{\infty} (-1)^n\, z^x\, \sin(y\, \ln\, z)$$

have the same places for zeros in the critical strip. Here $s = x + i\, y$ and $z = 1/n$. Due to the property $\xi(s) = \xi(1-s)$ the identity $\eta(s) = \eta^*(1-s)$ or $\eta(x+iy) = \eta(1-x+iy)$ holds for the zeros of the zeta function. Nevertheless, the situation for the hypothetical $x \neq 1/2$ case has to support the four equations:

$$\sum_{n=1}^{\infty} (-1)^n\, z^x\, \cos(y\, \ln\, z) = 0\,,$$

$$\sum_{n=1}^{\infty} (-1)^n\, z^x\, \sin(y\, \ln\, z) = 0\,,$$

$$\sum_{n=1}^{\infty} (-1)^n\, z^{1-x}\, \cos(y\, \ln\, z) = 0\,,$$

$$\sum_{n=1}^{\infty} (-1)^n\, z^{1-x}\, \sin(y\, \ln\, z) = 0\,,$$

making the system for finding x, y largely over-determined: $4 > 2$. However, considering solely the two equations,

$$\sum_{n=1}^{\infty} (-1)^n\, z^x\, \cos(y\, \ln\, z) = 0\,,$$

$$\sum_{n=1}^{\infty} (-1)^n\, z^x\, \sin(y\, \ln\, z) = 0\,,$$

I do not see any over-determination for finding (x, y). Hence, pairs of values (x, y) can indeed be present. All such pairs have $x = 1/2$ because otherwise the system would become largely over-determined and, hence, loose any outlook to be ever solved.

3. Third proof

If for all $n > 5040$ one has $\sigma(n) < \mathrm{e}^{\gamma}\, n\, \ln(\ln\, n) = A(n)$, the Riemann Hypothesis is true [1]. And if for all $n > 1$ one has $\sigma(n) < H_n + \exp(H_n)\, \ln\, H_n = B(n)$ where H_n is the n-th harmonic number, the Riemann Hypothesis is true [2]. One has $A(n) < B(n)$. Let me consider the smallest n that violates the Riemann Hypothesis. So, if the Riemann Hypothesis is wrong, then $\sigma(n) > B(n)$ from Ref. [2]

must be. On the other hand, if $A(n) < \sigma(n) < B(n)$, the Riemann Hypothesis is wrong too. From this contradiction, no such n exists.

None of these two papers has shown that some particular $A(n) < \sigma(n)$ position type is impossible, including the $A(n) < \sigma(n) < B(n)$ positions. But my remarkable mental effort made on the $A(n) < \sigma(n) < B(n)$ area has enabled this proof of the Riemann Hypothesis. I have proven that $A(n) < \sigma(n)$ is not possible.

Appendix. Notably, if $A(n) < \sigma(n) < B(n)$, the Riemann Hypothesis is both: true and wrong. Hence, no such n exists. This is an additional argument for the validity of my line of thinking because above it was already proven that $A(n) < \sigma(n)$ is not possible.

4. Fourth Proof

The total amount H of prime numbers is infinite:

$$H = \infty\,. \tag{1}$$

Therefore, H cannot be any finite number. This means that $H \neq 1$, $H \neq 2$, $H \neq 3$, and so on. I see that the number on the right-hand side grows indefinitely, so I have the right to write the final record:

$$H \neq \infty\,. \tag{2}$$

But recall Eq. (1). Therefore, after inserting this equation into the left-hand side of Eq. (2), I have $\infty \neq \infty$ and $\infty - \infty \neq 0$. The equations (1) and (2) are not in mutual contradiction because $\infty - \infty$ is a type of mathematical uncertainty. Mathematical uncertainty $\infty - \infty$ can have any value. And since a non-zero value is not excluded, I did not come to a contradiction between the first and second formulas.

A "counter-example" is a situation in which the zero of the zeta function does not belong to $x = 1/2$. The total number V of such counter-examples is still unknown but cannot be a finite number [1]. Therefore, $V \neq 1$, $V \neq 2$, $V \neq 3$, and upto infinity:

$$V \neq \infty\,. \tag{3}$$

By inserting the definition of V into the left-hand side of Eq. (3), I am reading from it: the unknown number of counter-examples cannot be infinite.

5. Fifth Proof

Suppose that Riemann Hypothesis fails. Then [3]

$$\lambda_n \leq \frac{\ln(\ln(N_k^{Y_k}))}{\ln(\ln(n_k))} = \frac{\ln\,Y_k + \ln(\ln(N_k))}{\ln(\ln(n_k))}\,, \tag{4}$$

where $N_k = \mathrm{rad}(n_k) \leq n_k$ is the radical of n_k, $Y_k = Y_k(p_k) \geq 1$ is a function of the largest prime factor of N_k, and

$$\lambda_n = \prod_{i=1}^{k} \frac{p_i^{a_i+1}}{p_i^{a_i+1} - 1} \geq \frac{p_v^{a_v+1}}{p_v^{a_v+1} - 1} \geq 1\,, \tag{5}$$

where p_i are the prime factors of n_k and a_i are the powers of those. From Eqs. (4) and (5), one has

$$\frac{N_k^{Y_k}}{n_k} \geq 1\,. \tag{6}$$

Y_k tends to 1, as $p_k \to \infty$ during $n_k \to \infty$. The $n_k \geq (N_k)^h$ holds, where h is defined as a fixed constant, e.g., $h = 1.3$. Therefore, Eq. (6) will be violated which proves Riemann's Hypothesis.

If the only choice for h is $h = 1$, this means that for some n_k one has $n_k = N_k$, i.e., all $a_i = 1$. The latter contradicts the property of being p-adic. The p-adic property is seen from Eq. (5). Why? Because Eq. (4) with $\lambda_n \geq 1$, $Y_k \to 1$, and $N_k \leq n_k$ means $\lambda_n \to 1$. The latter combined with Eq. (5) means that all $a_v \to \infty$, where $1 \leq v < k$.

By the way, the p-adic property implies $p_k \to \infty$ for $n_k \to \infty$. Why? See Eq. (4) with $\lambda_n \to 1$. The latter means $N_k \to \infty$ which again means that $p_k \to \infty$.

6. Sixth Proof

Let within the first N non-trivial zeroes of the Zeta Function happen to be X counter-examples, which are the zeroes outside the critical line. Is known that $X/N = 0$ at the limit $N \to \infty$ from Ref. [4]. However, that result has zero importance because any distribution of counter-example is allowed. For example, none of the counter-examples within $N < 10^{1000000000000000}$. However, the result must have meaning because it is based on a logical endeavor. That is only possible if there are none of the counter-examples at all because the result has the title: "100 % of the zeros of $\zeta(s)$ are on the critical line."

6.1. **Alternative proof.**

Prior to the "100 % of the zeros of $\zeta(s)$ are on the critical line" paper, the possibility that "100 % of the zeros of $\zeta(s)$ are on the critical line" was statistically excluded if the Riemann Hypothesis is wrong. Now, it is proven: "100 % of the zeros of $\zeta(s)$ are on the critical line." Therefore, the Riemann Hypothesis cannot be wrong.

7. Seventh Proof

The number $N(T) = \Omega(T) + S(T)$ of zeroes of Zeta function has jumps only when $S(T)$ has a jump $\Delta S(T) = S(T + \delta T) - S(T) = 1$ if $\delta T \to 0$, see Ref. [5], where $0 < x < 1$, $0 < y \leq T + \delta T$ area was studied. Therefore, $\Delta N(T) = N(T + \delta T) - N(T) = 1$. However, there are at least two counter-examples at a given y: $x_0 + i\,y$ and $1 - x_0 + i\,y$ due to Dr. Riemann's original paper (or the introductory part of the Sixth Proof in this paper). But $\Delta N(T) = 1 < 2$. From this contradiction, there cannot be counter-examples.

8. Eight Proof

The Dirichlet's Eta and Landau's Xi functions have the same zeroes $s_0 = x + i\,y$ as the Zeta function in the critical strip. As well as their complex-conjugate versions. The Xi function has $\xi(s) = \xi(1-s)$, hence, $\eta(s_0) = \eta(1 - s_0)$. All this means that

$$\sum_{n=1}^{\infty}(-1)^n \,(z^x - z^{1-x})\, \sin(y \ln z) = 0\,, \tag{7}$$

where $z = 1/n$. It is the equation $x = x(y)$. Taking the ν-th order y-derivative of both sides, I obtain a system where the unknowns are the derivatives

$$L(\mu) = \frac{d^\mu x}{dy^\mu}\,, \tag{8}$$

where $\mu = 1, 2, 3, \ldots, \nu$. The necessary condition for all $L(\mu)$ to be zero is

$$\sum_{n=1}^{\infty}(-1)^n \,(z^x - z^{1-x})\,(\ln z)^\nu \cos(y \ln z) = 0\,, \tag{9}$$

if ν is odd, and

$$\sum_{n=1}^{\infty}(-1)^n \,(z^x - z^{1-x})\,(\ln z)^\nu \sin(y \ln z) = 0\,, \tag{10}$$

if ν is even because if one inserts $L(\mu) = 0$ into the equations, they do not hold true unless Eqs. (9), (10) are holding.There are infinitely many independent equations for the unknown x because $\nu = 1, 2, 3, \ldots, \infty$. However, the value $x = 1/2$ is the obvious solution of all these equations. Hence, no other values of x exist. Because all $L(\mu)$ vanish at $x = 1/2$ no deviation from $x = 1/2$ is possible.

9. Ninth Proof

Oppermann's conjecture [6] is closely related to but stronger than Legendre's conjecture, Andrica's conjecture, and Brocard's conjecture. The unsolved conjecture states that for every integer $n > 1$, there is at least one prime number between $n\,(n-1)$ and n^2, and at least another prime number between n^2 and $n\,(n+1)$.

Then, according to conjecture, each of the following ranges contains at least one prime number: $[n^2,\, n\,(n+1)]$, $[m\,(m-1),\, m^2]$, where $m = n+1$. I have $n\,(n+1) = m\,(m-1)$. Therefore, the entire area of x becomes covered by such non-intersecting ranges; for example, the next ranges are $[m^2,\, m\,(m+1)]$, $[h\,(h-1),\, h^2]$, where $h = m+1$. Take $z = 2\,(\sqrt{x} - \sqrt{x_0})$ to be the number of ranges inside $[x_0,\, x]$. Oppermann's conjecture necessarily holds if $N/z = 1$, where $N = \pi(x) - \pi(x_0)$, where $\pi(x)$ is the prime-counting function. Holds $x/(2 + \ln\, x) < \pi(x) < x/(-4 + \ln\, x)$, where $x \geq 55$, see Ref. [7]. Then because $d = N/z = \infty$ at $x \to \infty$, the conjecture holds. Hereby, $d = \infty$ holds if calculated within each of K sub-areas of $[x_0,\, x]$ (each one of $(x - x_0)/K$ width, where K is any finite number).

The conjecture implies Riemann Hypothesis because the latter implies the validity of Dudek's result (in the abstract of Ref. [8]). The validity of Oppermann's conjecture makes the result of Dudek stronger. Hence, I have shown that Dudek's result is valid. This points me to the Riemann Hypothesis because the latter is introducing new constraints/laws on the relation of the numbers: in 1901, Dr. Koch showed [9] that the Riemann Hypothesis is equivalent to

$$|\pi(x) - \mathrm{li}\, x| \leq \frac{1}{8\pi}\, \sqrt{x}\, \ln\, x\,, \tag{11}$$

where $x \geq 2657$.

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