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Research Article

Patterns of Squares Around an Arbitrary Triangle

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H. Ebisui and J.C.G. Notrott studied patterns of squares around respectively a right and an arbitrary triangle, thus generalizing the Pythagorean theorem. We construct a new pattern of squares around an arbitrary triangle, based on the four (hinged) squares theorem, using simple vector constructions to avoid trigonometric calculations. This gives rise to some known or unknown number sequences, with new applications on a geometric pattern of squares.

Generalizations of the Pythagorean Theorem

The Pythagorean theorem for a right-angled triangle is the most well-known theorem. There are many generalizations, such as Japanese mathematician H. Ebisui's 'Pythagorean fivefold theorem'. J.C.G. Notrott had published an even more general result earlier (see [1] and [2]), but he did so in the magazine Pythagoras, in Dutch, so one can assume both discoveries were made independently.

Ebisui considered a right-angled triangle ΔABC (in red on Fig. 1) with the right angle in C and squares on the sides a_1, b_1, c_1. The Pythagorean theorem states that $c_1^2 = a_1^2 + b_1^2$. If a ring of (blue) squares with sides A_1, B_1, C_1 is built on the convex hull of these squares, then $5\,C_1^2 = A_1^2 + B_1^2$. Another ring of squares with sides a_2, b_2, c_2 on the convex hull of these squares will again satisfy the Pythagorean theorem, and the next one with sides A_2, B_2, C_2 will again satisfy $5\,C_2^2 = A_2^2 + B_2^2$. And so on.

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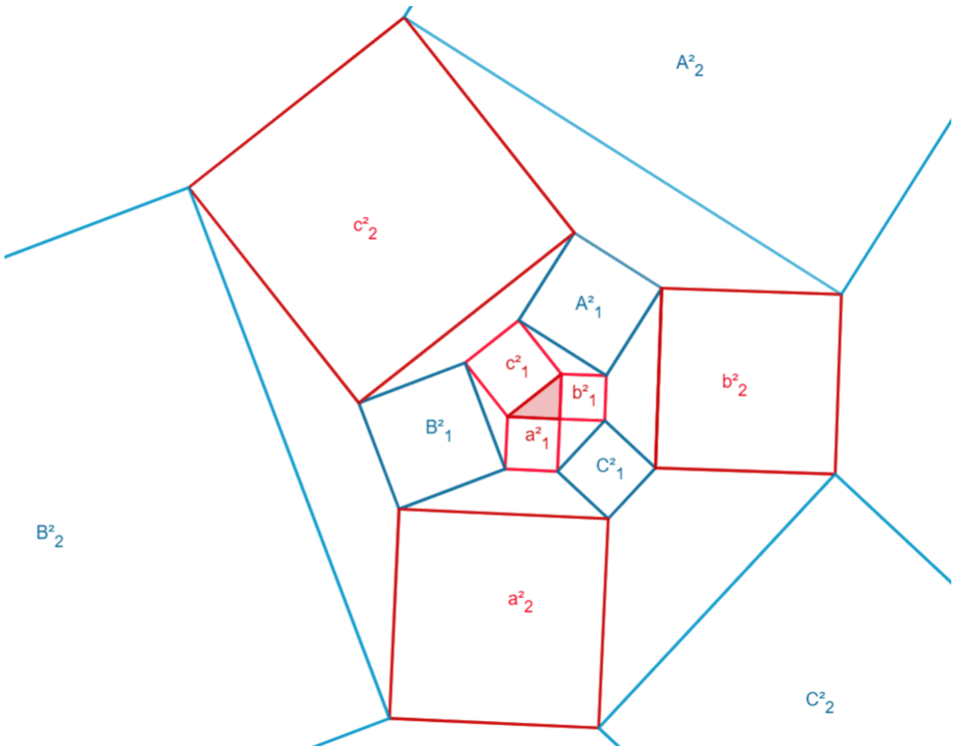

Figure 1. A right-angled triangle (red) and several rings of squares around it, alternately satisfying the Pythagorean theorem $c_i^2 = a_i^2 + b_i^2$ or the Pythagorean fivefold theorem $5C_i^2 = A_i^2 + B_i^2$

Notrott considered an arbitrary triangle and constructed squares around it the same way (see Fig. 2). Using similar notations, the sums of the areas of the squares of the consecutive rings are:

$A_1^2 + B_1^2 + C_1^2 = 3\left(a_1^2 + b_1^2 + c_1^2\right),$

$a_2^2 + b_2^2 + c_2^2 = 16\ \left(a_1^2 + b_1^2 + c_1^2\right),$

$A_2^2 + B_2^2 + C_2^2 = 75\ \left(a_1^2 + b_1^2 + c_1^2\right), \ldots$

which gives rise to the sequence 1, 3, 16, 75, 361, 1728, 8281, … (A005386 in the Online Encyclopedia of Integer Sequences). Of course, for a right triangle, where $C^2{}_1 = c^2{}_1 = a^2{}_1 + b^2{}_1$, Notrott's result becomes Ebisui's 'fivefold theorem'.

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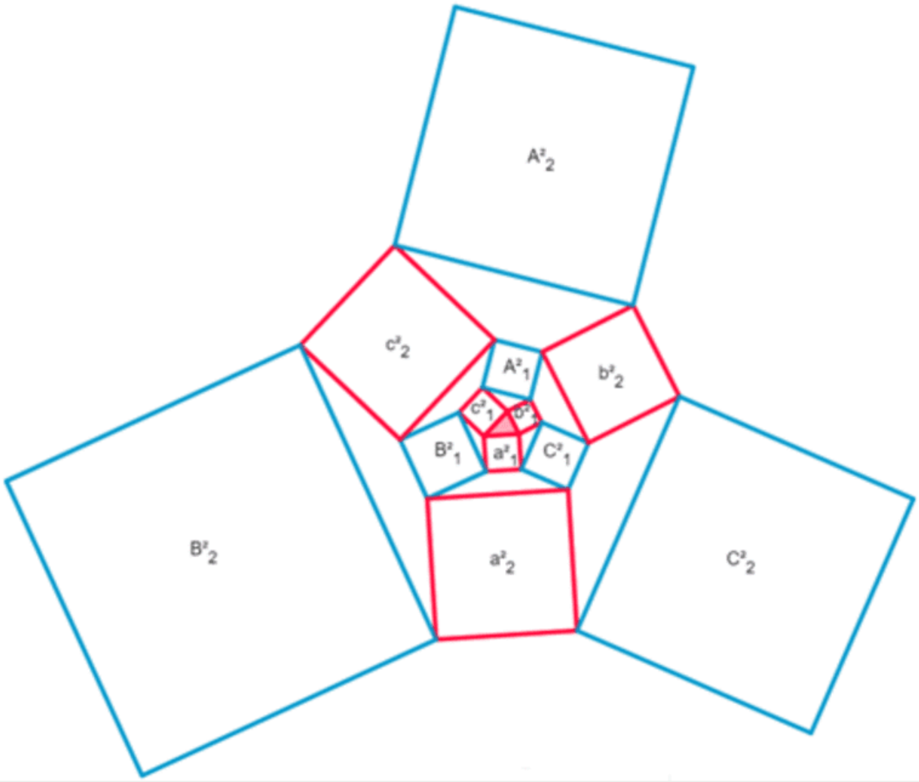

Figure 2. Rings of squares around an arbitrary triangle

A proof of an even more general theorem for arbitrary triangles was given by Long Huynh Huu (see [3]). He shows that any linear relation that holds between the areas of the squares of the first ring will also be valid for the areas of the squares of the third, fifth, ... ring. And any linear relation that holds for the areas of the squares of the second ring will also hold for those of the fourth, sixth, ... ring.

To a Pattern of Four (Hinged) Squares around an Arbitrary Triangle

There are still other patterns to be discovered in the squares around an arbitrary triangle $\Delta\, ABC$. Let's start with four (red) squares instead of three (see Fig. 3). Denote $\overrightarrow{CB} = \vec{a}$, $\overrightarrow{AC} = \vec{b}$, $\overrightarrow{BA} = \vec{c}$, $\overrightarrow{DG} = \vec{d}$ and the vectors obtained by rotating them 90° clockwise respectively by $\vec{a'}$, $\vec{b'}$, $\vec{c'}$ and $\vec{d'}$. Note this implies $\left(\vec{a'}\right)' = -\vec{a}$. These notations allow the use of simple vector constructions and avoid trigonometric calculations as used in [4].

Thus, $\vec{c} = -(\vec{a} + \vec{b})$, $\vec{d} = \vec{b'} - \vec{a'}$ and $\vec{c'} = -(\vec{a'} + \vec{b'})$, $\vec{d'} = \vec{a} - \vec{b}$. Moreover, since the vectors $\vec{a}$ and $\vec{a'}$ have the same length, as well as the vectors $\vec{b}$ and $\vec{b'}$, and the same angle between them, we have that $\vec{a'} \bullet \vec{b'} = \vec{a} \bullet \vec{b}$. From $\vec{c} \bullet \vec{c'} = 0 = \left(\vec{a} + \vec{b}\right) \bullet \left(\vec{a'} + \vec{b'}\right)$ $= \vec{a} \bullet \vec{a'} + \vec{a} \bullet \vec{b'} + \vec{b} \bullet \vec{a'} + \vec{b} \bullet \vec{b'} = \vec{a} \bullet \vec{b'}, + \vec{b} \bullet \vec{a'}$

it follows that $\vec{a} \bullet \vec{b'} = -\vec{b} \bullet \vec{a'}$.

The areas of the squares with sides c and d are:

$c^2 = \vec{c} \bullet \vec{c} = \left(\vec{a} + \vec{b}\right) \bullet \left(\vec{a} + \vec{b}\right) = a^2 + 2\,\vec{a} \bullet + b^2$

$d^2 = \vec{d} \bullet \vec{d} = \left(\vec{b'} - \vec{a'}\right) \bullet \left(\vec{b'} - \vec{a'}\right) = b'^2 - 2\,\vec{a'} \bullet \vec{b'} + a'^2$

$= a^2 - 2\,\vec{a} \bullet \vec{b} + b^2$, so that: $c^2 + d^2 = 2(a^2 + b^2)$. Hence, "*the sum of the areas of the squares on the non-equal sides of two triangles with two equal sides and supplementary*

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enclosed angles is double the sum of the areas of the squares on the equal sides". This result is known as the four (hinged) squares theorem (we added the word 'hinged' to avoid confusion with Langrange's four squares theorem).

Now, we construct four new squares by connecting vertices of the previous set. Then:

$$\vec{a_1} = \vec{b'} - \vec{c'} = \vec{a'} + 2\,\vec{b'}$$

$$\vec{b_1} = \vec{d'} - \vec{b} = \left(\vec{a} - \vec{b}\right) - \vec{b} = \vec{a} - 2\,\vec{b},$$

$$\vec{c_1} = \vec{c'} - \vec{a'} = -2\,\vec{a'} - \vec{b'},$$

$$\vec{d_1} = -\vec{a} - \vec{d'} = -\vec{a} - \left(\vec{a} - \vec{b}\right) = -2\,\vec{a} + \vec{b},$$

so that:

$$\overrightarrow{a'_1} = -\vec{a} - 2\,\vec{b},$$

$$\overrightarrow{b'_1} = = \vec{a'} - 2\,\vec{b'},$$

$$\overrightarrow{c'_1} = 2\,\vec{a} + \vec{b},$$

$$\overrightarrow{d'_1} = -2\,\vec{a'} + \vec{b'}.$$

Using these expressions, the sums of the areas of opposite squares are equal since:

$$a_1^2 + d_1^2 = \left(\vec{a'} + 2\,\vec{b'}\right) \bullet \left(\vec{a'} + 2\,\vec{b'}\right) + \left(\vec{b} - 2\,\vec{a}\right) \bullet \left(\vec{b} - 2\,\vec{a}\right)$$
$$= a'^2 + 4\,b'^2 + 4\,\vec{a'} \bullet \vec{b'} + 4\,a^2 + b^2 - 4\,\vec{a} \bullet \vec{b} = 5a^2 + 5\,b^2$$

and

$$b_1^2 + c_1^2 = \left(\vec{a'} - 2\,\vec{b'}\right) \bullet \left(\vec{a'} - 2\,\vec{b'}\right) + \left(\vec{b} + 2\,\vec{a}\right) \bullet \left(\vec{b} + 2\,\vec{a}\right)$$
$$= a'^2 + 4\,b'^2 - 4\,\vec{a'} \bullet \vec{b'} + b^2 + 4\,a^2 + 4\,\vec{a} \bullet \vec{b} = 5a^2 + 5\,b^2$$

Hence, $a_1^2 + d_1^2 = b_1^2 + c_1^2$.

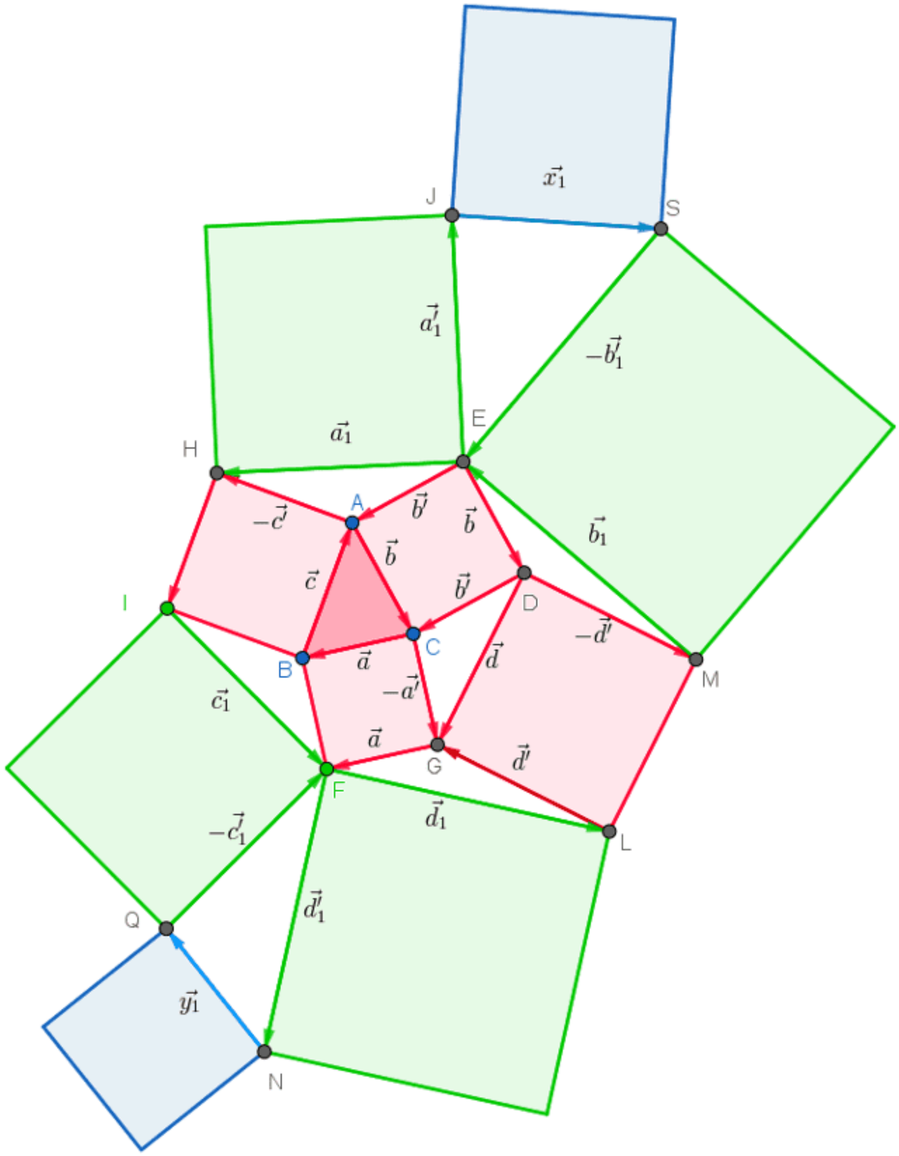

Figure 3. A new pattern of squares around an arbitrary triangle

Before building the next set of four squares, we first create two auxiliary squares on the vertices of the squares with sides a_1 and b_1, and on those with sides c_1 and d_1. The vectors associated with its sides can be expressed as:

$$\vec{x_1} = \overrightarrow{b'_1} - \overrightarrow{a'_1} = \vec{a} + \vec{a'} + 2\,\vec{b} - 2\,\vec{b'} \quad \text{and}$$

$$\vec{y_1} = \overrightarrow{c'_1} - \overrightarrow{d'_1} = 2\,\vec{a} + 2\,\vec{a'} + \vec{b} - \vec{b'}.$$

Consequently,

$$x_1^2 = \left(\vec{a} + \vec{a'} + 2\,\vec{b} - 2\,\vec{b'}\right) \bullet \left(\vec{a} + \vec{a'} + 2\,\vec{b} - 2\,\vec{b'}\right)$$

$$= \left(\vec{a} + \vec{a'}\right)^2 + 4\left(\vec{b} - \vec{b'}\right)^2 + 4\left(\vec{a} + \vec{a'}\right) \bullet \left(\vec{b} - \vec{b'}\right)$$

$$= 2\,a^2 + 8\,b^2 + 4\left(\vec{a} \bullet \vec{b} + \vec{a'} \bullet \vec{b} - \vec{a} \bullet \vec{b'} - \vec{a'} \bullet \vec{b'}\right)$$

$$= 2\,a^2 + 8\,b^2 + 8\,\vec{a'} \bullet \vec{b},$$

and similarly, $y_1^2 = 8\,a^2 + 2\,b^2 + 8\,\vec{a'} \bullet \vec{b}$.

The last part of these expressions can be reformulated in terms of the area $A_{\Delta ABC}$ of the original triangle ΔABC.

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Note that the angles between the vectors $\overrightarrow{a'}$ and $\overrightarrow{b}$ and the angle between the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ sum up to 270° in the case of an acute triangle or differ by 90° in the case of an obtuse triangle so that:

$$\overrightarrow{a'} \bullet \overrightarrow{b} = ab\cos\left(\overrightarrow{a'}, \overrightarrow{b}\right) = -a\,b\,\sin\left(\overrightarrow{a}, -\overrightarrow{b}\right) = -2\,A_{\Delta ABC}$$

It follows that:

$x_1^2 = 2\,a^2 + 8\,b^2 - 16\,A_{\Delta ABC}$,

$y_1^2 = 8\,a^2 + 2\,b^2 - 16\,A_{\Delta ABC}$,

and so $x_1^2 + y_1^2 = 10\left(a^2 + b^2\right) - 32A_{\Delta ABC}$.

The next ring of squares is constructed with a vertex of a square of the previous one and a vertex of an auxiliary square. And this procedure can be continued (see Fig. 4). Labelling the vectors associated with the blue auxiliary squares as $\overrightarrow{x}_i$ and $\overrightarrow{y}_i$, and those associated with the other squares as $\overrightarrow{a}_i$, $\overrightarrow{b}_i$, $\overrightarrow{c}_i$ and $\overrightarrow{d}_i$ yields:

$$\overrightarrow{a_2} = \overrightarrow{a}_1 + \overrightarrow{x'}_1 = 2\,\overrightarrow{a'} - \overrightarrow{a} + 4\,\overrightarrow{b'} + 2\,\overrightarrow{b}$$

$$\overrightarrow{b_2} = \overrightarrow{b}_1 - \overrightarrow{x'}_1 = 2\,\overrightarrow{a} - \overrightarrow{a'} - 4\,\overrightarrow{b} - 2\,\overrightarrow{b'}$$

$$\overrightarrow{x_2} = -\overrightarrow{a'}_2 + \overrightarrow{x}_1 + \overrightarrow{b'}_2 = 4\,\overrightarrow{a'} + 4\overrightarrow{a} - 8\,\overrightarrow{b'} + 8\,\overrightarrow{b} = 4\,\overrightarrow{x_1}$$

Hence, $x_2^2 = 16\,x_1^2$ and

$$\overrightarrow{b'}_2 - \overrightarrow{a'}_2 = \overrightarrow{b'}_1 - \overrightarrow{a'}_1 + 2\,\overrightarrow{x}_1 = 3\,\overrightarrow{x}_1$$

Similarly,

$$\overrightarrow{c_2} = 2\,\overrightarrow{a} - 4\overrightarrow{a'} - \overrightarrow{b} - 2\,\overrightarrow{b'}$$

$$\overrightarrow{d_2} = 2\,\overrightarrow{a'} - 4\,\overrightarrow{a} + \overrightarrow{b'} + 2\,\overrightarrow{b}$$

$$\overrightarrow{y_2} = 8\,\overrightarrow{a'} + 8\,\overrightarrow{a} - 4\,\overrightarrow{b'} + 4\,\overrightarrow{b} = 4\,\overrightarrow{y_1}$$

So that the sum of the areas of opposite squares equals:

$$a_2^2 + d_2^2 = b_2^2 + c_2^2 = 25\left(a^2 + b^2\right) - 64A_{\Delta ABC}$$

In the next paragraph, we will generalize this pattern.

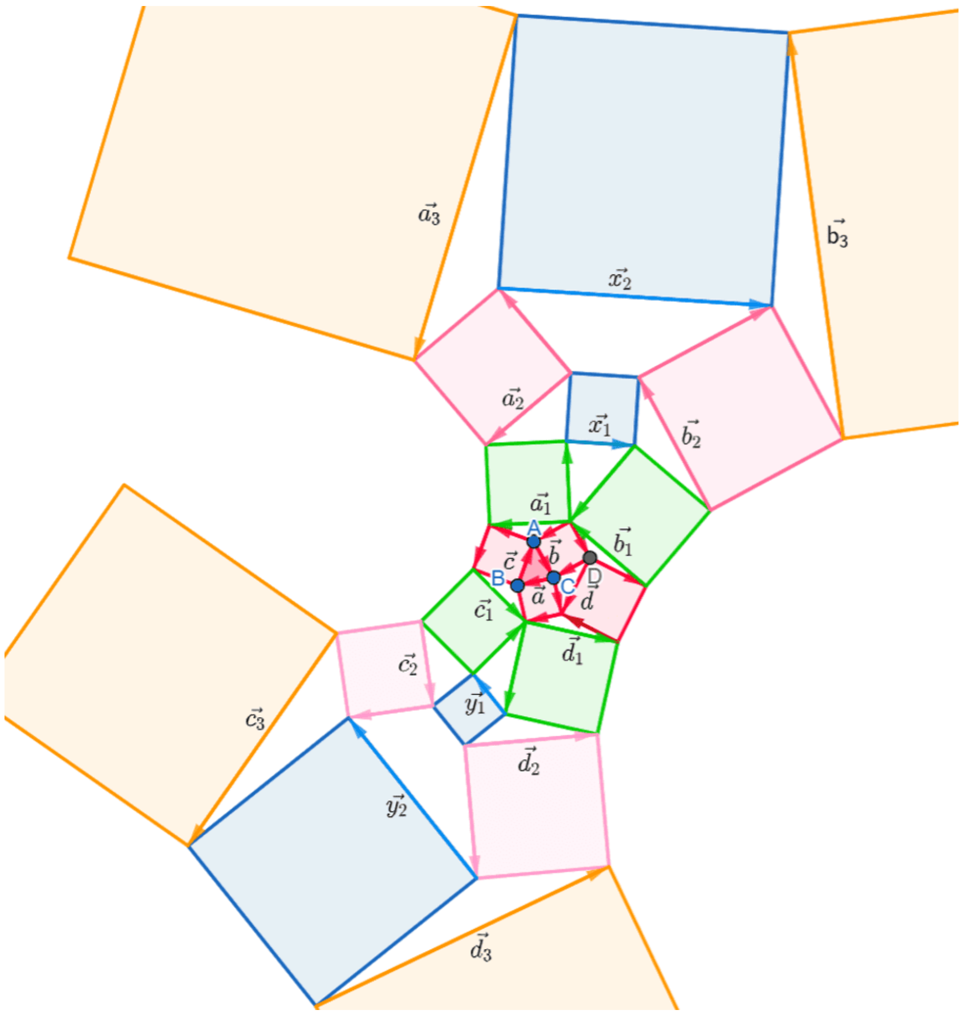

Figure 4. A pattern of squares around an arbitrary triangle

Generalization of the Pattern of Four (Hinged) Squares around an Arbitrary Triangle

For $i \geq 2$:

$$\vec{a}_i = \vec{a}_{i-1} + \vec{x'}_{i-1}$$

$$\vec{b}_i = \vec{b}_{i-1} - \vec{x'}_{i-1}$$

$$\vec{x}_i = \vec{x}_{i-1} + \vec{b'}_i - \vec{a'}_i$$

so that

$$\vec{a}_i + \vec{b}_i = \vec{a}_{i-1} + \vec{b}_{i-1} = \vec{a}_1 + \vec{b}_1 = \vec{a} + \vec{a'} - 2\vec{b}.$$

$$+ 2\vec{b'}$$

Rotating 90° clockwise yields:

$$\vec{a'}_i = \vec{a'}_{i-1} - \vec{x}_{i-1}$$

$$\vec{b'}_i = \vec{b'}_{i-1} + \vec{x}_{i-1}$$

It follows that

$\vec{a'}_i + \vec{b'}_i = \vec{a'}_{i-1} + \vec{b'}_{i-1} = \vec{a'}_1 + \vec{b'}_1 = -\vec{a} + \vec{a'}$ an $-2\vec{b} - 2\vec{b'}$

$$\vec{b'}_i - \vec{a'}_i = 2\vec{x}_{i-1} + \left(\vec{b'}_{i-1} - \vec{a'}_{i-1}\right)$$

Hence,

$$\vec{x}_i = \vec{x}_{i-1} + \vec{b'}_i - \vec{a'}_i = 3\vec{x}_{i-1} + \left(\vec{b'}_{i-1} - \vec{a'}_{i-1}\right)$$

Now we turn to the sequence of the $\overrightarrow{x}_i$ and the sequence of the $\overrightarrow{b'}_i - \overrightarrow{a'}_i$. We claim that:

$\overrightarrow{x}_i = k_i \overrightarrow{x}_1$ with $k_1 = 1,\ k_2 = 4$ and $k_i = 4\,k_{i-1} - k_{i-2}$ for $i \geq 3$,

$\overrightarrow{b'}_i - \overrightarrow{a'}_i = l_i \overrightarrow{x}_1$ with $l_1 = 1,\ l_2 = 3$ and $l_i = 4\,l_{i-1} - l_{i-2}$ for $i \geq 3$. These sequences correspond to sequences A001353 and A001835 of the Online Encyclopedia of Integer Sequences. Both series are extensively commented, but the geometric applications given here seem new.

From the previous calculations, it follows that the relations hold for $n = 1,\ 2$.

For $n \geq 3$, we have by induction on n:

$$\overrightarrow{b'}_i - \overrightarrow{a'}_i = 2\,\overrightarrow{x}_{i-1} + \left(\overrightarrow{b'}_{i-1} - \overrightarrow{a'}_{i-1}\right) = 2\,k_{i-1}\,\overrightarrow{x}_1 + l_{i-1}\,\overrightarrow{x}_1 = (2\,k_{i-1} + l_{i-1})\,\overrightarrow{x}_1 = l_i\,\overrightarrow{x}_1$$

and

$$\overrightarrow{x}_i = \overrightarrow{x}_{i-1} + \left(\overrightarrow{b'}_i - \overrightarrow{a'}_i\right) = (k_{i-1} + l_i)\,\overrightarrow{x}_1 = k_i\,\overrightarrow{x}_1,$$

so that

$l_i = 2\,k_{i-1} + l_{i-1}$ (1) and $k_i = k_{i-1} + l_i$ (2) for $i \geq 3$.

Now, using (1), (2):

$$\begin{aligned} l_{i+1} &= 2\,k_i + l_i && (1)\\ &= 2\,k_{i-1} + 2\,l_i + l_i && (2)\\ &= 3\,l_i + l_i - l_{i-1} && (1)\\ &= 4\,l_i - l_{i-1} \end{aligned}$$

In a similar way, we find that:

$$k_{i+1} = k_i + l_{i+1} = k_i + l_i + 2\,k_i = 3\,k_i + l_i = 3\,k_i + k_i - k_{i-1} = 4\,k_i - k_{i-1}$$

which proves the result.

This implies that the areas of the auxiliary squares are:

$x_i^2 = k_i^2\,x_1^2 = k_i^2(2\,a^2 + 8\,b^2 - 16\,A_{\Delta ABC})$

In an analogous way:

$$\overrightarrow{c}_i = \overrightarrow{c}_{i-1} - \overrightarrow{y'}_{i-1}$$

$$\overrightarrow{d}_i = \overrightarrow{d}_{i-1} + \overrightarrow{y'}_{i-1}$$

$$\overrightarrow{y}_i = \overrightarrow{y}_{i-1} + \overrightarrow{c'}_i - \overrightarrow{d'}_i$$

$$\overrightarrow{c'}_i - \overrightarrow{d'}_i = l_i\,\overrightarrow{y}_1$$

$$\overrightarrow{y}_i = k_i\,\overrightarrow{y}_1$$

$$y_i^2 = k_i^2\,y_1^2 = k_i^2(2\,a^2 + 8\,b^2 - 16\,A_{\Delta ABC})$$

where the sequence of the k_i and the sequence of the l_i are the same as in the case of the $\overrightarrow{x}_i$ and the $\overrightarrow{b'}_i - \overrightarrow{a'}_i$.

Now let's turn to the sequence of the sums of the areas of opposite squares. We claim that $\overrightarrow{a}_i = \overrightarrow{a}_1 + (k_1 + k_2 + \ldots + k_{i-1})\,\overrightarrow{x'}_1$. Obviously, the relation holds for $i = 2$. For $i \geq 2$, we have, by induction on i that:

$$\begin{aligned} \overrightarrow{a}_i = \overrightarrow{a}_{i-1} + \overrightarrow{x'}_{i-1} &= \overrightarrow{a}_1 + (k_1 + k_2 + \ldots + k_{i-2})\,\overrightarrow{x'}_1 + k_{i-1}\overrightarrow{x'}_1\\ &= \overrightarrow{a}_1 + (k_1 + k_2 + \ldots + k_{i-1})\,\overrightarrow{x'}_1 \end{aligned}$$

Similarly,

$$\overrightarrow{b}_i = \overrightarrow{b}_1 - (k_1 + k_2 + \ldots + k_{i-1})\,\overrightarrow{x'}_1$$

$$\overrightarrow{c}_i = \overrightarrow{c}_1 - (k_1 + k_2 + \ldots + k_{i-1})\,\overrightarrow{y'}_1$$

$$\overrightarrow{d}_i = \overrightarrow{d}_1 + (k_1 + k_2 + \ldots + k_{i-1})\,\overrightarrow{y'}_1$$

It follows that

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$$\vec{a}_i^{\,2} + \vec{d}_i^{\,2} = a_1^2 + d_1^2 + (k_1 + k_2 + \ldots + k_{i-1})^2 \left(x_1^2 + y_1^2 \right) + 2 (k_1 + k_2 + \ldots + k_{i-1}) \left(\vec{a}_1 \bullet \vec{x'}_1 + \vec{d}_1 \bullet \vec{y'}_1 \right)$$

where

$$\vec{a}_1 \bullet \vec{x'}_1 + \vec{d}_1 \bullet \vec{y'}_1 = \left(\vec{a'} + 2 \vec{b'} \right) \bullet \left(-\vec{a} + \vec{a'} + 2 \vec{b} + 2 \vec{b'} \right) + \left(-2 \vec{a} + \vec{b} \right) \bullet \left(-2 \vec{a} + 2 \vec{a'} + \vec{b} + \vec{b'} \right)$$

$$= 5\, a^2 + 5\, b^2 - 16\, A_{\Delta ABC}$$

Using the above expression and those for $a_1^2 + d_1^2$ and $x_1^2 + y_1^2$ derived previously, yields:

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$$
\begin{aligned}
a_i^2 + d_i^2 \\
= \Big[1 + 2\left(k_1 + k_2 + \ldots + k_{i-1}\right) \\
+ 2\left(k_1 + k_2 + \ldots + k_{i-1}\right)^2 \Big]\left(5a^2 + 5b^2\right) \\
- \Big[\left(k_1 + k_2 + \ldots + k_{i-1}\right) + \left(k_1 + k_2 + \ldots + k_{i-1}\right)^2 \Big]\ 32\ A_{\Delta ABC}
\end{aligned}
$$

This expression can be rewritten as:

$$a_i^2 + d_i^2 = 5\ r_i\left(a^2 + b^2\right) - 32\ s_i\ A_{\Delta ABC}$$

where the sequence of the r_i and the sequence of the s_i are given by:

$r_i = 1 + 2\left(k_1 + k_2 + \ldots + k_{i-1}\right)$ for $i \geq 2$ and $r_1 = 1$ $+ 2\left(k_1 + k_2 + \ldots + k_{i-1}\right)^2$

$s_i = \left(k_1 + k_2 + \ldots + k_{i-1}\right) + \left(k_1 + k_2 + \ldots + k_{i-1}\right)^2$ for $i \geq 2$ and $s_1 = 0$.

The sequence of the r_i is 1, 5, 61, 3241, 11705, 163021, ... which is not yet in the Online Encyclopedia of Integer Sequences. The sequence of the s_i is 0, 2, 30, 420, 5852, 81510, ... and corresponds to the sequence known as A217855 in the Online Encyclopedia of Integer Sequences. Again, this series has no geometric application yet.

In a similar way, $b_i^2 + c_i^2 = 5\ r_i\left(a^2 + b^2\right) - 32\ s_i\ A_{\Delta ABC}$.

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