Qeios

Inadequacies of Sommerfeld's Front Velocity Definition

Steffen Kühn
steffen.kuehn@aurinovo.de

30 Jul 2024

***Abstract*—Current practice defines the front velocity of a signal as the limit of the phase velocity for infinitely high frequency. However, the present article provides evidence that the propagation velocities of signal fronts for input signals of nonzero temporal duration can result from the phase velocities in the low-frequency range. In conclusion, although the impulse response propagates at the phase velocity for infinitely high frequency, this is not generally true for the step response.**

***Index Terms*—front velocity, phase velocity, signal velocity**

I. Introduction

The front velocity is defined in the scientific literature as the limit of the phase velocity $v_p(\omega \to \infty) = c$ (e.g., [1]–[4]). However, this article mathematically demonstrates by means of a causal counterexample that the current definition of the front velocity is incorrect. In fact, only the speed of the impulse response of a transmission line is limited by c. The speed of the step response, in contrast, can be dominated by the phase velocities in the low-frequency range.

At a first glance, this aspect seems to be a contradiction, because the step response is the convolution of the impulse response with the unit step function. However, this article shows that no paradox exists, and the step response could move faster than c if the transmission line had phase velocities faster than c.

The definition of the front velocity as the limit of the phase velocity originates from a paper published in 1914 by the well-known theoretical physicist Arnold Sommerfeld [5]. His mathematical reasoning is essentially based on the discontinuity in a sine signal when the signal is turned on. This discontinuity consists of arbitrarily high frequencies which propagate with phase velocity $v_p(\omega \to \infty) = c$. On this basis, he concluded that a front, i.e., a sudden change in the signal level, likewise cannot propagate faster than at a speed of c.

These considerations continue to represent the foundation of what is considered the current state of the art [2], [6]. Often this definition is used to shortcut a discussion or to argue that a transmission of information at a speed faster than light in vacuum is principally impossible. This might be the case, but the reason is probably that there are no transmission media that have phase velocities that exceed the speed of light.

This seems to be in contradiction with the term *superluminal phase velocity* which can occasionally be found in the literature [3], [7]–[9]. We note that this refers to phase velocities of partially standing waves, i.e., waves in which a part of the wave is moving in one direction and another part is moving in the opposite direction. Such partially standing waves appear when electromagnetic waves are reflected from surfaces or molecules in the transmission medium. The phase velocities of such partially standing waves are not one-way phase velocities and do not represent true phase velocities.

II. Starting point

To make it evident that Sommerfeld's front velocity definition is inadequate, we assume that a signal $s_i(t)$ with the corresponding Fourier transform

$$\hat{s}_i(\omega) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} s_i(t)\, e^{-j\,\omega\,t} \mathrm{d}t \tag{1}$$

is applied to the input at location $x = 0$ of a transmission line with the transfer function $\hat{h}(\omega)$. We further assume without loss of generality that the input signal is normalized so that the total energy $\mathrm{E}\{s_i(t)\}$ is unity and consequently that equation

$$\mathrm{E}\{s_i(t)\} := \int_{-\infty}^{+\infty} s_i(t)^2\, \mathrm{d}t = 1 \tag{2}$$

is satisfied.

Let the transfer function of the transmission line be

$$\hat{h}(\omega) = e^{-j\,\omega\,x/v_p(\omega)}, \tag{3}$$

with $v_p(\omega)$ being the phase velocity at a given angular frequency ω and x being the location of the measurement point at the transmission line. As can be seen, this transfer function has unity gain for all frequencies and thus represents an all-pass filter. Note that for physically reasonable phase velocities, condition $v_p(\omega) = v_p(-\omega)$ must apply.

The effect of the transmission line on the input signal can be obtained by applying the inverse Fourier transform to the product of the signal spectrum $\hat{s}_i(\omega)$ and the transfer function $\hat{h}(\omega)$, i.e., by calculating

$$s_o(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{s}_i(\omega)\, \hat{h}(\omega)\, e^{j\,\omega\,t} \mathrm{d}\omega. \tag{4}$$

https://doi.org/10.32388/7QNDLJ.2

By substituting the transfer function (3) into (4), we obtain the output signal

$$s_o(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{s}_i(\omega)\, e^{j\omega\left(t - x/v_p(\omega)\right)} \mathrm{d}\omega \tag{5}$$

of the transmission line for the applied input signal $s_i(t)$. As equation (5) clearly shows, the output signal $s_o(t)$ consists simply of running waves superimposed and weighted with the signal spectrum $\hat{s}_i(\omega)$ of the input signal $s_i(t)$. Thus, equation (5) is consistent with intuitive understanding.

III. Energy conservation

For the following it is important that the overall energy of the output signal $s_o(t)$ is preserved at any location $x > 0$. This aspect can be easily shown by using Parseval's theorem, which gives the equation

$$\mathrm{E}\{s_o(t)\} = \int_{-\infty}^{+\infty} |\hat{s}_i(\omega)\, \hat{h}(\omega)|^2\, \mathrm{d}\omega = \int_{-\infty}^{+\infty} |\hat{s}_i(\omega)|^2\, |\hat{h}(\omega)|^2\, \mathrm{d}\omega. \tag{6}$$

Because $|\hat{h}(\omega)| = 1$, we obtain

$$\mathrm{E}\{s_o(t)\} = \int_{-\infty}^{+\infty} |\hat{s}_i(\omega)|^2\, \mathrm{d}\omega = \mathrm{E}\{s_i(t)\} = 1. \tag{7}$$

Consequently, a transmission line with the transfer function (3) does not change the energy of the signal.

IV. Specific phase velocity function

Let us now assume a specific and very simple phase velocity function:

$$v_p(\omega) = \begin{cases} u, & |\omega| \le \omega_u \\ c, & \text{otherwise.} \end{cases} \tag{8}$$

Herein, all phase velocities for $|\omega| \le \omega_u$ are equal to u but beyond that are equal to c.

Substituting the phase velocity function (8) into equation (5) yields the output signal

$$s_o(t) = \left[s_i\left(t - \frac{x}{c}\right) - \alpha\left(t - \frac{x}{c}\right)\right] + \alpha\left(t - \frac{x}{u}\right), \tag{9}$$

with the auxiliary function

$$\alpha(\zeta) := \frac{1}{\sqrt{2\pi}} \int_{-\omega_u}^{+\omega_u} \hat{s}_i(\omega)\, e^{j\omega\zeta}\, \mathrm{d}\omega. \tag{10}$$

As expected, the output signal $s_o(t)$ apparently consists of two components

$$s_c(t) := s_i\left(t - \frac{x}{c}\right) - \alpha\left(t - \frac{x}{c}\right). \tag{11}$$

and

$$s_u(t) := \alpha\left(t - \frac{x}{u}\right) \tag{12}$$

moving at velocities c and u, respectively.

V. Ideal rectangular pulse as input

We now analyze the propagation of an ideal rectangular pulse with variable duration τ:

$$s_i(t) = \frac{1}{\sqrt{\tau}} \left(\Theta(t) - \Theta(t - \tau)\right). \tag{13}$$

Here, Θ is the Heaviside step function. As can be verified, the condition (2) holds for this signal, i.e., the overall energy is 1. Of note, for $t < 0$ the signal is exactly 0, and for $t = 0$, it has an infinitely steep slope. The signal is therefore completely causal and contains infinitely high frequencies. Note that the function (13) can represent both very short impulses ($\tau \to 0$) and a single change of the signal level ($\tau \to \infty$).

The Fourier transform $\hat{s}_i(\omega)$ can be easily calculated:

$$\begin{aligned} \hat{s}_i(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} s_i(t)\, e^{-j\omega t} \mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi\tau}} \int_{0}^{\tau} e^{-j\omega t} \mathrm{d}t \\ &= \frac{j\left(e^{-j\omega\tau} - 1\right)}{\sqrt{2\pi\tau}\,\omega}. \end{aligned} \tag{14}$$

The term $\alpha(\zeta)$, defined by equation (10), can we obtain by substituting equation (14) and computing the integral. We get

$$\alpha(\zeta) = \frac{1}{\pi\sqrt{\tau}} \left(\mathrm{Si}\left(\omega_u \zeta\right) - \mathrm{Si}\left(\omega_u (\zeta - \tau)\right)\right), \tag{15}$$

with

$$\mathrm{Si}(\zeta) := \int_0^{\zeta} \frac{\sin(t)}{t}\, \mathrm{d}t \tag{16}$$

beeing the sine integral. The output signal $s_o(t) = s_c(t) + s_u(t)$ is now determined for all locations x. For the signal component that propagates at the speed u, we obtain

$$s_u(t) = \frac{\mathrm{Si}\left(\omega_u\left(t - \frac{x}{u}\right)\right) - \mathrm{Si}\left(\omega_u\left(t - \tau - \frac{x}{u}\right)\right)}{\pi\sqrt{\tau}}. \tag{17}$$

Figure 1 shows the waveform of the signal power $s_o(t)^2$ for example parameters at different distances x from the input of the transmission line.

We can also calculate the energy $\mathrm{E}\{s_u(t)\}$ that is transported by the signal component $s_u(t)$. Because of Parseval's theorem, the energy is given by

$$\mathrm{E}\{s_u(t)\} = \int_{-\infty}^{+\infty} |\hat{s}_u(\omega)|^2\, \mathrm{d}\omega. \tag{18}$$

As can be seen from equations (10) and (12),

$$\hat{s}_u(\omega) = \left(\Theta(\omega + \omega_u) - \Theta(\omega - \omega_u)\right)\, \hat{s}_i(\omega)\, e^{-j\omega x/u}, \tag{19}$$

i.e.,

$$\mathrm{E}\{s_u(t)\} = \int_{-\omega_u}^{+\omega_u} |\hat{s}_i(\omega)|^2\, \mathrm{d}\omega = \int_{-\omega_u}^{+\omega_u} \hat{s}_i(\omega) \cdot \hat{s}_i(\omega)^*\, \mathrm{d}\omega. \tag{20}$$

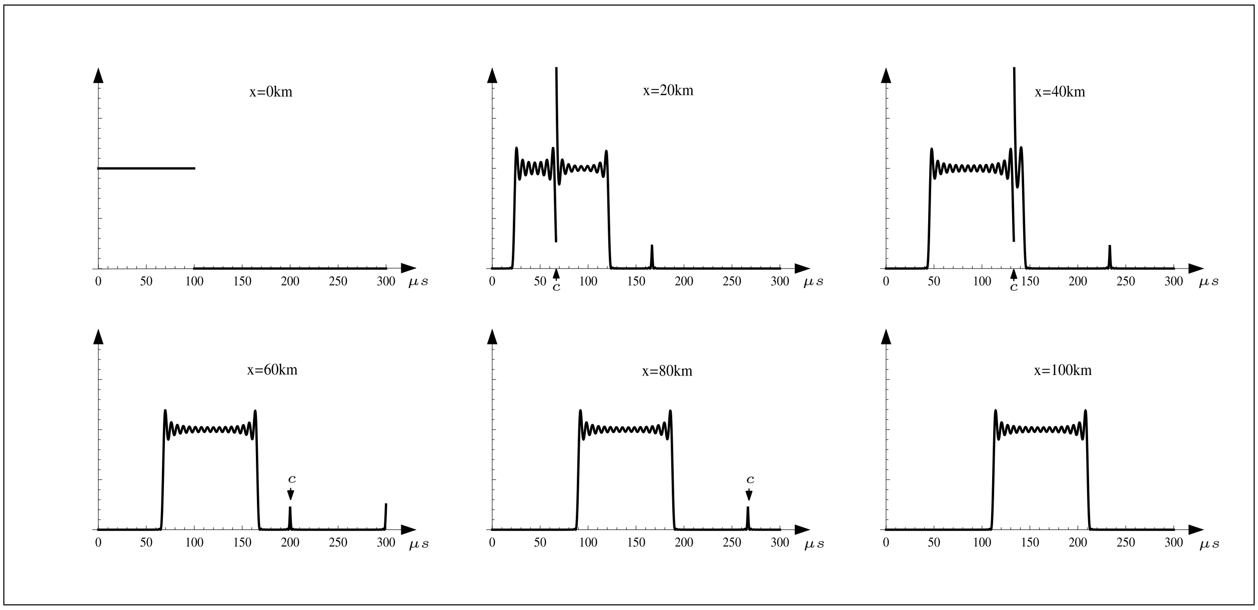

Fig. 1. The figure shows the waveform of the energy $s_o(t)^2$ of the signal at different distances x from the input of the transmission line. In this example, $u = 3\,c$, $\tau = 100\,\mu$s and $\omega_u = 1\,$MHz. As can be seen, the rectangular pulse propagates with velocity $3\,c$. The high-frequency components, in contrast, have only the velocity c and form small glitches, which are increasingly left behind with increasing distance (as an aid, the locations that can be reached by a signal with velocity c are marked by small arrows). The low frequency part of the signal $s_u(t)$ clearly does not have an infinitely steep slope. However, this does not mean that there is no clearly recognizable front.

By substituting equation (14) we get

$$\begin{aligned} \mathrm{E}\{s_u(t)\} &= \int_{-\omega_u}^{+\omega_u} \frac{\left(e^{-j\omega\tau}-1\right)}{\sqrt{2\pi}\,\tau\,\omega}\,\frac{\left(e^{j\omega\tau}-1\right)}{\sqrt{2\pi}\,\tau\,\omega}\,\mathrm{d}\omega \\ &= \int_{-\omega_u}^{+\omega_u} \frac{1-\cos(\omega\,\tau)}{\pi\,\omega^2\,\tau}\,\mathrm{d}\omega. \end{aligned} \tag{21}$$

Evaluating the integral yield

$$\mathrm{E}\{s_u(t)\} = \frac{2\,(\cos(\tau\,\omega_u) + \tau\,\omega_u\,\mathrm{Si}(\tau\,\omega_u) - 1)}{\pi\,\tau\,\omega_u}. \tag{22}$$

As equation (22) reveals, the energy of this signal component depends only on the product $\tau\,\omega_u$. For $\tau \to \infty$, $\mathrm{E}\{s_u(t)\}$ becomes equal to 1. Therefore, in this case, all the energy of the signal moves with velocity u, and the component with velocity c disappears.

For finite τ, i.e., for true rectangular signals, a major part of the energy can be contained in the signal part that propagates with velocity u. For example, for $\tau\,\omega_u = 1000$ the energy $\mathrm{E}\{s_u(t)\}$ is approximately 0.99937. Even for $\tau\,\omega_u = 2\,\pi$, there is still a significant amount of energy in the signal component $s_u(t)$, because in this case $\mathrm{E}\{s_u(t)\} \approx 0.90282$. For $\tau \to 0$, however, $\mathrm{E}\{s_u(t)\}$ becomes 0 and we can conclude that very short impulses can only propagate at the speed c.

VI. Discussion

As has become apparent, the transmission line defined by equation (8) behaves strangely. If we were to attempt to measure the impulse response, we would find that the impulse propagates with velocity c along the transmission line. We would not notice that there is an additional part that propagates with velocity u, because the energy of that part is close to 0.

However, if we were to measure the step response, we would observe that although the front loses steepness because of the low-pass filtering, the front needs only the time x/u and not the time x/c to reach the location x. However, we would not notice that in addition, there is a part that propagates with velocity c, because that part contains practically no energy.

Thus, one might be inclined to state that the impulse response propagates with velocity c, but the step response propagates with velocity u. This conclusion seems to diretly contradict the rule that the step response can be represented as the convolution of the unit step or Heaviside step function with the impulse response. Therefore, several questions arise:

1) Is this effect an artifact of the non-causality of the studied transmission line?
2) How can this paradox be resolved?
3) How should we define the front velocity correctly?

Question 1 can be answered quickly by imagining the total transmission line as a parallel network of two separate transmission lines, as shown in figure 2. In this model, one transmission line transmits the low-frequency part of the input signal, and the other transmits the high-frequency part by placing *causal* low-pass and high-pass filters before each line. Subsequently, the delay behavior is modeled with ideal delay elements with delay constants x/u and x/c. These ideal delay elements are also causal, because they do not produce any dispersion.

As can easily be seen, the complete transmission line is causal. Nevertheless, nothing changes the fact that the limit of the

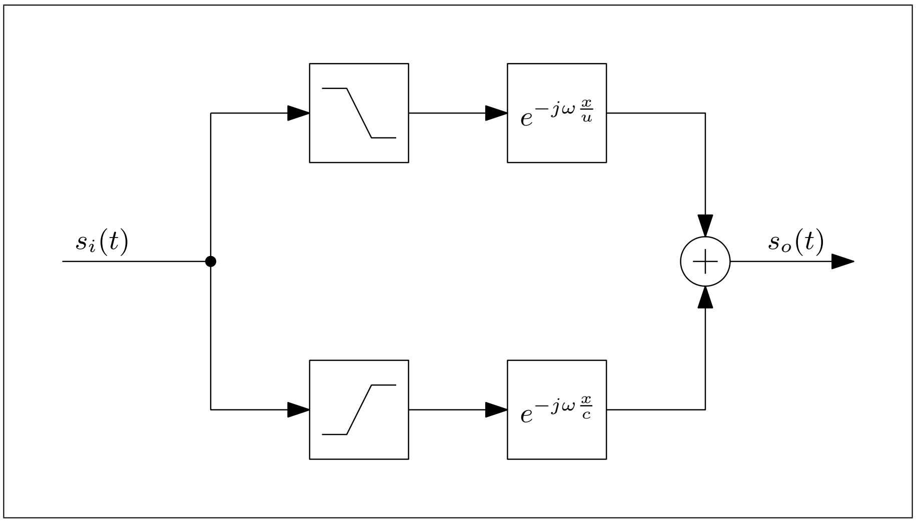

Fig. 2. This transmission line is causal, because all components are causal: the upper channel is a series of a causal low-pass filter and a causal transmission line in which all phase velocities have the same value, u. The lower channel is a series of a causal high-pass filter and a causal transmission line in which all phase velocities have the value c.

phase velocity for $\omega \to \infty$ is c and that the impulse response propagates with velocity c, while the step response moves mainly with velocity u. For causal filters, the existence of a mixed region where both transmission lines are conductive is not important for this argument.

To further illustrate this statement, we consider the two most simple known causal filters, namely resistor-capacitor circuits. The transfer function of the low-pass filter is

$$\hat{h}_L(\omega) = \frac{1}{1 + j\omega RC}. \tag{23}$$

The transfer function of the complementary high-pass filter is

$$\hat{h}_H(\omega) = \frac{j\omega RC}{1 + j\omega RC}. \tag{24}$$

The total transfer function corresponding to figure 2 is then

$$\hat{h}(\omega) = \hat{h}_L(\omega)\, e^{-j\omega \frac{x}{u}} + \hat{h}_H(\omega)\, e^{-j\omega \frac{x}{c}}. \tag{25}$$

It is not difficult to calculate the corresponding step response, that means the output signal

$$s_o(t) = \Theta\left(t - \frac{x}{u}\right)\left(1 - e^{-\frac{1}{RC}\left(t - \frac{x}{u}\right)}\right) + \Theta\left(t - \frac{x}{c}\right) e^{-\frac{1}{RC}\left(t - \frac{x}{c}\right)} \tag{26}$$

of the input signal

$$s_i(t) = \Theta(t). \tag{27}$$

As can be easily seen, $s_o(t)$ is perfectly causal and consists of two parts with two fronts, namely

- one with an infinitely steep slope which moves with velocity c (Fig. 3, dotted line) and
- one which rises only with $1/(RC)$ but propagates with velocity u (Fig. 3, solid line).

But the limit of the phase velocity for $\omega \to \infty$ is obviously equal to c since $\hat{h}_L(\omega \to \infty) = 0$. Nevertheless, there is a signal part that is propagating at velocity u independently of this limit. This, however, demonstrates once more that

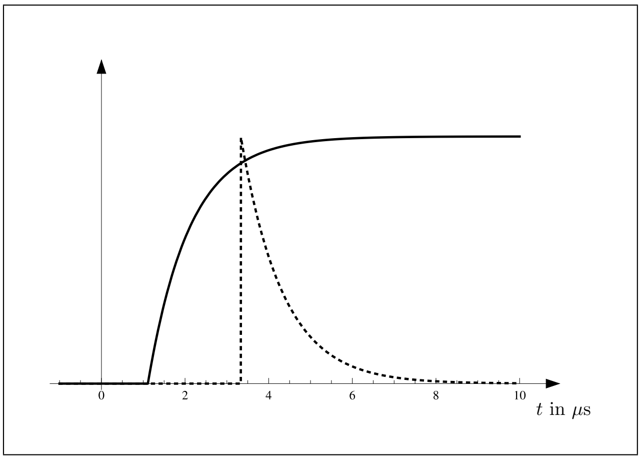

Fig. 3. Parts of signal $s_o(t)$ in equation (26) at $x = 1000$m for $RC = 1\mu$s and $u = 3c$. It is obvious that the signal is causal, but the step response moves faster than the impulse response.

Sommerfeld's front velocity definition is definitely inadequate and incorrect, since the term "front" suggests that this part is in *any case* the fastest. For $u > c$, however, this is not true.

Figure 2 also answers question 2: the apparent paradox can be resolved because in measuring the impulse response, essentially only the properties of the lower channel are effective, whereas in measuring the step response, essentially only the upper channel has an effect. Mathematically, however, the system response for an rectangular pulse of variable duration is given by equation (9), which contains components that move with velocity u and components that propagate with velocity c. In conclusion, measuring only the impulse response or only the step response may be insufficient, because important information may be missing in both.

Finally question 3 remains regarding how to define the front velocity. First of all, it has become obvious that the phase velocity for infinitely high frequency represents *not* the upper limit for the propagation speed of signal fronts. Instead, it is only an impulse response velocity and does not necessarily provide information about the speed at which the step response or information propagates. It seems reasonable to suppose here that the true front velocity is given by the fastest phase velocity. However, there are further arguments to be considered. Therefore, the question 3 needs to be investigated in more detail and cannot be answered conclusively at this point.

VII. Summary

The article has demonstrated that it is impossible to be certain that the phase velocity for infinitely high frequency represents the upper limit for the propagation speed of information in a transmission line. The reasoning that led to this false assumption is essentially based on the justified claim that a real transmission line must always have a causal impulse response. However, Sommerfeld failed to realize that a real transmission line could also be a network. As shown by a simple and causal example, it is possible that in such a network the energy of an impulse moves through a different part of the network than the main part of the energy of a rectangular pulse. For this reason, it is not generally possible to deduce the propagation speed of the step response from the propagation speed of the impulse response.

VIII. Acknowledgments

The author thanks Max Minh Tran, Qeios and the reviewers of Qeios for their support and comments.

References

[1] J. L. Leander, "On the relation between the wavefront speed and the group velocity concept," *J. Acoust. Soc. Am*, vol. 100, no. 6, 1996. [Online]. Available: https://doi.org/10.1121/1.417249

[2] P. W. Milonni, *Fast light, slow light and left-handed light*. CRC Press, 2004.

[3] R. Chiao, "Superluminal phase and group velocities: A tutorial on sommerfeld's phase, group, and front velocities for wave motion in a medium, with applications to the "instantaneous superluminality" of electrons," 2011. [Online]. Available: https://arxiv.org/abs/1111.2402

[4] W. Withayachumnankul, B. M. Fischer, B. Ferguson, B. R. Davis, and D. Abbott, "A systemized view of superluminal wave propagation," *Proceedings of the IEEE*, vol. 98, no. 10, pp. 1775–1786, 2010.

[5] A. Sommerfeld, "Über die Fortpflanzung des Lichtes in dispergierenden Medien," *Annalen der Physik*, vol. 44, no. 4, 1914.

[6] T. Sauter, "Superluminal signals: an engineer's perspective," *Physics Letters A*, 2001. [Online]. Available: https://doi.org/10.1016/S0375-9601(01)00183-9

[7] G. B. Malykin and E. A. Romanets, "Superluminal motion (review)," *Optics and Spectroscopy*, vol. 112, no. 6, pp. 920–934, 2012.

[8] S. Hrabar, I. Krois, I. Bonic, and A. Kiricenko, "Ultra-broadband simultaneous superluminal phase and group velocities in non-foster epsilon-near-zero metamaterial," *Appl. Phys. Lett.*, vol. 102, no. 054108, 2013.

[9] M. S. Kulya, V. A. Semenova, V. G. Bespalov, and N. V. Petrov, "On terahertz pulsed broadband gauss-bessel beam free-space propagation," *Scientific Reports*, vol. 8(1):1390, 2018.