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Horizon and curvature

Alain Haraux

Sorbonne Université, Université Paris-Diderot SPC, CNRS, INRIA,

Laboratoire Jacques-Louis Lions, LJLL, F-75005, Paris, France.

e-mail: `alain.haraux@sorbonne-universite.fr`

Abstract

This paper is devoted to a strange looking question: is it possible to deduce the shape of a smooth convex set by measuring at each point the distance of the horizon standing at a fixed height h? The question is surprisingly difficult and we only have partial results.

Key words: convex sets, curvature, distance

1 Introduction

It is well known that a smooth planar curve with constant non-zero curvature is a circle. An analogous property can be stated for a surface in $\mathbb{R}^3$. The property of constant curvature can be used to verify the spherical shape of the Earth by measurements made on Earth itself. Such measurements are in a sense more convincing than arguments relying on exterior objects such as the direction of sunlight, used by Erathostenes to prove at least locally the curved character of the planet. They can at least theoretically be made global.

A simple calculation, detailed in section 2 below, shows a direct relationship between the radius of the Earth and the distance of the horizon. But what if we do not know in advance that the planet is approximately spherical? Can we really estimate (locally or globally) the curvature by measuring the distance of the horizon? The present paper is devoted to some partial results and one basic question in this direction.

Recovering the shape from horizon measurements enters the category of so called "inverse problems" and usually this kind of problems is not easy. Ruling out absolute flatness of the Earth or more generally a planet by local measurements is the easiest part: on a plane, the distance of horizon is infinite from any point, and more generally for a subdomain of a plane, the horizon in any direction coincides with a boundary point of the domain. So, in theory, if the domain is perfectly flat, walking towards the horizon should not allow new objects to appear. But the adepts of flatness theory (which can also be considered as a challenge to well recognized certitudes), do not usually claim absolute flatness. And to contradict that theory, we need much stronger arguments.

The plan of this paper is as follows: section 2 is devoted to the case of a sphere, in section 3 we state the main definitions useful for stating the results, in section 4 we indicate how a bound on the curvature radius of the boundary allows to localize a convex set. Section 5 is devoted to the construction of examples showing that a global upper bound of the horizon distance does not imply anything on the curvature radius. Sections 6 and 7 contain the main positive results of this short note. Finally, Section 8 contains some remarks and observations on related questions of interest.

2 Radius of the Earth and distance of the horizon

In this section, we consider a perfectly spherical planet (which may be the Earth) with radius R and we want to compute the distance of horizon seen from any point of the Earth located at a distance h from the ground. First it is immediate, due to rotation invariance, that the distance of the horizon in any direction is the same, let us denote it by $H(h) := H$. To compute H, let us consider a point O on the ground and the point A on the vertical half-line starting from O such that $OA = h$. We consider any plane Π containing the vertical segment OA and the disk D, intersection of the planet with the plane Π. Let T be the contact point of one of the two tangents at $\Gamma = \partial D$ passing through point A. Then $AT = H$. Denoting by C the center of the planet, We have $TC = R = OC$, then $AC = R + h$ and by the Pythagorean theorem applied in the triangle ATC, we infer

$$(R + h)^2 = H^2 + R^2.$$

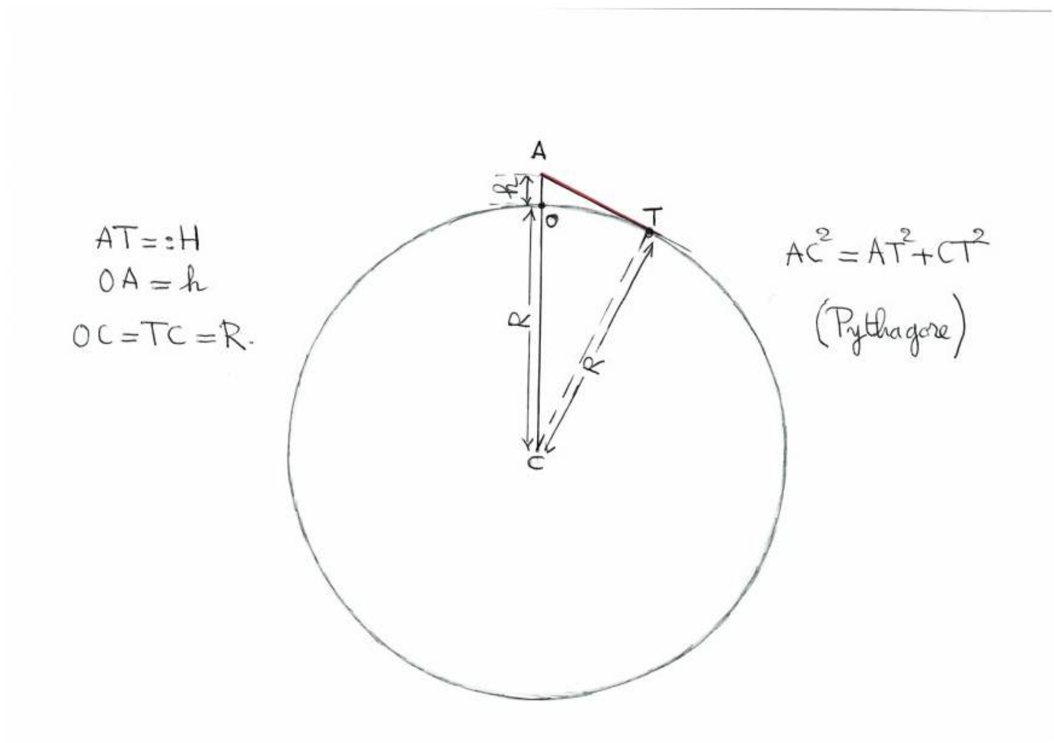

Fig. 1: Spherical planet

By expanding the square on the left, we get

$$R^2 + 2Rh + h^2 = H^2 + R^2,$$

yielding

$$H^2 = 2Rh + h^2.$$

Therefore

$$H = \sqrt{2Rh + h^2} \tag{2.1}$$

and

$$R = \frac{H^2 - h^2}{2h}. \tag{2.2}$$

For instance on Earth where $R \sim 6400\,km$, for a man of height $1.75m$ with eyes at approximately $1.65\,m$ from the ground, the calculation gives $H \sim 4.6\,km$. This is quite consistent with what we observe when there is no obstacle to limit our vision. And from an airplane, when If $h = 10\,km$, we end up with

$$H \sim \sqrt{128000} \sim 358\,km$$

which is also consistent with the size of the landscape that we can see from an airplane when the wheather allows that.

3 Some definitions

Before attacking the inverse problem mentioned in the introduction, we need to define precisely what we mean by the horizon on a given hypersurface of $\mathbb{R}^N$ $(N \geq 2)$ at an exterior point. It is

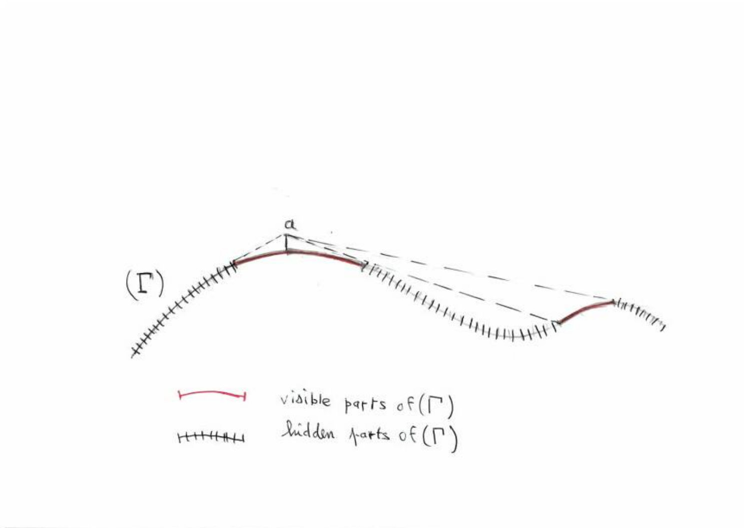

Fig. 2: Disconnected visible set.

clear that if the hypersurface is bounded, the distance to the horizon will be finite everywhere, and that the converse is false. Also, the notion is only of interest if the hypersurface is in fact the boundary of a solid domain, so that we cannot "see" the interior points. And finally, even for a curve in $\mathbb{R}^2$, the definition of the horizon is problematic if the curvature changes sign, since in a given direction the set of points which can be seen from the exterior point may be disconnected. (see Figure 2)

For this reason, we shall restrict ourselves to the case of the boundary of a convex set. Let us consider a closed convex set $K \subset \mathbb{R}^N$ and its boundary $\Sigma = \partial K$. Given a point $a \notin K$ we consider the smallest closed convex cone C with vertex a containing K. It is given by the formula

$$C = a + \overline{\bigcup_{\lambda > 0} \lambda(K - a)}.$$

Then we can define the *horizon set* $\mathcal{H}(a, K)$ at a relative to K by

$$\mathcal{H}(a, K) = \partial C \cap \Sigma.$$

If $N = 2$, the set ∂C is made of two straight lines and if K is strictly convex, $\mathcal{H}(a, K)$ is a pair of points. On the other hand, if Σ contains some flat parts, $\mathcal{H}(a, K)$ is more generally the union of two separate line segments, some of which possibly reduced to one point, depending on the position of a, see Figure 3.

If $N = 3$, the set ∂C is a conic surface with vertex a and if K is strictly convex, $\mathcal{H}(a, K)$ is a curve. On the other hand, if Σ contains some flat parts, $\mathcal{H}(a, K)$ can become very different from a curve. We shall not try here to describe the most general situation in $\mathbb{R}^3$, since convex sets in 3 dimensions can already be rather complicated, see Figure 4.

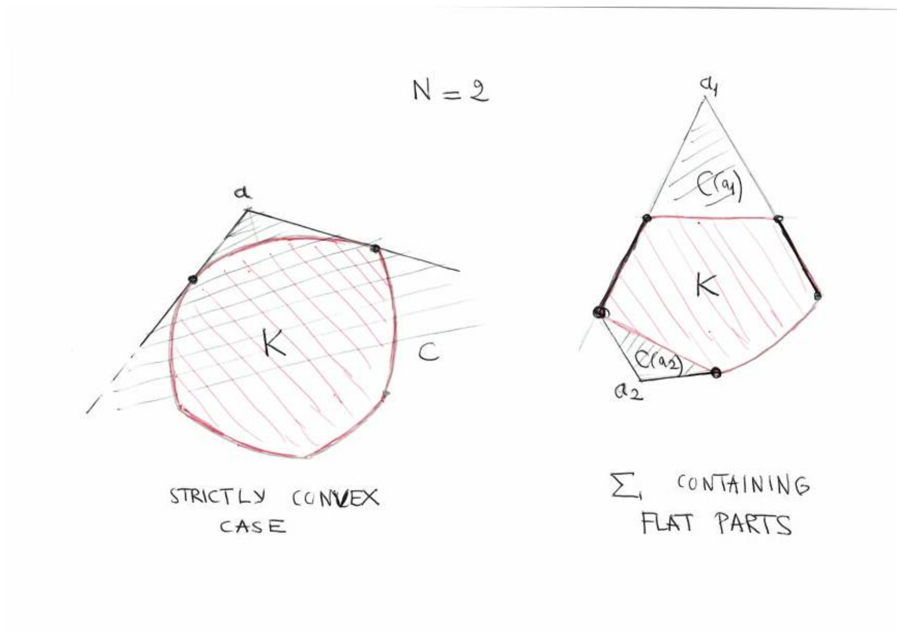

Fig. 3: N=2

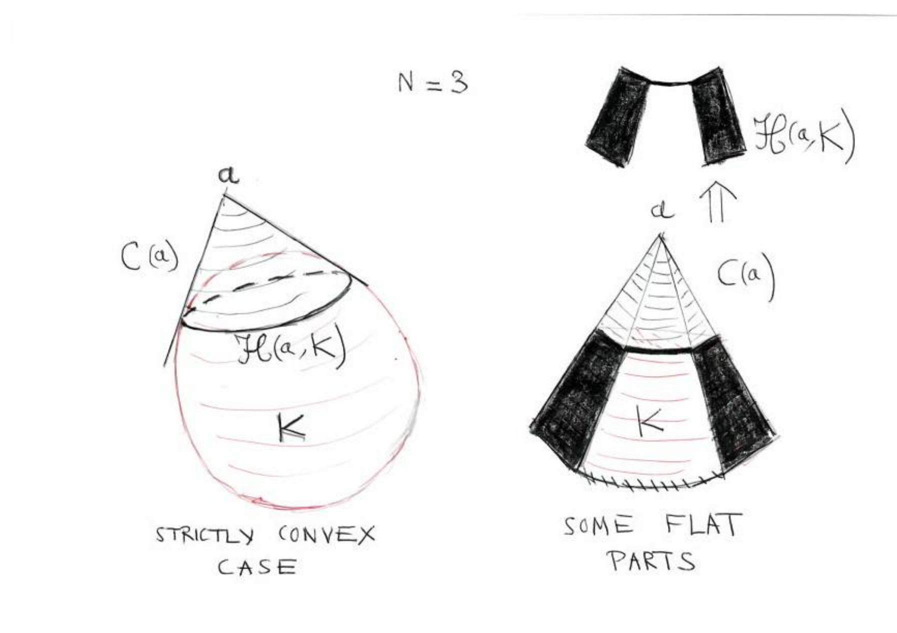

Fig. 4: N=3

In the sequel, the most useful notion will be the horizon distance from a point $a \notin K$ in a given direction. When K is a strictly convex C^1 domain of $\mathbb{R}^N$ which contains interior points, so that it does not reduce to dimension $N-1$, let us consider any half-plane P whose edge coincides with the normal at K emanating from a. Then $\mathcal{H}(a, K) \cap P$ reduces to a point $T = T(a, K, P)$, which is the contact point of K with the unique tangent at Σ contained in P and passing through a. We set

$$H(a, K, P) := dist(a, T(a, K, P)) = aT.$$

In 2 dimensions, there are only two half-planes P with edge Oa that can be written P^+ and P^-, with P^+ corresponding to the right if we take the following system of axes : the origin O is the projection of a on K, the vertical axis $\overrightarrow{Oy}$ is the half-line in the direction $\overrightarrow{Oa}$ given by outgoing normal through a and we chose an horizontal orientation to fix $\overrightarrow{Ox}$. In N dimensions, each half-plane P is characterized by its unit vector of origin O orthogonal to $\overrightarrow{Oa}$.

The main objective of this note is to establish a relationship between the properties of the horizon distance from the points above a given point on Σ and the curvature radius at that point. But the first thing to investigate is which kind of information can be deduced from the properties of the curvature on Σ.

4 Curvature radius and confinement

A first question, natural for both planar curves and surfaces in $\mathbb{R}^3$, is the following: If the curvature radius at all points is bounded by a finite number R, can we say that the curve (resp. surface) is bounded? For the problem which we address here, we always work in a plane, so we just provide the simplest property in the direction.

Proposition 4.1. *Let K be a closed convex domain of $\mathbb{R}^3$, with C^2 boundary Γ. If the curvature radius of Γ is everywhere less than R, the curve is entirely contained in a disk of radius R.*

Proof. First, the curve has no inflexion points and as a consequence the curvature has a constant sign. At any given point A , the curve is entirely located in one of the two half-planes delimited by the tangent vector. Let us study the local situation by choosing the coordinate axes in such a way that the point A has coordinates $(0, R)$ and the equation of Γ becomes

$$y = f(x)$$

for $|x|$ small, with $f'(0) = 0$ and $f''(0) < 0$. For $x > 0$ small enough, the curvature radius is

$$\rho(x) = -\frac{(1 + y'^2)^{3/2}}{y''}.$$

Therefore

$$y'' = -\frac{1}{\rho(x)}(1 + y'^2)^{3/2} \leq -\frac{1}{R}(1 + y')^{3/2}.$$

Since the circle of center (0, 0) and radius R satisfies $y = z(x)$ with $z'(0) = 0$ and

$$z'' = -\frac{1}{R}(1 + z'^2)^{3/2},$$

the comparison principle implies that

$$\forall x \in (0, \tau(x)), \quad y'(x) \leq z'(x). \tag{4.1}$$

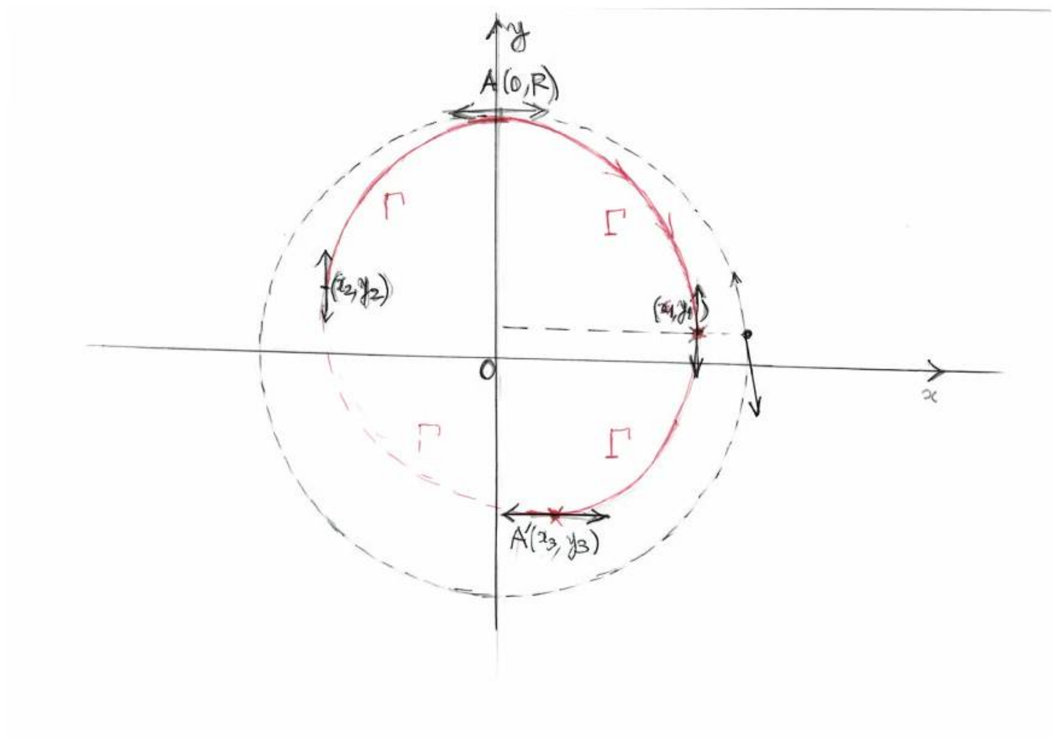

Fig. 5: Confinement.

Because $y(0) = z(0) = R$, we also infer $y(x) \leq z(x)$. From this we deduce that the arc of Γ on the right is interior to the quarter of disk with center (0, 0) and radius R, until the curve Γ crosses the x-axis for the first time. From (4.1) it follows that $|y'(x)|$ blows up for some value $x_1 \in (0, R]$. By rotating the axes by $+\frac{\pi}{2}$ so that $(y(x_1), x_1)$ plays the role of the initial point and proceeding backwards instead of forward, we can see that actually $y(x_1) \geq 0$. Now two remarks are in order: first the curve Γ is entirely contained in the half-plane $x \leq x_1$. Secondly, by doing the same argument on the left, we find a point $x_2 \in [-R, 0)$ such that $y(x_2) \geq 0$ and $|y'(x)|$ blows up at x_2. In particular the curve Γ is entirely contained in the vertical strip $x_2 \leq x \leq x_1$. Now we can reproduce the previous argument starting from the point $(y(x_1), x_1)$ and rotating the axes by $+\pi/2$, because $x_1 \leq R$ and the slope of the disk with center (0, 0) and radius R at the point of abscissa y_1 is nonnegative in the new coordinate system. Then the arc of Γ on the right starting from $(y(x_1), x_1)$ will remain in the second quarter of the disc until the curve Γ becomes horizontal, which provides a point $A' = (x_3, y_3)) \in \Gamma$ with $x_3 < x_1 < R$ such that the tangent at Γ at point A' is horizontal. And then the curve Γ is contained in the strip $y_3 \leq y \leq R$. In particular, Γ is bounded as a subset of the rectangle $R = [x_2, x_1] \times [y_3, R]$, see Figure 5.

To complete the proof we need to be a little bit more precise. Since we now know that Γ is compact, we can find two points (A, B) in Γ such that

$$AB = d(A, B) = \max_{(X,Y) \in \Gamma \times \Gamma} d(X, Y).$$

An immediate geometrical argument shows that the tangents to Γ at both points A and B are orthogonal to the segment AB. For instance if the tangent at B is not orthogonal to AB, taking a point C very close to B on the side of Γ where the tangent makes an angle $> \pi/2$, we see that

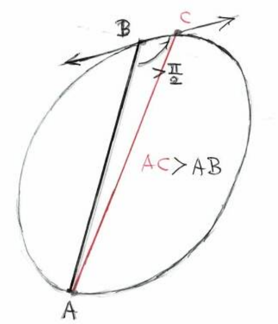

Fig. 6: Diameter.

$AC > AB$, contradicting the maximality, see Figure 6. The same argument applies to A. By choosing A as the starting point in the above construction, we can see easily that $B = A'$ and so A' belongs to the vertical axis. We observe that A' is the "bottom" of Γ while A is the top. Note that there is only one top and one bottom since the curvature never vanishes. Now, we can see that the "right" part of Γ, meaning by that its intersection with the half-plane $x \geq 0$ is contained in the right half-disk of radius R with center (0,0). The same argument applies, mutatis mutandis, to the left part of Γ, so that Γ is entirely contained in the disk of radius R with center (0,0).

□

Remark 4.2. *The result is probably still valid if Γ is any C^2 curve, not necessarily the boundary of a convex set. But the proof would presumably be more involved.*

Remark 4.3. *An analogous result is probably still valid in $\mathbb{R}^N$ for the boundary Σ of a C^2 convex set assuming all directional curvatures to be larger than $\frac{1}{R}$, since in that case each section by a plane would be contained in a disc of radius R. Taking A, B in Σ such that AB realizes the diameter of Σ, it is likely that some sphere of radius R tangent to Σ at A will enclose Σ.*

5 Horizon and curvature radius

Encouraged by the previous result, we might try to deduce a bound on the curvature radius from an upper bound of the horizon distance. Unfortunately, we have the following very strong negative results.

Proposition 5.1. *For any $H_0 > 0$, we can find a C^∞ convex domain D with completely flat parts (hence an infinite curvature radius at some points) and a positive h such that*

$$\forall M \in \partial D, \quad \max\{H_+(h, M), H_-(h, M)\} \leq 2H_0.$$

Proof. Let us consider an even piecewise affine concave function F defined on $[-NH_0, +NH_0]$ ($N \in \mathbb{N}, N > 0$) with $F(-NH_0) = F(NH_0) = 0$, $F(0) = \varepsilon > 0$ such that the slope jumps at all

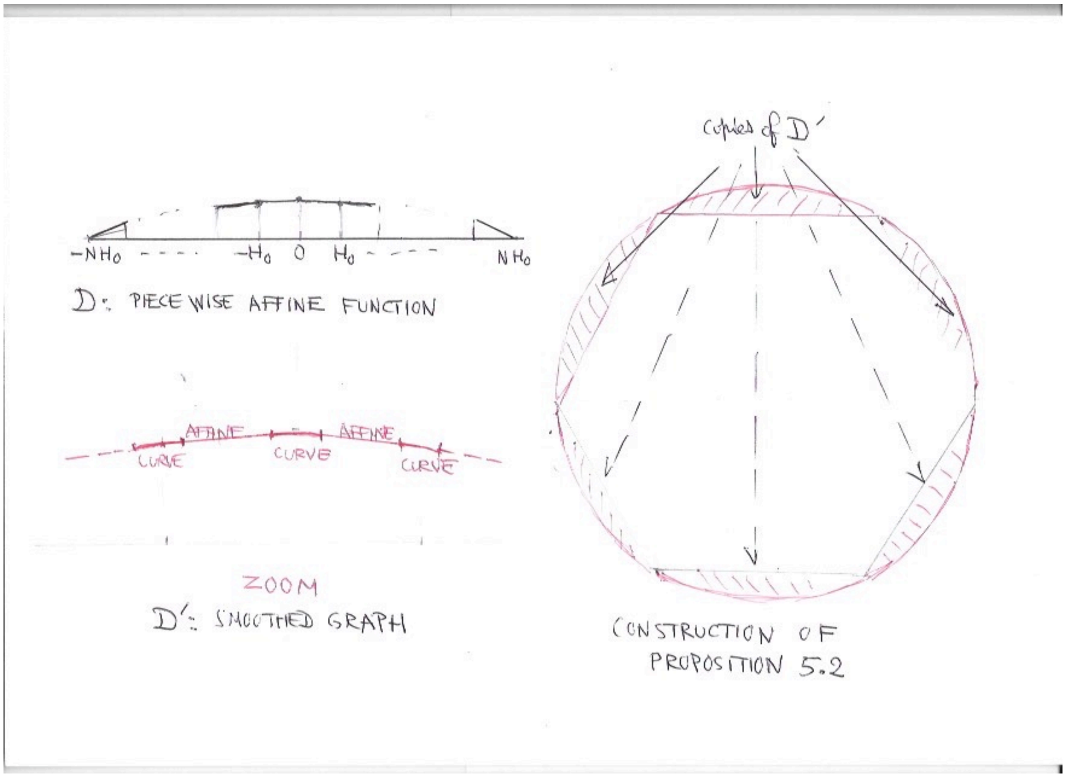

Fig. 7: Construction of the domains D, D' and D".

points kH_0, where k is an integer between $-(N-1)$ and $(N-1)$. For a point on the graph of F other than the corners, both left and right horizon distances relative to the convex set

$$S := \{(x, y)|x \in [-NH_0, +NH_0], 0 \le y \le F(x)\}$$

are less than the length of the corresponding face. By smoothing the corners, the distance of horizon for any point lying on the smoothed curve remains less than the maximum of lengths of the faces. But since the slope of the faces is less than $\frac{\varepsilon}{NH_0} > 0$, for ε small enough, that maximum is less than $2H_0$. And then for sufficiently small h we obtain the result since the horizon distance is continuous with respect to h. See Figure 7. $\square$

Proposition 5.2. *For any $H_0 > 0$, we can find a C^∞ strictly convex domain D" with arbitrarily large curvature radius at some boundary points and a positive h such that*

$$\forall M \in \partial D", \quad \max\{H_+(h, M), H_-(h, M)\} \le 2H_0.$$

Proof. Starting from the previous domain D, we can replace the flat parts by curved ones with an arbitrarily small curvature, getting a domain D'. Then we consider a large number of copies of D' in and dispose them around a circle. More precisely, we start with a regular P-gone with P very large, the size of the sides being equal to the length of D and place a copy of D' above each side of the P-gone. When D' is sufficiently flat, the resulting domain will still be convex. Finally we smooth the remaining corners without destroying convexity. We shall obtain this way a C^∞ strictly convex curve arbitrarily close to a circle for which the distance of the horizon for points very close to the curve will be as small as we wish, while the curvature radius at some points is arbitrarily large. We skip the details. See Figure 7. □

Remark 5.3. *The main point here is that the maximum of the curvature radius can be made as large as we wish for a small fixed height h, so that there a bound on H(h) does not imply anything on the curvature radius. In addition, the domain that we construct can be almost indistinguishable from a disk. A similar construction for a ball of* $\mathbb{R}^3$ *should be possible, but it may be more delicate since regular polyhedrons with a large number of faces do not exist! We should sacrifice at least partially the symmetry.*

6 A very partial result

Theorem 6.1. *Let* Γ *be* C^2 *curve delimiting a strictly convex compact domain with positive curvature everywhere. Then it happens that*

$$\lim_{h\to 0} H_{\pm}(h, M) = 0 \tag{6.1}$$

and we have more precisely

$$\lim_{h\to 0} \frac{H_{\pm}^2(h, M)}{h} = 2\rho(M) \tag{6.2}$$

where $\rho(M)$ *is the curvature radius at point* M*.*

Proof. In the sequel we shall write for simplicity $H_{\pm}(h, M) = H(h, M)$ since the proofs are identical in both positive and negative directions. The first result is immediate since assuming the contrary, by compactness and continuity we find a point $P \in \Gamma$ such that $H(0, P) > 0$, a contradiction with strict convexity. By a suitable choice of coordinates, we set $M = (0, 0)$ and we represent locally the curve Γ in an orthonormal frame by the equation $y = f(x)$ where $f(0) = f'(0) = 0$ and $f''(0) < 0$. With this convention we have

$$f''(0) = -\frac{1}{\rho(M)}.$$

We consider the right horizon point $T(x, f(x))$ (the calculation on the left will give the same result) and we set $H^2(h, M) = H$. We note first that

$$H^2 = x^2 + (h - f(x))^2 \tag{6.3}$$

and since the tangent at point (x, f(x)) contains the point (0, h) we have the basic formula

$$h = f(x) - xf'(x), \tag{6.4}$$

see Figure 8. In addition, by definition of $f''(0)$, we have since $f'(0) = 0$,

$$f'(x) = xf''(0) + o(x).$$

In particular,

$$h - f(x) = -x^2 f''(0) - xo(x) \tag{6.5}$$

yielding

$$(h - f(x))^2 = o(x^2). \tag{6.6}$$

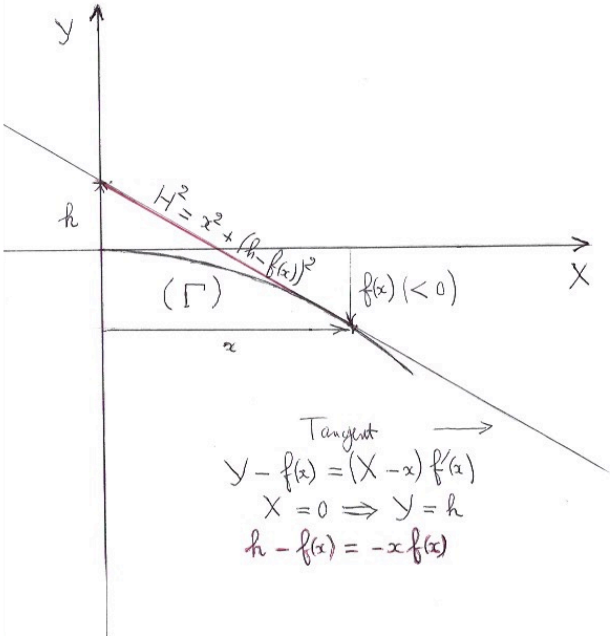

Fig. 8: Graph.

Therefore since $x^2 = x^2(h) \leq H^2(h)$ tends to 0 with h, we have as a consequence of (6.3) and (6.6)

$$\lim_{h \to 0} \frac{H^2(h)}{x^2(h)} = 1.$$

Now we have by Taylor's formula

$$f(x) = f(0) + xf'(0) + \frac{1}{2} f''(0)x^2 + o(x^2) = \frac{1}{2} f''(0)x^2 + o(x^2). \tag{6.7}$$

Adding (6.7) and (6.5) yields

$$h = -\frac{1}{2} f''(0)x^2 + o(x^2) \tag{6.8}$$

giving

$$H^2(h) \sim x^2(h) \sim 2\frac{h}{|f''(0)|} = 2\rho(M)h. \tag{6.9}$$

□

For the next result, we consider a C^2 surface S delimiting a strictly convex domain with positive curvature everywhere. For any $M \in S$ and P any half-plane containing the normal to S at M with edge consisting of that normal, we define $H(h, M, P)$ as the horizon in the direction P from the point $M + h\overrightarrow{n}$, where $\overrightarrow{n}$ is the outgoing normal unitary vector to S at M.

Corollary 6.2. *If for a C^2 surface S delimiting a strictly convex domain of $\mathbb{R}^3$ with positive curvature everywhere, it happens that $H(h, M, P)$ is independent of the point and the direction P for all h small enough, the surface is a sphere.*

Proof. Let M be any point of the surface S and Π any plane containing the normal to S at M. By applying Theorem 6.1 to $\Gamma(\Pi) := S \cap \Pi$, we obtain that $\Gamma(\Pi)$ has constant curvature.

Therefore by a well known result, $\Gamma(\Pi)$ is a circle (an entire circle and not just an arc since the horizon distance is positive in both directions above each point of the curve), and its radius does not depend on Π. It is easy to conclude by rotation with respect to the axis containing the normal to S at M. $\square$

Remark 6.3. *Each half-plane P with edge consisting of the normal to S at M can be characterized by the unit vector u of its intersection with the tangent plane $TM(S)$.The results can also be written in terms of u which represents roughly the aiming direction.*

Remark 6.4. *In the statement of Corollary 6.2, the hypothesis "for all h small enough" can be replaced by for a sequence h_n of height tending to 0.*

7 A reinforced condition involving the angle

If the convex set K is a ball with radius R, not only the distance H(h) of the horizon at height h is given by

$$H(h) = \sqrt{2Rh + h^2}$$

as shown in Section 2, but the angle θ of the tangent AT with the vertical is constant. On figure 1 it appears that

$$\cos\theta = \frac{H}{R+h} = \frac{2Hh}{H^2 + h^2} \tag{7.1}$$

and this defines perfectly the angle

$$\theta = (\overrightarrow{AO}, \overrightarrow{AT}) \in (0, \frac{\pi}{2}).$$

Conversely, we have the following proposition

Proposition 7.1. *If for a C^1 surface S delimiting a strictly convex domain of $\mathbb{R}^3$ with positive curvature everywhere, it happens that for some $h > 0$ we have*
1) $H(h, M, P) := H$ is independent of the point $M \in S$ and the direction P.
2) At all points M, the angle θ of the incoming normal unit vector $\overrightarrow{\nu(M)}$ at M with the vector $\overrightarrow{v(M,P)}$ joining $A(M) = M - h\overrightarrow{\nu(M)}$ with the horizon point in direction P satisfies

$$\cos\theta = \frac{2Hh}{H^2 + h^2}.$$

3) $\dfrac{\theta}{\pi} \notin \mathbb{Q}$.
Then S is a sphere with radius $R = \dfrac{H^2 - h^2}{2h}$.

Proof. Let M be any point of the surface S and Π any plane containing the normal to S at M. We consider the point $\omega := M + R\overrightarrow{\nu(M)}$ which does not depend on Π. We claim that in the plane Π, the circle Γ with center ω and radius R contains the right horizon point $T_+(h, M, P) := T$. Indeed if we consider the intersection ω' of the normal to AT at A with the vertical, then in the rectangular triangle $AT\omega'$ we have

$$A\omega' = \frac{H}{\cos\theta}. \tag{7.2}$$

Since the value chosen for R implies $H^2 = 2Rh + h^2$ and then

$$\cos\theta = \frac{2Hh}{H^2+h^2} = \frac{2Hh}{2Rh+2h^2} = \frac{H}{R+h},$$

now (7.2) gives $A\omega' = R + h$. In particular $\omega' = \omega$ and

$$A\omega = R + h.$$

Then we observe that

$$\sin^2\theta = \frac{(R+h)^2 - H^2}{(R+h)^2} = \left[\frac{R}{R+h}\right]^2$$

and $T\omega = A\omega \sin\theta = R$. Now if we consider the successive right horizon points at height h corresponding to $A_0 = M, T = A_1$, etc...we obtain an infinite sequence of points A_n belonging to $S \cap \Pi$ and to Γ, with

$$\theta_n = (\overrightarrow{\omega A}, \overrightarrow{\omega A_n}) = n\theta.$$

By using condition 3), we infer that the sequence A_n is dense in Γ. Because $S \cap \Pi$ is closed, we find $\Gamma \subset S \cap \Pi$. Since this is true for any plane Π, S contains the sphere Σ of center ω and radius R. Since S is the boundary of a closed convex set, the only possibility is that S is exactly equal to Σ. $\square$

Remark 7.2. *If, unfortunately,* $\frac{\theta}{\pi} \in \mathbb{Q}$ *, we are unable to conclude. And of course this happens for infinitely many values of h for a given R, even when S is a sphere. This restriction is a bit strange.*

Remark 7.3. *For this result we do not need the domain to be* C^2*, since the curvature radius is not needed in the proof.*

MAIN QUESTION. Is the result of Proposition 7.1 still true assuming only property 1)?

8 Additional remarks and topics of interest

Our results seem to be very partial, so it is interesting to try to understand where difficulties may come from.

8.1 Related questions

The main question raised in this paper is reminiscent of another similar looking global problem concerning convex subsets of $\mathbb{R}^2$ with constant width. In that case, there are many other solutions than disks, even with boundaries consisting of closed smooth algebraic curves, cf. e.g. [1]-[6].

8.2 To be investigated

We have shown that an upper bound on the curvature radius implies a bound on the diameter, hence in particular on the horizon distance. It might be of interest to refine the estimate on the maximum of the horizon distances as a function of the upper bound of the curvature radius and the height h. What would be the optimal inequality? One would expect something tending to 0 with h as in the case of a disk.

8.3 Which extensions can we expect?

The big question is whether it is sufficient to assume the constancy of $H_\pm(h, M, P)$ for some fixed $h > 0$. This does not follow from the technique of theorem 6.1, and it looks quite intricate, since the main surprise in this study is the counterexample of Subsection 5, showing that an upper bound of $H(h)$ gives essentially nothing. We might expect that if for all h small enough, $H_\pm(h, M, P)$ varies between $C(h)$ and $kC(h)$ for some $k > 1$, then the curvature radius also varies between two constants.

8.4 A possible source of difficulty

For the statement of Theorem 6.1 and Corollary 6.2, we need to assume that Γ (resp Σ) is C^2, while the hypothesis on the horizon makes sense for a C^1 manifold. This is because we use in the proof the curvature radius which is a C^2 notion. This might mean that either the result is false assuming only the constancy of $H_\pm(h, M, P)$ for some fixed $h > 0$, or that the method has be be modified, although the involvement of curvature seems to be the right tool to prove circularity or sphericity ... except in the framework of Proposition 7.1!

8.5 The measurement problem

As was pointed out to me by L. Simonot and L. Dettwiller, evaluating the distance of the horizon on the earth is not at all straightforward, even if we use a laser beam. The main point is that the atmosphere can distort light rays, the effect being due to refraction phenomena of thermal origin. For instance in an airplane flying at an altitude of 10 000 meters, when the surrounding temperature is about - 40°C, while the ground is heated at +30°C, the distorsion would correspond, according to the figures given to me by L. Dettwiller, to a virtual altitude differing from the real one by $70 \times 8.6 \sim 600$ meters, a relative variation of 6 % providing a relative error of 3% on the measure of H. This is not negligible, and when we deal with boats or walking people, the relative mistake will probably reach larger values, up to 10%, as a consequence of local variations of temperature and the complex heat transmission phenomena between the ground, the atmosphere and oceans. So if we really are to check the constancy of the horizon distance with great accuracy, corrector methods will have to be used. For more details, cf. e.g. [7]-[10]. It is clear that distortion of light becomes an even more important point in the framework of Proposition 7.1

8.6 Stability

All the partial results of this note are pure mathematical results based on exact hypotheses. In practice any measure is subject to incertitudes, even without the phenomenon described just above. So, whichever will be the final result, the stability question shall become prominent after the main question is solved.

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References

[1] M. Bardet, T. Bayen, On the degree of the polynomial defining a planar algebraic curves of constant width. arXiv:1312.4358 (2013).

[2] T. Bayen, J.-B. Hiriart-Urruty, Objets convexes de largeur constante (en 2D) ou d'épaisseur constante (en 3D): du neuf avec du vieux, *Ann. Sci. Math. Quebec.* **36**, 1 (2012), 17–42.

[3] W. Blaschke, Konvexe Bereiche gegebener konstanter Breite und kleinsten Inhalts, *Math. Ann.* **76**, (1915) 504–513.

[4] G.D. Chakerian, Sets of constant width, *Pacific.J. Maths* **19**, 1 (1966), 13–21.

[5] H.G. Eggleston, A proof of Blaschke's theorem on the Reuleaux Triangle, *Quart. J. Math.* **3** (1952), 296-7.

[6] S. Rabinowitz, A polynomial curve of constant width. *Missouri Journal of Mathematical Sciences.* **9** (1) (1997), 23–27.

[7] W.E. Knowles Midldleton, Bouguer, Lambert, and the theory of horizontal visibility. *Isis* **5**1, 2 (1960), 145-149 .

[8] L. Dettwiller, La discussion par Kummer d'une quadrature sur la réfraction astronomique : commentaire historique, *C. R. Phys.* **23** (2022), n° S1, Online first.

[9] L. Dettwiller, L'invariant de Bouguer et ses conséquences : commentaire historique, *C. R. Phys.* **23** (2022), n° S1, Online first.

[10] L. Dettwiller, Les développements de Lambert : commentaire historique, *C. R. Phys.* **23** (2022), n° S1, Online first.