Research Article

Encoding Sequences in Intuitionistic Real Algebra

Miklós Erdélyi-Szabó[1]

1. Alfréd Rényi Institute of Mathematics, Budapest, Hungary

We show that in the presence of random Kripke's schema choice sequences can be recursively encoded in intuitionistic real algebra.

1. Introduction

The consistency of the full version of RR-KS - Relativised Random Kripke's Schema (introduced in the paper [1]) with the usual axiom system of intuitionistic real analysis remains open, only the consistency of a significantly weaker form when relativisation applies to decidable species is proved (see the Appendix of this paper). In [1] it was shown that the interpretability of second-order Heyting arithmetic in intuitionistic real algebra follows from this axiom system. In this paper we sidestep the problem of consistency to get a related result - the interpretability of full second order arithmetic in intuitionistic real algebra using the axiom system from [2]. In fact, if the first order structure of natural numbers is definable in an intuitionistic real algebra (as in the real algebraic structures of the models described in [3] and in [4] - the definability there follows from R-KS and the usual axioms, for the details see [2]), then so is second order arithmetic. Since the treatment here is (mostly) axiomatic (as was in the gappy [1]) the result parallels the results in [5] and [2] where we moved from proving the interpretability of the natural number structure in a model of intuitionistic real algebra to proving interpretability from an axiom system, moving away from the pecularities of a given structure. In [6] we have shown an encoding of true second order arithmetic in models, here we give an encoding using an axiom system. The point is that the coding will be direct, no Gödel numbering of syntax is needed.

From [1] let us recall the following. We shall use the language L_1 from [3]. It contains two sorts of variables – m, n, k, etc. ranging over the elements of ω, and α, β, etc. ranging over choice sequences.

 doi.org/10.32388/7IMXXE

We also have the equality symbol $=$. It will be used in atomic formulas of the form $t = t'$ or $\xi(t) = t'$ where t and t' are terms of natural-number sort and ξ is of choice sequence sort.

In [6] we defined randomized Kripke's schema as the following axiom schema of second-order arithmetic:

$$\text{R-KS}(\varphi) \equiv \exists\beta[\, (\exists n(\beta(n) > 0) \to \varphi) \wedge (\neg\exists n(\beta(n) > 0) \to \neg\varphi) \wedge \\ \forall k > 0(\neg\exists n(\beta(n) = k) \to \varphi \vee \neg\varphi) \wedge \forall k > 0\forall n((\beta(n) = k) \to \forall m \geq n(\beta(m) = k))\,]$$

where φ is a formula that does not contain the choice sequence variable β free.

Also in [6] we have proved the following.

i. The models of intuitionistic second order arithmetic described in [3] and in [4] are models of R-KS.
ii. From a standard axiom system augmented with R-KS the definability in the language LOR of ordered rings of the set of natural numbers follows: there is a formula $N(x)$ with one free variable on the language of ordered rings such that from the axioms $\exists k \in \mathbb{N}^+(x = k) \equiv N(x)$ follows in two-sorted intuitionistic predicate calculus with equality.

The folowing is an immediate corollary.

Proposition 1. *The ordered semiring structure of natural numbers has a uniform definition in the real algebraic part of any model of the axiom system T from [3] (also in [4]) with R-KS added. From this follows that each recursive function has such a uniform definition. The restriction to $\{x|N(x)\}$ of the formula defining the recursive function/relation in the structure of natural numbers works.* □

Also, the real algebraic structure is defined in intuitionistic second order arithmetic: In pages 134-135 [7] Vesley considers a species R of *real-number generators*: $\xi \in R$ (also denoted by $R(\xi)$) if and only if the sequence $2^{-x}\xi(x)$ $(x \in \omega)$ of diadic fractions is a Cauchy-sequence with $\forall k\exists x\forall p|2^{-x}\xi(x) - 2^{-x-p}\xi(x+p)| < 2^{-k}$, i.e. if and only if $\forall k\exists x\forall p 2^k|2^p\xi(x) - \xi(x+p)| < 2^{x+p}$. Note that any choice sequence with range in the set $\{0, 1\}$ is a real number generator (with the corresponding real in the closed unit interval).

Equality, ordering, addition and multiplication on R are also defined. (cf. also pages 20-21 [8]). and the definitions can be extended readily to polynomials of choice sequences.

ξ is a *global real-number generator* just in case $R(\xi)$ holds. We shall use the letters f, g, u etc. to range over global real-number generators and we shall use the defined quantifiers

doi.org/10.32388/7IMXXE

$$\exists^R u\theta :\equiv \exists u(R(u) \wedge \theta); \forall^R u\theta :\equiv \forall u(R(u) \to \theta)$$

also (definable) quantification $\exists^{R,2}$ and $\forall^{R,2}$ over real number generators with range in the set $\{0,1\}$

and when we have the definition $N(x)$ of the natural numbers

$$\exists^{\mathbb{N}^+} u\theta :\equiv \exists u(N(u) \wedge \theta); \forall^{\mathbb{N}^+} u\theta :\equiv \forall u(N(u) \to \theta)$$

For each natural number n there is a corresponding global real-number generator f_n (also denoted by n in the context of real numbers only) defined as follows: $f_n(l) = m$ iff $m = n2^l$. Then

$$f_n f = \underbrace{f + \cdots + f}_{n}$$

2. Encoding choice sequences

We encode choice sequences as real numbers in the closed unit interval such a way that atomic formulas of the form $\beta(m) = k$ could be translated into a formula $[\beta, m, k]$ in the language of ordered rings. Also the set of codes will be definable, a code will be unique (with respect to equality of real numbers) and each code will be a code of a unique choice sequence (see Lemma (7). and Lemma (?). below).

Some notation: $\langle . , . \rangle$ will denote a recursive injective and surjective pairing function.

The code C_ξ of a choice sequence ξ will be a real number with generator a $0-1$ sequence C'_ξ such that

i. $C'_\xi(2\langle m, k\rangle) = 1$ iff $\xi(m) = k$ $(m, k \in \omega)$.
ii. $C'_\xi(2p+1) = 0$ $(p \in \omega)$.

Note that each real in the interval $[0,1]$ has a $0-1$ real number generator.

The second (technical) condition is needed to make sure that any real number is a code of at most one sequence and to be able to deduce properties of a $0-1$ reaql number generator from the properties of the corresponding real (see Lemma 2).

Now some details. First of all we have to express $C'_\xi(p) = 0$ with a formula on the languge LOR of ordered rings involving the real numbers $C_\xi \in [0,1]$ and p with $N(p)$. The next lemma gives the required translation.

For a $0-1$ sequence generator x' let x be the corresponding real. Then $x'(p) = 0$ means that in the diadic expansion of x there is no $1/2^p$.

doi.org/10.32388/7IMXXE

Lemma 2. *In this lemma we assume that the $0-1$ sequence representation of a real number z is not eventually 1, ie. $\forall m \exists n > m(z(n) = 0)$ (standardrepresentation).*

i. *$x'(m) = 0$ for all $m > p$ iff $\exists^+ q(x \cdot 2^p = q)$.*

ii. *Assume $x'(p) = 0$ and $x'(i) = 0$ for some $i > p$. Let $y_p = \Sigma_{i=1}^{p-1} x'(i) \cdot 1/2^i$ a (rational) real. Then $\exists^+ q(y_p \cdot 2^{p-1} = q)$ and $0 \leq x - y_p < 1/2^p$.*

iii. *If for some y $\exists^+ q(y \cdot 2^{p-1} = q)$ and $0 \leq x - y < 1/2^p$ and $x'(i) = 0$ for some $i > p$, then $x'(p) = 0$. In this case for the standard representation of y we have $y'(j) = x'(j)$ for all $j < p$.*

Proof.

ii. $x - y_p = \Sigma_{i \geq p} x'(i) \cdot 1/2^i \geq 0$

Since $x'(p) = 0$, $x - y_p = \Sigma_{i>p} x'(i) \cdot 1/2^i < \Sigma_{i>p} 1/2^i = 1/2^p$. $\square$

2. First note that x' being a $0-1$ choice sequence implies that $x'(k) = 0$ or $x'(k) = 1$ for all k. By (i) y is of the form $y = \Sigma_{i=1}^{p-1} y'(i) \cdot 1/2^i$ for an appropriate representation y'.

If $j < p$ is the least index with $y'(j) \neq x'(j)$ then if $y'(j) < x'(j)$ we have $x - y \geq 1/2^j - \Sigma_{m=j+1}^{p-1} 1/2^m + 1/2^p > 1/2^p$. If $y'(j) > x'(j)$ then $x - y = -1/2^j + \Sigma_{m>j} x'(m) \cdot 1/2^m < -1/2^j + 1/2^j = 0$, since $x'(i) = 0$ for some $i > p$.

If for all $i < p$ $y'(i) = x'(i)$ and $x'(p) = 1$, then $x - y \geq 1/2^p$. $\square$

To apply the lemma we need to take care of the condition on the "returning zeros". For a $0-1$ sequence x' corresponding to the real number x let $[x', p]$ denote the LOR-fromula equivalent to

$$\exists y \exists^+ q(y \cdot 2^{p-1} = q) \wedge 0 \leq x - y < 1/2^p$$

Lemma 3. *$\forall m x'(2m+1) = 0$ and $x'(p) = 0$ iff $\forall m[x', 2m+1] \wedge [x', p]$.*

Proof. First assume that $\forall m x'(2m+1) = 0$ then by Lemma 2.(ii) and (iii) for all p $x'(p) = 0$ iff $[x, p]$ taking care of the left to right direction. Next assume that $\forall m[x', 2m+1]$ but there is a smallest m such that $x'(2m+1) = 1$. Again, by Lemma 2.(iii) for all $i > 2m+1$, $x'(i) = 1$. Thus $x \geq 2 \cdot 1/2^{2m+1} = 1/2^{2m}$. From $[x', 2m+1]$ there is y such that $\exists^{\mathbb{N}^+} q(y \cdot 2^{2m} = q)$ and $0 \leq x - y < 1/2^{2m+1}$ so by Lemma 2.(iii) we have $y'(j) = x'(j)$ for all $j \leq 2m$. Then $x - y = \Sigma_{q>2m} 1/2^q = 1/2^{2m}$ a contradiction. $\square$

Definition 4. *For a choice sequence ξ let ξ' be the $0-1$ sequence (real number generator) with $\xi(k) = m$ (decidable) iff $\xi'(\langle k, m\rangle) = 1$ (decidable). The code of an atomic formula of the form $\xi(k) = m$ is*

doi.org/10.32388/7IMXXE

the LOR-formula $C(\xi, k, m)$ equivalent to $\forall p[\xi', 2p+1] \wedge [\xi', 2\langle k, m\rangle]$.

A $0-1$ sequence (real number generator) x is a code ($CODE(x)$) iff $\forall m[x, 2m+1]$, so the set of codes is definable in real algebra by a LOR formula.

For a choice sequence α and a $0-1$ sequence (real number generator) u, u is the code of α, denoted as $C(\alpha, u)$ if $CODE(u) \wedge \forall k \forall m(\alpha(k) = m \leftrightarrow C(u, k, m))$. $\square$

From the above definitions and lemmas follows that each choice sequence has a unique code and a code corresponds to a unique choice sequence:

Corollary 5. $\forall \alpha \exists^{R,2}! u C(\alpha, u) \wedge \forall^{R,2} u(CODE(u) \rightarrow \exists! \alpha C(\alpha, u))$ $\square$

Definition 6. *For each choice sequence variable ξ let ξ' be a designated variable ranging over $0-1$ sequences. For each L_1-formula θ we define its LOR translation $\tau(\theta)$ as follows.*

- *If θ is first order atomic, $\tau(\theta) := \theta$*
- *$\tau(\xi_i(t_1) = t_2)$ is the LOR formula equivalent to $C(\xi', t_1, t_2)$*
- *$\tau(\psi_1 \circ \psi_2) := \tau(\psi_1) \circ \tau(\psi_2)$ for $\circ = \wedge, \vee, \rightarrow$*
- *$\tau(\neg \psi_1) := \neg \tau(\psi_1)$*
- *$\tau(\exists x \psi_1)$ is the LOR formula equivalent to $\exists^{\mathbb{N}^+} x(\tau(\psi_1))$*
- *$\tau(\forall x \psi_1)$ is the LOR formula equivalent to $\forall^{\mathbb{N}^+} x(\tau(\psi_1))$*
- *$\tau(\exists \xi_i \psi_1)$ is the LOR formula equivalent to $\exists^{R,2} \xi_i'(CODE(\xi_i') \wedge \tau(\psi_1)(\xi_i'))$*
- *$\tau(\forall \xi_i \psi_1)$ is the LOR formula equivalent to $\forall^{R,2} \xi_i'(CODE(\xi_i') \rightarrow \tau(\psi_1)(\xi_i'))$*

Proposition 7. *To ease the notation we use only one free sequence (natural number) variable. With the notation of Definition 6.*

$$\forall \xi \forall^{R,2} \xi'(C(\xi, \xi') \rightarrow (\theta(\xi) \leftrightarrow \tau(\theta)(\xi')))$$

Proof. By formula induction on the complexity of θ. The interesting cases are the second order quantifier cases. The first order atomic case is immediate, the case for $\tau(\xi_i(t_1) = t_2)$ follows from Corollary 5. and the definitions. The first order quantifier cases can be handled similarly as in [6].

For the universal quantifier case first assume that $\forall \xi \psi_1$ and $CODE(\xi')$. By Corollary 5. there is a unique ξ such that $C(\xi, \xi')$ holds and from $\forall \xi \psi_1$ we also have $\psi_1(\xi)$. Then by the induction hypothesis $\tau(\psi_1)(\xi')$, so $\forall^{R,2} \xi'(CODE(\xi') \rightarrow \tau(\psi_1))$, ie. $\tau(\forall \xi \psi_1)$ holds.

Next assume $\tau(\forall \xi \psi_1)$, ie. $\forall^{R,2} \xi'(CODE(\xi') \rightarrow \tau(\psi_1))$, we have to prove $\forall \xi \psi_1$. By Corollary 5. for any ξ there is a unique ξ' such that $C(\xi, \xi')$ and since from this $CODE(\xi')$ follows, we have $\tau(\psi_1)(\xi')$ by our assumption. We can apply the induction hypothesis to get $\psi_1(\xi)$.

For the existential quantifier case first assume that $\exists\xi\psi_1$. For such ξ, by Corollary 5. there is ξ' with $C(\xi,\xi')$, so $CODE(\xi')$ holds. By the induction hypothesis $\tau(\psi_1)(\xi')$ and we have $\exists^{R,2}\xi'(CODE(\xi')\wedge\tau(\psi_1)(\xi'))$ ie. $\tau(\exists\xi\psi_1)$.

Finally assume $\tau(\exists\xi\psi_1)$, ie. $\exists^{R,2}\xi'(CODE(\xi')\wedge\tau(\psi_1)(\xi'))$. By Corollary 5. for a witness ξ' there is (a unique) ξ such that $C(\xi,\xi')$ holds. By the induction hypothesis $\psi_1(\xi)$ and we are done. □

The next theorem is immediate.

Theorem 8. *For an L_1-sentence θ, $\tau(\theta)$ is an equivalent (in the axiom system T from* [3] *with R-KS added) LOR-sentence recursively obtained from θ.* □

Notes

Mathematics Subject Classification: 03-B20, 03-F25, 03-F35, 03-F60.

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Declarations

Funding: No specific funding was received for this work.

Potential competing interests: No potential competing interests to declare.