Quaternionic Bekenstein-Sanders Guage Fields for TeVeS

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Abstract

Treating the Bekenstein-Sanders field B_μ, for which $B_\mu B^\mu = -1$ as a gauge field requires that the field be non--Abelian. This structure was worked out in a previous publication by Horwitz, Gershon and Schiffer, where an equivalent Kaluza-Klein metric was found for an extended ($5D$) spacetime. In this paper, we study a quaternionic formulation of this theory with quaternionic gauge fields and quaternionic wave functions (as discussed in two seminal books by S.L. Adler), thereby establishing a connection between quaternionic quantum mechanics and general relativity.

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I. Introduction

The Bekenstein-Sanders [1] tensor-vector-scalar theory of gravitation (TeVeS) has been shown to account for the galactic rotation curves, lensing, and other cosmological phenomena (see review of Skordis [2]) without the significant presence of dark matter.[1]

It has recently been shown [3], that there is an invariant Hamiltonian formalism for the TeVeS theory, achieved by a conformal transformation, for which the essential Bekenstein-Sanders field B_μ , satisfying $B_\mu B^\mu = -1$, emerges as a gauge field (see also [4] for the many body case).Since the normalization condition $B_\mu B^\mu = -1$ must be maintained under gauge transformations, it is necessary that the field B_μ be non-Abelian, similar to a Yang-Millls [5] field.

The interesting possibility that the field B_μ can be represented as a quaternionic field is investigated in this paper. This possibility would imply that the quantum mechanical wave functions for which B_μ is the gauge field are also quaternionic, as discussed by [6][7].

In the following we will be working in the framework of the embedding of the relativistic quantum theory[8] in the curved space of Einstein's general relativity [9][10]. The vectors and tensors we shall discuss, and local partial derivatives are well-defined in the local tangent space at each point.

The dynamics of such quaternionic wave functions has been discussed by Adler[11] using *trace dynamics*, thereby opening up a possibly fruitful field relating quaternionic quantum mechanics and general relativity.

II. Quaternionic Non-Abelian Gauge

As discussed in [6], the quaternionic wave function $\psi(x)$ may undergo left and right gauge transformations

$$\psi \rightarrow \bar{\psi} = \omega\psi\omega', \quad (2.1)$$

with $\omega\omega^* = \omega'\omega'^* = 1$, where $\{\,^*\}$ is the quaternion conjugate, for complex units $\{e_i\}$, $i = 1, 2, 3$ and $e_1e_2e_3 = -1$, $e_i^2 = -1$ and cyclic, $(e_ie_j)^* = e_je_i$, $i \neq j$. The prime indicates the left gauge.

The covariant derivative [6] is defined by $(\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$, and indices are raised and lowered by the Minkowski metric $\eta_{\nu_\mu} = \{-1, +1, +1, +1\})$

$$\mathrm{D}_\mu\psi = \partial_\mu\psi + B_\mu\psi - \psi B'_\mu, \quad (2.3)$$

with $B_\mu{}^* = -B_\mu, B'_\mu{}^* = -B'_\mu$. Under gauge transformations of the form(2.1),

$$\begin{aligned} B_\mu &\rightarrow \omega B_\mu\omega^* + \omega\partial_\mu\omega^* \\ B'_\mu &\rightarrow \omega' B'_\mu\omega'^* + \omega'\partial_\mu\omega'^* \end{aligned} \quad (2.4)$$

Differentiating $\omega\omega^* = 1$, we see that

$$\omega\partial_\mu\omega^* = -\partial_\mu\omega\omega^* = -(\omega\partial_\mu\omega^*)^* \quad (2.5)$$

so the additional terms in equations (2.4) are pure quaternion imaginary.

Under the general gauge transformation

$$\begin{aligned} \bar{\mathrm{D}}_\mu\bar{\psi} &= \partial_\mu(\omega\psi\omega'^*) + (\omega B_\mu\omega^* - \partial_\mu\omega\omega^*)\omega\psi\omega'^* \\ &-(\omega\psi\omega'^*)(\omega' B'_\mu\omega'^* + \omega'\partial_\mu\omega'^*) \\ &= \partial_\mu(\omega\psi\omega'^*) + (\omega B_\mu\psi\omega'^* - \partial_\mu\omega\psi\omega'^*) \\ &-\omega\psi B'_\mu\omega'^* - \omega\psi\partial_\mu\omega'^* \\ &= \omega\partial_\mu\psi\omega'^* + \omega B_\mu\psi\omega'^* - \omega\psi B'_\mu\omega'^* \\ &= \omega\mathrm{D}_\mu\psi\omega'^*, \end{aligned} \quad (2.6)$$

showing that the covariant derivative of ψ transforms under gauge transformations in the same way as ψ [6].

We will be primarily interested in the left gauge in the following (because of the structure of the quantum quaternionic scalar product, as we shall see) but a similar argument is effective for the right sided gauge as well.

It is essential for the Bekenstein-Sanders results that, as mentioned above, under gauge transformations, the relation

$$B_\mu B^\mu = -1, \quad (2.7)$$

requires that the gauge field be non-Abelian. The proof that there is a class of gauge transformations which preserves (2.7) can most easily be carried out for infinitesimal gauge transformations. With the help of(2.5), we may write the transform of (2.7) as

$$\begin{aligned} B_\mu B^\mu &\rightarrow (\omega B_\mu \omega^* + \omega \partial_\mu \omega^*)(\omega B^\mu \omega^* - \partial^\mu \omega \omega^*) \\ &= \omega B_\mu B^\mu \omega^* - \partial_\mu \omega B^\mu \omega^* + \omega B_\mu \partial^\mu \omega^* - \partial_\mu \omega \partial^\mu \omega^* \end{aligned} \quad (2.8)$$

The first term on the right provides the necessary -1, so we must show that

$$-\partial_\mu \omega B^\mu \omega^* + \omega B_\mu \partial^\mu \omega^* - \partial_\mu \omega \partial^\mu \omega^* = 0. \quad (2.9)$$

Moreover, since

$$(\partial_\mu \omega B^\mu \omega^*)^* = -\omega B_\mu \partial^\mu \omega^* \quad (2.10)$$

we have, from (2.9), the requirement

$$2 Re \partial_\mu \omega B^\mu \omega^* - \partial_\mu \omega \partial^\mu \omega^* = 0. \quad (2.11)$$

We now show that there exist solutions for this nonlinear relation by studying infinitesimal local gauge transformations of the form (ϵ real and small), for a neighborhood of some x^μ,

$$\omega = 1 + \epsilon v, \quad (2.12)$$

with v pure quaternion imaginary, so that

$$\omega \omega^* = (1 + \epsilon v)(1 - \epsilon v) = 1 + O(\epsilon^2).$$

Now, substituting (2.12) into (2.11), one finds, to $O(\epsilon^2)$, that we must have

$$Re B_\mu \partial^\mu v = 0 \quad (2.13).$$

Since B_μ is timelike, there is a (local) Lorentz frame for which only its time component is non-zero; in this frame,

$$Re B_0 \partial^0 v = 0. \quad (2.14)$$

For

$$\begin{aligned} B_0 &= e_1 b_1 + e_2 b_2 + e_3 b_3 \\ \partial^0 v &= e_1 \partial^0 v_1 + e_2 \partial^0 v_2 + e_3 \partial^0 v_3 \end{aligned} \quad (2.15)$$

from which it follows that

$$ReB_0\partial^0 v = -\Sigma_{i=1}^3 b_i\partial^0 v_i. \quad (2.16))$$

It is therefore necessary and sufficient (by successive infinitesimal transformations), that in this local frame, the quaternionic parts of the time deivative of the infinitesimal gauge transformation be orthogonal to the quaternionic vector part of B_0. Since (2.12) is invariant under local Lorentz transformations, this result implies that(2.13) must be valid as well, at any point in the manifold, implying that there is a class of gauges that leaves $B_\mu B^\mu = -1$.[2]

Although the quaternionic wave function has the property that it can carry left or right gauge trnsformations, it will be convenient (and sufficient for our present purposes) to use the left gauge[3].

III. Quaternionic Kaluza-Klein Theory.

Consider a local single particle gauged Hamiltonian of the form

$$K = \frac{1}{2m} g^{\mu\nu}(x)(p_\mu - \epsilon B_\mu(x))(p_\nu - \epsilon B_\nu(x)) + \Phi(x), \quad (3.1)$$

where Φ is a (real-valued) world scalar field, K is quaternion real, $g^{\mu\nu}$ is the (real-valued) Einstein metric, p_μ is quaternion imaginary (discussed in [6]), and B_μ is the quaternionic Bekenstein-Sanders field. We define, as in[3], a conformally modified metric

$$\hat{g}^{\mu\nu} = g^{\mu\nu}\frac{K}{K - \Phi} \quad (3.2)$$

Since $g^{\mu\nu}$ is real-valued, we may cancel K from both sides, and multiply by $(K - \Phi)$ to show the equivalence between (3.2) and (3.1).

Defining, as in [3], [1],

$$\frac{K}{K - \Phi} \equiv e^{-2\phi}, \quad (3.3)$$

the Hamiltonian

$$K_K = \frac{1}{2m}\tilde{g}^{\mu\nu} p_\mu p_\nu, \quad (3.4)$$

for [1]

$$\tilde{g}^{\mu\nu} = e^{-2\phi}(g^{\mu\nu} + B^\mu B^\nu) - e^{2\phi}B^\mu B^\nu, \quad (3.5)$$

$$K_K == e^{-2\phi}g^{\mu\nu}p_\mu p_\nu - 2sinh2\phi B^\mu B^\nu \quad (3.6)$$

is equivalent to (3.1), generating the same equations of motion[8][12].

We argue here that for $p_\mu = q\partial_\mu$, with q imaginary quaternionic [6],

$$\begin{aligned} [q, \omega] &= 0 \\ [B_\mu, q]. &= 0 \end{aligned} \quad (3.7)$$

The first of (3.7) is implied by the requirement

$$\begin{aligned} (p_\mu - \bar{B}_\mu)\bar{\psi} &= (q\partial_\mu - \bar{B}_\mu)\omega\psi, \\ &= \omega(q\partial_\mu - B_\mu)\psi \end{aligned} \quad (3.8)$$

or

$$q(\partial_\mu\omega)\psi + q\omega\partial_\mu\psi - \bar{B}_\mu\omega\psi = \omega q\partial_\mu\psi - B_\mu\psi. \quad (3.9)$$

The gauge condition

$$\bar{B}_\mu = (q\partial_\mu\omega)\omega^{-1} + \omega B_\mu\omega^{-1} \quad (3.10)$$

follows if $[q, \omega] = 0$, so that the $\partial_\mu\psi$ term cancels on both sides.

Furthermore, since q is constant, and we take it to commute with ω,

$$[\bar{B}_\mu, q] = \omega[B_\mu, q]\omega^{-1}; \quad (3.11)$$

if we start with $(B_\mu)_{initial} = 0$, it follows from (3.10) that a first gauge step to $(\bar{B}_\mu)_{next}$ will still commute with q. This condition is maintained for any sequence of ω's (commuting with q), and, therefore, for any B_μ constructed in this way.

We may now define a Kaluza-Klein metric [12]

$$g^{AB} = \begin{pmatrix} g^{\mu\nu} & B^\mu \\ B^\mu & g^{55} \end{pmatrix}. \quad (3.12)$$

If we take [3]

$$p_5 = -\frac{p_\mu B^\mu}{g^{55}}(1 \pm \sqrt{1 - 2g^{55}sinh2\phi}, \quad (3.13)$$

then

$$K_K = \frac{1}{2m}g^{AB}p_Ap_B. \quad (3.14)$$

Wesson [13] and Kaluza [12] chose $g^{55} = const$, but in our context, it may be taken to be zero.

Conclusions

We have discussed a quaternionic formulation of the Bekenstein-Sanders[1] TeVeS gravitational theory. It was shown in [3] that this theory can be derived by a conformal transformation from a Hamiltonian form on a curved space[10], for which the Bekenstein-Sanders vector field B_μ is a non-Abelian gauge field. We give here a quaternionic formulation suggested by this structure. We proved for this quaternionic formulation (as well as provided a missing proof for the Yang Mills form [3]) that there is a set of gauge transformations that preseves the Bekenstein-Sanders condition$B_\mu B^\mu = -1$. It has been shown [4] that one can construct a theory for $N \geq 2$ particles in such a TeVeS theory, suggesting that a rigorous statistical mechanics could be developed (see also Giordino *et al.* cited in [2]).

Since the wave functions in the Hilbert space, carrying the non-Abelian quaternionic guage, are quaternionic, as dynamical variables they may satisfy the *race dynamics* developed by Adler [11], opening a subject for future research, relating quaternionic quantum mechanics to general relativity.

Appendix I

Proof for existence of gauges preserving $B_\mu B^\mu = -1$ for standard Yang-Mills theory.

For standard Yang-Mills theory [3] (result stated but not proved there), under gauge transformation,

$$\begin{aligned} \underline{B}_\mu \underline{B}^\mu &= \left(\omega B_\mu \omega^* - \frac{i}{\epsilon}\frac{\partial \omega}{\partial x^\mu}\omega^*\right)\left(\omega B^\mu \omega^* - \frac{i}{\epsilon}\frac{\partial \omega}{\partial x_\mu}\omega^*\right) \\ &= \omega B_\mu B^\mu \omega^* - \frac{i}{\epsilon}\left[\frac{\partial \omega}{\partial x^\mu} B^\mu \omega^* + \omega B_\mu \omega^* \frac{\partial \omega}{x^\mu}\omega^*\right] \\ &\quad - \frac{1}{\epsilon^2}\frac{\partial \omega}{\partial x^\mu}\omega^* \frac{\partial \omega}{\partial x_\mu}\omega^*. \end{aligned} \tag{I.1}$$

Now, differentiating $\omega^* \omega = 1$ (as above),

$$\omega^* \frac{\partial \omega}{\partial x^\mu}\omega^* = -\frac{\partial \omega^*}{\partial x^\mu}, \tag{I.2}$$

we find from $(I.1)$ that

$$\bar{B}_\mu \bar{B}^\mu = -1 - \frac{i}{\epsilon}\left[\frac{\partial \omega}{\partial x_\mu} B_\mu \omega^* - \omega B_\mu \frac{\partial \omega^*}{\partial x_\mu}\right] + \frac{1}{\epsilon^2}\frac{\partial \omega}{\partial x^\mu}\frac{\partial \omega^*}{\partial x_\mu} \quad (I.3)$$

Now,

$$\left(-\omega B_\mu \frac{\partial \omega^*}{\partial x_\mu}\right)^* = +\frac{\partial \omega}{\partial x_\mu} B_\mu \omega^* \quad (I.4)$$

so that, to maintain the relation $\bar{B}_\mu \bar{B}^\mu = -1$, we must have

$$2Re\left[\frac{\partial \omega}{\partial x_\mu} B_\mu \omega^*\right] = \frac{1}{\epsilon}\frac{\partial \omega}{\partial x^\mu}\frac{\partial \omega^*}{\partial x_\mu} \quad (I.5)$$

In order to analyze this relation, we first study the infinitesimal gauge, for $v^* = -v$,

$$\omega = 1 + \eta v. \quad (I.6)$$

Substutiting into $(I.5)$, one finds the condition, to first order,

$$\left\{\frac{\partial \omega}{\partial x_\mu}, B_\mu\right\} = 0. \quad (I.7)$$

Now, choose a local Lorentz frame for which $B_\mu \rightarrow B_0$, so that our condition becomes

$$\left\{\frac{\partial \omega}{\partial x_0}, B_0\right\} = 0. \quad (I.8)$$

For the Yang-Millls fields, we may represent

$$\frac{\partial \omega}{\partial x_0} = ia_0 + i\Sigma_{i=1}^3 a_i \tau_i, \quad (I.9)$$
$$B_0 = ib_0 + i\Sigma_{i=1}^3 b_i \tau_i$$

where a_0, a_i, b_0, b_i are real numbers , τ_i Pauli mmatrices. To satisfy $(I.8)$, we must have $a_0 = b_0 = 0$. What remains is the condition

$$\Sigma_i^3 a_i b_i = 0, \quad (I.10)$$

closely analogous to what was obtained in (2.15) for the quaternionic theory.

Appendix II

Quaternionic Hilbert Space Scalar Product and Left Gauge

We take the quaternionic Hilbert space scalar product to satisfy

$$\begin{aligned}(af, g) &= a(f, g) \\ (f, ag) &= (f, g)a^*\end{aligned} \quad \text{(II.1)}$$

Then, to pass to Dirac wave function representation, we use the spectral representation of the x^μ operator

$$\int dE(x) = \int |x><x| d^4x = I, \quad \text{(II.2)}$$

where the integration is in the same sense as in[14]. Then, for (conjugate of the usual form)

$$f(x) = <f|x>, \quad \text{(II.3)}$$

we have,

$$(f, g) = \int <f|x><x|g> d^4x = \int f(x)g(x)^* d^4x, \quad \text{(II.4)}$$

With our convention $(II.1)$, we have $<af|x> = a<f|x>$ so that

$$\int <af|x><x|g>^* d^4x = \int a<f|x><x|g>^* d^4x = a(f, g). \quad \text{(II.5)}$$

For the left gauge $\omega(x)$, it then follows that

$$<\omega f|x> = \omega(x)(f|x> = \omega(x)f(x), \quad \text{(II.6)}$$

and

$$(\omega f, g) = \int \omega(x)f(x)g(x)^* d^4x, \quad \text{(II.7)}$$

the left gauge, as we have used in the text.

Using the choice of linearity $(fa, g) = a^*(f, g)$, with $<x|f> = f(x)$, we would have $<x|f\omega> = <x|f>\omega(x) = f(x)\omega(x)$, the alternative right gauge.

Footnotes

1 Similar results have been obtained by Yahalom[15] using retarded forces carried by gravitational waves[16]. Although gravitational waves emerge from Einstein's equations with a special choice of gauge (to harmonic coordinates) for spacetime, the prediction of physically observable phenomena is independent of the choice of gauge, as for the choice of Lorentz gauge in electromagnetism. The retardation theory of Yahalom is therefore completely covariant. Our study here is motivated by the interesting connection between the TeVeS theory and non-Abelian gauge fields.

2 We show in Appendix I that an analogous proof can be given for the standard Yang-Mills[5] formulation followed in [3].

3 The representation of wave functions in configuration space and linearity of scalar products are discussed in Appendix

II [6], [7]

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