Horizon and curvature

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Abstract

This short paper is devoted to a strange looking question: is it possible to deduce the shape of a smooth convex set by measuring at each point the distance of the horizon standing at a fixed height h? The question is surprisingly difficult and we only have partial results.

Key words: convex sets, curvature, distance

1 Introduction

It is well known that a smooth planar curve with constant non-zero curvature is a circle. An analogous property can be stated for a surface in $\mathbb{R}^3$. The property of constant curvature can be used to verify the spherical shape of the earth by measurements made on earth itself. Such measurements are in a sense more convincing than arguments relying on exterior objects such as the direction of sunlight, used by Erathostenes to prove at least locally the curved character of the planet. They can at least theoretically be made global.

A simple calculation, detailed in section 2 below, shows a direct relationship between the radius of the earth and the distance of the horizon. But what if we do not know in advance that the planet is approximately spherical? Can we really estimate (locally or globally) the curvature by measuring the distance of the horizon? This simply looking question is the object of the present paper.

2 Radius of the earth and distance of the horizon.

In this section, we consider a perfectly spherical planet (which may be the earth) with radius R and we want to compute the distance of horizon seen from any point of the earth located at a distance h from the ground. First it is immediate, due to rotation invariance, that the distance of the horizon in any direction is the same, let us denote it by $H(h) := H$. To compute H, let us consider a point O on the ground and the point A on the vertical half-line starting from O such that $OA = h$. We consider any plane Π containing the vertical segment OA and the disk D, intersection of the planet with the plane Π. Let T be the contact point of one of the two tangents at $\Gamma = \partial D$ passing through point A. Then $AT = H$. Denoting by C the center of the planet, We have $TC = R = OC$, then $AC = R + h$ and by the Pythagorean theorem applied in the triangle ATC, we infer

$$(R + h)^2 = H^2 + R^2$$

By expanding the square on the left, we get

$$R^2 + 2Rh + h^2 = H^2 + R^2$$

yielding

$$H^2 = 2Rh + h^2$$

Therefore

$$H = \sqrt{2Rh + h^2} \tag{1}$$

and

$$R = \frac{H^2 - h^2}{2h}. \tag{2}$$

For instance on earth where $R \sim 6400\,km$, for a man of height $1,75m$ with eyes at approximately $1,65m$ from the ground, the calculation gives $H \sim 4,6\,km$. This is quite consistent with what we observe when there is no obstacle to limit our vision. And from a plane, when If $h = 10\,km$, we end up with

$$H \sim \sqrt{128000} = 358\,km$$

which is also consistent with the size of the landscape that we can see from a plane when the wheather allows it.

3 Recovering the shape from horizon measurements.

This is a so called "ìnverse problem" and usually this kind of problems is not easy.

3.1 Ruling out flatness

Ruling out absolute flatness of the earth or more generally a planet by local measurements is the easiest part: on a plane, the distance of horizon is infinite from any point, and more generally for a subdomain of a plane, the horizon in any direction coincides with a boundary point of the domain. So, in theory, if the domain is perfectly flat, walking towards the horizon should not allow new objects to appear.

Unfortunately, the adepts of flatness theory (which can also be considered as a challenge to well recognized certitudes), do not usually claim absolute flatness. And to contradict that theory, we need much stronger arguments.

3.2 Curvature radius and confinement

A first question, natural for both planar curves and surfaces in $\mathbb{R}^3$, is the following: If the curvature radius at all points is bounded by a finite number R, can we say that the curve (resp. surface) is bounded? The answer is positive and we even have the following sharper property :

Proposition 3.1. *If the curvature radius of a C^2 planar curve Γ without stationary points is everywhere less than R, the curve is entirely contained in a disk of radius R.*

Proof. First, the curve has no inflexion point and a consequence the curvature has a constant sign. At any given point, the curve is entirely located in one of the two half-planes delimited by the tangent vector. Let us study the local situation by choosing the coordinate axes in such a way that the point has coordinates (0, R) and the equation of Γ becomes

$$y = f(x)$$

for $|x|$ small with $f'(0) = 0$ and $f''(0) < 0$. For $x > 0$ small enough, the curvature radius is

$$\rho(x) = -\frac{(1 + y'^2)^{3/2}}{y''}$$

Therefore

$$y'' = -\frac{1}{\rho(x)}(1 + y'^2)^{3/2} \leq -\frac{1}{R}(1 + y')^{3/2}$$

Since the circle of center (0,0) and radius R satisfies $y = z(x)$ with

$$z'' = -\frac{1}{R}(1 + z'^2)^{3/2}$$

the comparison principle implies that

$$\forall x \in (0, \tau(x)), \quad y'(x) \leq z'(x)$$

Because $y(0) = z(0) = R$, we also infer $y(x) \leq z(x)$ and it is not difficult to deduce that the arch of Γ on the right is interior to the quarter of disk with center (0, 0) and radius R, until the curve Γ crosses the x-axis for the first time. But actually, $|y'(x)|$ blows up for some value $x_1 \leq R$. By rotating the axes by $+\frac{\pi}{2}$ so that $(x_1, y(x_1))$ plays the role of the initial point and proceeding backwards instead of forward, we can see that actually $y(x_1) > 0$. The rest of the proof is rather straighforward, following a sequence of steps analogous to the first one. Since the curve might be a spiral, an infinite number of steps may be necessary for the proof to be complete. □

3.3 Horizon and curvature radius

Encouraged by the previous result, we might try to deduce a bound on the curvature radius from an upper bound of the horizon distance. Unfortunately, we have the following very strong negative result.

Proposition 3.2. *For any $H_0 > 0$, we can find a C^∞ convex domain D with completely flat parts (hence an infinite curvature radius) and a positive h such that*

$$\forall M \in \partial D, \quad H(h, M) \leq 3H_0$$

Proof. Let us consider an even piecewise affine concave function F defined on $[-NH_0, +NH_0)]$ (N positive integer) with $F(-L) = F(L) = 0$, $F(0) = \varepsilon > 0$ such that the slope jumps at all points kH_0, where k is an integer between $-(N-1)$ and $(N-1)$. For a point on the graph of F, the horizon relative to the convex set

$$S := \{(x, y) | x \in [-NH_0, +NH_0], 0 \leq y \leq F(x)\}$$

is either reduced to 0 for summits, or to the length of the face for the other points. By smoothing the corners, the distance of horizon for a point lying on the smoothed curve remains less than the maximum of lengthes of the faces. But since the slope of the faces is less than $\frac{\varepsilon}{NH_0} > 0$, for ε small enough, that maximum is less than $2H_0$. And then for sufficiently small h we obtain the result since the horizon distance is continuous with respect to h. □

3.4 A very partial result

Theorem 3.3. *Let Γ be C^2 curve delimiting a strictly convex compact domain with positive curvature everywhere. Then if it happens that*

$$\lim_{h \to 0} H(h, M) = 0$$

we have more precisely

$$\lim_{h \to 0} \frac{H^2(h, M)}{h} = 2\rho(M)$$

where $\rho(M)$ is the curvature radius at point M.

Proof. By a suitable choice of coordinates, we set $M = (0, 0)$ and we represent locally the curve Γ in an orthonormal frame by the equation $y = f(x)$ where $f(0) = f'(0) = 0$ and $f''(0) < 0$. With this convention we have

$$f''(0) = -\frac{1}{\rho(M)}$$

We consider the right horizon point $T(x, f(x))$ (the calculation on the left will give the same result) and we set $H^2(h, M) = H$. We note first that

$$H^2 = x^2 + (h - f(x))^2$$

and we have the two basic formulas

$$h = f(x) - xf'(x)$$

$$f'(x) = xf''(0) + o(x)$$

In particular,

$$h - f(x) = -x^2 f''(0) - xo(x) \tag{3}$$

yielding

$$(h - f(x))^2 = o(x^2)$$

So since $x^2 = x^2(h) \leq H^2(h)$ tends to 0 with h, we have

$$\lim_{h \to 0} \frac{H^2(h)}{x^2(h)} = 1$$

Now we have by Taylor's formula

$$f(x) = f(0) + xf'(0) + \frac{1}{2} f''(0)x^2 + o(x^2) = \frac{1}{2} f''(0)x^2 + o(x^2) \tag{4}$$

Adding (4) and (3) yields

$$h = -\frac{1}{2} f''(0)x^2 + o(x^2) \tag{5}$$

giving

$$H^2(h) \sim x^2(h) \sim 2\frac{h}{|f''(0)|} = 2\rho(M)h. \tag{6}$$

□

Corollary 3.4. *If for a C^2 surface S delimiting a strictly convex domain with positive curvature everywhere, it happens that $H(h, M, u)$ is independent of the point and the direction u in the tangent plane $T_M(S)$ for all h small enough and in addition, H(h) tends to 0 with h, the surface is a sphere.*

Proof. Let M be any point of the surface S and Π any plane containing the normal to S at M. By applying Theorem 3.3 to $\Gamma(\Pi) := S \cap \Pi$, we obtain that $\Gamma(\Pi)$ has constant curvature. Therefore by a well known result, $\Gamma(\Pi)$ is a circle (an entire circle and not just an arc since the horizon distance is positive in both directions above each point of the curve), and its radius does not depend on Π. It is easy to conclude by rotation with respect to the axis containing the normal to S at M. □

4 Concluding remarks

The results of this short note seem to be very partial, so it is interesting to try to understand where difficulties may come from.

4.1 Related questions

The main question raised in this paper is reminiscent of another similar looking global problem concerning convex subsets of $\mathbb{R}^2$ with constant width. In that case, there are many other solutions than disks, even with boundaries consisting of closed smooth algebraic curves, cf. e.g. [1]-[6].

4.2 Which extensions can we expect?

The big question is whether it is sufficient to assume the constancy of $H(h, M, u)$ for some fixed $h > 0$. This does not follow from the technique of theorem 3.3, and it looks quite intricate, since the main surprise in this study is the counterexample of Subsection 3.3, showing that an upper bound of $H(h)$ gives essentially nothing. We might expect that if for all h small enough, $H(h, M, u)$ varies between $C(h)$ and $kC(h)$ for some $k > 1$, then the curvature radius also varies between two constants.

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