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Vector-Matrix Reversal Operation

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Abstract

An in-depth description of a forgotten matrix operation, the reversal operator, is performed. The properties of such an operation are also given. Ancillary descriptions of matrix regions that are not often used, like the anti-diagonal, are also discussed.

Keywords

Reversal operator. Reverse of a vector. Reverse of a matrix. Reversal Invariance. Reversal matrix.

1. Introduction

The author is aware of a large amount of literature in the form of books related to structure, manipulation, and solving equations involving matrices and vectors (considered here as row or column matrices). Thus, the aim of the present study cannot be an exhaustive bibliographic description. However, a literature résumé will be given in any way to provide a biased point of view, which will be dealt with here.

Perhaps the readers have in mind the basic definitions of Linear Algebra both in teaching volumes [1, 2] and in specialized treatises like the old book of Wilkinson [3], a compendium of the Linear Algebra techniques for computational purposes, with the two-volume practical treatise [4] mainly based on the previous Wilkinson's book. The exhaustive study of Durand [5] is of similar interest, where the most interesting Linear Algebra problems are studied and solved. One can obtain more recent information in [6].

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Surely, the readers will know the words reversal, reverse, and reversing, among other meanings and uses, in certain aspects of time description [7,8,9] they are often encountered. However, as far as the present author knows, such wording is scarcely used enough in mathematics [10], if not at all, as some operation to manipulate matrices in a Linear Algebra context.

This paper describes how a reversal operator acting on matrices and vectors might be defined, by what means one can set the reversal of a matrix, transform matrix elements upon reversal, the structure of reverse matrices, and the relations with other well-known operators and operations acting over and belonging to matrix algebra.

The present paper is structured so that first, one can set the vector reversal and its properties and purpose. After this initial setup, one can discuss the structure of square matrix elements as a step to define reversal in them. At this stage, one will describe the concept of anti-diagonal and the new matrix regions that one can add to the already known ones. After this, one discusses several aspects of matrix reversal, including the definition of a reversal matrix, which reverses a vector via a simple matrix product. This study continues with the action of the reversal operator upon the matrix product, the determinant of a matrix, and the matrix inverse.

2. Vector Reversal

2.1. Definition and symbols of row and column vectors

Let's suppose an *N*-dimensional vector space $V_N(\mathbb{Q})$, where one has chosen the field $\mathbb{Q}$ instead of $\mathbb{R}$, stressing the computational background of vector-matrix operations developed in this study.

Choosing to represent a vector, noted by $\langle \boldsymbol{a}|$, in the form of a row vector, then one can write:

$$\langle \boldsymbol{a}| = (a_1, a_2, a_3, \dots a_N) \in V_N(\mathbb{Q}),$$

one selects vectors this way instead of the equivalent column ket symbols because this row form is easier to write explicitly than the equivalent column dual counterpart. Such consideration becomes easy to accept, taking into account that there is a straightforward relation between both representations involving the dual vector space $V_N^*(\mathbb{Q})$ and the transposition of row into column vectors:

$$\forall \langle \boldsymbol{a}| \in V_N(\mathbb{Q}) \rightarrow \exists |\boldsymbol{a}\rangle = \langle \boldsymbol{a}|^T \in V_N^*(\mathbb{Q}).$$

2.2. Reversal Operator Definition

The *reverse* of any vector belonging to the space $V_N(\mathbb{Q})$ is defined and noted as:

$$\forall\langle\boldsymbol{a}| \in V_N(\mathbb{Q}) \rightarrow \exists\langle\boldsymbol{a}|^R = (a_N, a_{N-1}, a_{N-2}, \ldots, a_1) \in V_N(\mathbb{Q}).$$

Therefore, a superscript R on the right side of the vector symbol represents the reversal operator.

Also, if necessary, one can use the following equivalent notations to denote the reversed vectors under the reversal operator: $\langle\boldsymbol{a}|^R = \langle\boldsymbol{a}^R|$.

2.3. Properties

With this definition above, the reversal of a vector is a linear operation similar to the transposition or the conjugation, as one can easily find:

1) $\forall\{\langle\boldsymbol{a}|, \langle\boldsymbol{b}|\} \subset V_N(\mathbb{Q}): (\langle\boldsymbol{a}| + \langle\boldsymbol{b}|)^R = \langle\boldsymbol{a}|^R + \langle\boldsymbol{b}|^R$
2) $\lambda \in \mathbb{Q} \wedge \langle\boldsymbol{a}| \in V_N(\mathbb{Q}): (\lambda\langle\boldsymbol{a}|)^R = \lambda\langle\boldsymbol{a}|^R$
3) $\forall\langle\boldsymbol{a}| \in V_N(\mathbb{Q}): (\langle\boldsymbol{a}|^R)^R = \langle\boldsymbol{a}|$
4) $\forall\langle\boldsymbol{a}| \in V_N(\mathbb{Q}): (\langle\boldsymbol{a}|^T)^R = (\langle\boldsymbol{a}|^R)^T = |\boldsymbol{a}\rangle^R$

Any vector: $\langle\boldsymbol{i}| \in V_N(\mathbb{Q})$, whenever: $\langle\boldsymbol{i}|^R = \langle\boldsymbol{i}|$, then one can call it reversal invariant. For instance, the unity vector $\langle\mathbf{1}| = (1,1,1,\ldots,1)$ and all its homothecies are reversal invariant in any vector space dimension.

2.4. Reversal and Inward Product of Vectors

Concerning the inward product [11] of two vectors, defined as:

$$\forall\{\langle\boldsymbol{a}|, \langle\boldsymbol{b}|\} \subset V_N(\mathbb{Q}): \langle\boldsymbol{p}| = \langle\boldsymbol{a}| * \langle\boldsymbol{b}| = \{p_I = a_I b_I | I = 1, N\} \in V_N(\mathbb{Q}),$$

then, the reversal operation acts like an operation distributed within the inward product:

$$(\langle\boldsymbol{a}| * \langle\boldsymbol{b}|)^R = \langle\boldsymbol{a}|^R * \langle\boldsymbol{b}|^R.$$

2.5. Reversal and Scalar Product

Thus, the scalar product of two vectors is invariant under reversal:

$$\langle\boldsymbol{a}|\boldsymbol{b}\rangle = \langle\langle\boldsymbol{a}| * \langle\boldsymbol{b}|\rangle = \sum_{I=1}^{N} a_I b_I \wedge$$

$$\langle \boldsymbol{a}|\boldsymbol{b}\rangle^R = \left\langle \langle \boldsymbol{a}| * \langle \boldsymbol{b}| \right\rangle^R = \langle \langle \boldsymbol{a}|^R * \langle \boldsymbol{b}|^R \rangle = \sum_{I=N}^{1} a_I b_I = \sum_{I=1}^{N} a_{N-I+1} b_{N-I+1} \Rightarrow$$

$$\langle \boldsymbol{a}|\boldsymbol{b}\rangle = \langle \boldsymbol{a}|\boldsymbol{b}\rangle^R$$

2.6. Reversal and Euclidean Norms

Therefore, the Euclidean norm of a vector is invariant upon reversal because:

$$\langle \boldsymbol{a}|\boldsymbol{a}\rangle = \sum_{I=1}^{N} a_I^2 = \sum_{I=N}^{1} a_I^2 = \sum_{I=1}^{N} a_{N-I+1}^2 = \langle \boldsymbol{a}^R|\boldsymbol{a}^R\rangle = \langle \boldsymbol{a}|\boldsymbol{a}\rangle^R.$$

2.7. Half-reversal Euclidean Norms

However, there is the possibility to define the *half-reversal* Euclidean norm of a vector, like:

$$\langle \boldsymbol{a}|\boldsymbol{a}^R\rangle = \langle \boldsymbol{a}^R|\boldsymbol{a}\rangle = \sum_{I=1}^{N} a_I a_{(N-I+1)} = \left\langle \langle \boldsymbol{a}| * \langle \boldsymbol{a}^R| \right\rangle = \left\langle \langle \boldsymbol{a}^R| * \langle \boldsymbol{a}| \right\rangle.$$

An example illustrates this interesting property:

$$\langle \boldsymbol{a}| = (1 \quad 2 \quad 3) \Rightarrow \langle \boldsymbol{a}|\boldsymbol{a}\rangle = 14$$

$$\langle \boldsymbol{a}^R| = (1 \quad 2 \quad 3)^R = (3 \quad 2 \quad 1) \Rightarrow \langle \boldsymbol{a}^R|\boldsymbol{a}\rangle = \left\langle \langle \boldsymbol{a}^R| * |\boldsymbol{a}\rangle \right\rangle = 10.$$

2.8. Invariance of Higher Order Norms

One might write higher-order norms as complete sums of inward power vectors.

Using the following definition of the inward power of any vector:

$$\forall \langle \boldsymbol{a}| \in V_N(\mathbb{Q}) \wedge \forall P \in \mathbb{Q}: \langle \boldsymbol{a}|^{[P]} = (a_1^P, a_2^P, a_3^P, \dots a_N^P),$$

when the power is attached to a natural number $\forall P \in \mathbb{N}$, then the P-th order norm of the vector can be defined as:

$$\boldsymbol{N_P}[\langle \boldsymbol{a}|] = \sum_{I=1}^{N} |a_I^P| = \left\langle |\langle \boldsymbol{a}|^{[P]}| \right\rangle.$$

Such a definition is invariant by vector reversal, as one can write:

$$\boldsymbol{N_P}[\langle \boldsymbol{a}^R|] = \sum_{I=1}^{N} |a_{N-I+1}^P| = \left\langle \left(\left| \langle \boldsymbol{a}^R|^{[P]} \right| \right) \right\rangle = \boldsymbol{N_P}[\langle \boldsymbol{a}|].$$

3. Restructuration of Square Matrices: Definition of the A-diagonal Elements.

One of the most used structures of square $(N \times N)$ matrices is defining the diagonal and the subdiagonal elements parallel to it. This possibility gives rise to particular matrix types, such as triangular, tridiagonal, and band matrices.

As it is well-known, the diagonal of a matrix corresponds to elements starting at the position $(1,1)$ and ending at the position (N,N); that is, the index set, which one can describe as: $\{(I,I)|I=1,N\}$; or it is of interest to define the diagonal element set as:

$$Diag(\boldsymbol{A}) = (a_{11}; a_{22}; \ldots; a_{NN}) = (a_{II}|I = 1, N).$$

Among other possibilities not often employed, alternative structures exist within the elements of square $(N \times N)$ matrices. One can define the *anti-diagonal* (or shortly: *a-diagonal*), corresponding to the elements starting at the position $(1, N)$ and ending at the position $(N, 1)$; that is, in this case, the set of elements with indices: $\{(I, N-I+1)|I=1,N\}$, or using a similar definition to the diagonal:

$$A - Diag(\boldsymbol{A}) = \left(a_{1N}; a_{2(N-1)}; \ldots; a_{N1}\right) = \left(a_{I(N-I+1)}|I = 1, N\right).$$

Following the programming rules of some high-level languages like Python, one can construct a better description of a matrix's a-diagonal with the index range $\{0,N\}$. In doing so, one can set the matrix dimension to $\left((N+1)\times(N+1)\right)$. However, using this indexing possibility, one can write the a-diagonal set of indices like: $\{(I,N-I)|I=0,N\}$.

Such procedures permit the identification of new matrix regions as anti-diagonal and sub-anti-diagonals and also identify matrix anti-triangles, which will be in the upper and lower regions seen concerning the anti-diagonal. One can also augment matrix classification because of the presence of these matrix regions. Therefore, one can talk about anti-diagonal matrices, anti-triangular matrices, etc.

4. The Vector Indices of an A-diagonal of a Square Matrix

To characterize the role of the a-diagonal of a matrix even better, one can transform the a-diagonal row indices into a vector. The a-diagonal column indices are contained in the reversal on the row indices vector. That is:

$$\{(I, N-I)|I = 0, N\} \equiv \{\langle \boldsymbol{r}|, \langle \boldsymbol{r}|^R\} \rightarrow$$
$$\langle \boldsymbol{r}| = (0,1,2,\ldots N) \equiv \{I|I = 0, N\}$$
$$\langle \boldsymbol{r}|^R = (N, N-1, N-2, \ldots 1,0) \equiv \{N - I|I = 0, N\}$$

5. Tensor Sum of Two Vectors Indices

An interesting numerical subproduct of this set of vector indices $\{\langle \boldsymbol{r}|, \langle \boldsymbol{r}|^{\boldsymbol{R}}\}$, which one can use to discuss the Goldbach conjecture [12,13] and the Fermat theorem [14], is the structure of what can be called the tensor sum of them.

One can easily define the tensor sum of two indices as:

$$\boldsymbol{S}_N = \langle \boldsymbol{r}| \oplus \langle \boldsymbol{r}|^R \Rightarrow \forall I, J:$$
$$S_{IJ} = I + J \equiv r_I + r_J^R = r_I + r_{(J-I)} = I + (J - I) = J,$$

whenever the range of both subindexes starts at 0. As a result of this construct, the a-diagonal and sub-a-diagonals of an index tensor sum bear the column index used as a unique element.

Therefore, one can say that the sub-a-diagonals of the tensor sum matrices $\boldsymbol{S}_N$ of arbitrary dimension contain the natural number of the starting column. The a-diagonal of $\boldsymbol{S}_N$ contains the associated dimension number N.

For example, choosing $N = 4$, then the matrix $\boldsymbol{S}_4$ will have the form:

$$\boldsymbol{S}_4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{pmatrix}.$$

6. Reversal Matrix

One can define the *reversal matrix* $\boldsymbol{R}$ as a matrix similar to a diagonal matrix in the form of an a-diagonal one. It has been previously defined as the exchange matrix [15], but the naming using the reversal adjective is better within the present paper's ideas. In this case, the matrix $\boldsymbol{R}$ is a null matrix with a unit principal a-diagonal. The elements in the a-diagonal consist of matrix elements perpendicular to the diagonal and made of 1s. One can define this structure as:

$$\boldsymbol{R}_N = A - Diag(\boldsymbol{R}_N) = \left\{ r_{N;IJ} = \delta_{I(N-I+1)} | I = 1, N \right\} \rightarrow$$
$$\boldsymbol{R}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The reversal matrix can be used to reverse arbitrary vectors of the adequate dimension. Concerning the row vectors, the transformation acts on the right side of the vector. One can write:

$\forall\langle\boldsymbol{a}| \in V_N(\mathbb{Q}): \langle\boldsymbol{a}|^R = \langle\boldsymbol{a}|\boldsymbol{R}_N.$

Also, the matrix $\boldsymbol{R}_N$ is involutory, or self-inverse, thus:

$\boldsymbol{R}_N^2 = \boldsymbol{I}_N.$

7. Reversal of a Matrix

Once one knows that matrix elements can be reordered into an isomorphic row (or column) vector, one can consider any matrix reversal procedure a trivial algorithm. Then, one can reduce matrix reversal to the algorithm of a vector reversal. But it is interesting to discuss matrix reversal as an internal matrix operation on the same footing as conjugation, transposition, and inversion.

The action of the reversal operator on an arbitrary matrix can be defined through an algorithm as follows: first, the reversal operator acts on the set of matrix rows or columns, reversing it; second, it reverses every resultant row or column.

For $(M \times N)$ matrices, one can write using a column decomposition:

$$\boldsymbol{A} = \{a_{IJ}|I = 1, M; J = 1, N\} \equiv \left(|\boldsymbol{a}_J\rangle | J = 1, N\right) \Rightarrow$$
$$\boldsymbol{A}^R = \left(|\boldsymbol{a}_J\rangle | J = 1, N\right)^R = \left(|\boldsymbol{a}_J\rangle^R | J = N, 1\right)$$

One can define the same algorithm for a matrix representation in the row form.

For higher dimensionalities, like in the case of hypermatrices or high-order tensors, whose elements can be supposed to contain matrices in turn, the reversal operator acts reversing the order of submatrices first, then reversing the order of the lower representation submatrices until the above action on matrices and vectors is found.

A simple example can be easily given:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \left(\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \right) \Rightarrow$$

$$\boldsymbol{A}^R = \left(\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}^R \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}^R \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}^R \right) = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}$$

If a row decomposition is chosen for the matrix above, then one can write:

$$\boldsymbol{A} = \begin{pmatrix} (\boldsymbol{1} & \boldsymbol{2} & \boldsymbol{3}) \\ (\boldsymbol{4} & \boldsymbol{5} & \boldsymbol{6}) \\ (\boldsymbol{7} & \boldsymbol{8} & \boldsymbol{9}) \end{pmatrix} \Rightarrow \boldsymbol{A}^{\boldsymbol{R}} = \begin{pmatrix} (\boldsymbol{7} & \boldsymbol{8} & \boldsymbol{9})^{\boldsymbol{R}} \\ (\boldsymbol{4} & \boldsymbol{5} & \boldsymbol{6})^{\boldsymbol{R}} \\ (\boldsymbol{1} & \boldsymbol{2} & \boldsymbol{3})^{\boldsymbol{R}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{9} & \boldsymbol{8} & \boldsymbol{7} \\ \boldsymbol{6} & \boldsymbol{5} & \boldsymbol{4} \\ \boldsymbol{3} & \boldsymbol{2} & \boldsymbol{1} \end{pmatrix}$$

Therefore, the reversal of a matrix leaves the matrix dimension invariant.

8. Reversal of a Matrix Product

The reversal in the sum and product by a scalar or the inward product of matrices behaves like the already discussed formalism in the vectors because of the isomorphic reasons already given. However, reversal over matrix multiplication shall be studied in detail.

One can say that:

$$\boldsymbol{P} = \boldsymbol{AB} \Rightarrow \boldsymbol{P}^R = (\boldsymbol{AB})^R \equiv \boldsymbol{A}^R\boldsymbol{B}^R$$

A simple example provides initial information:

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix} \wedge \boldsymbol{B} = \begin{pmatrix} \boldsymbol{1} & \boldsymbol{2} \\ \boldsymbol{3} & \boldsymbol{4} \end{pmatrix} \to \boldsymbol{AB} = \begin{pmatrix} \boldsymbol{a} + \boldsymbol{3b} & \boldsymbol{2a} + \boldsymbol{4b} \\ \boldsymbol{c} + \boldsymbol{3d} & \boldsymbol{2c} + \boldsymbol{4d} \end{pmatrix} \to$$

$$\boldsymbol{A}^R = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \wedge \boldsymbol{B}^R = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \wedge (\boldsymbol{AB})^R = \begin{pmatrix} 2c + 4d & c + 3d \\ 2a + 4b & a + 3b \end{pmatrix}$$

$$\boldsymbol{A}^{\boldsymbol{R}}\boldsymbol{B}^{\boldsymbol{R}} = \begin{pmatrix} \boldsymbol{4d} + \boldsymbol{2c} & \boldsymbol{3d} + \boldsymbol{c} \\ \boldsymbol{4b} + \boldsymbol{2a} & \boldsymbol{3b} + \boldsymbol{a} \end{pmatrix} \Rightarrow (\boldsymbol{AB})^{\boldsymbol{R}}$$

Thus, the matrix reversal seems to be distributive concerning the matrix product.

However, this characteristic shall be generally demonstrated. To do this, one can write:

$$[\boldsymbol{AB}]_{IJ} = \sum_{K=1}^{N} a_{IK} b_{KJ} \to$$

$$[\boldsymbol{AB}]_{IJ}^R = [\boldsymbol{AB}]_{(N-I+1)(N-J+1)} = \sum_{K=1}^{N} a_{(N-I+1)(K)}\, b_{(K)(N-J+1)} =$$

$$\sum_{K=1}^{N} a_{(N-I+1)(N-K+1)}\, b_{(N-K+1)(N-J+1)} = [\boldsymbol{A}^R \boldsymbol{B}^R]_{IJ}$$

A less entangled demonstration can be obtained, realizing that in a matrix product, between two compatible matrices, each product element is just the scalar product of a row of the left side matrix by the right side column.

That is, one can write:

$$[\boldsymbol{AB}]_{IJ} = \langle \boldsymbol{a}_I | \boldsymbol{b}_J \rangle \rightarrow$$
$$[\boldsymbol{A}^R \boldsymbol{B}^R]_{IJ} = \left\langle \langle \boldsymbol{a}_I |^R * |\boldsymbol{b}_J \rangle^R \right\rangle = \left\langle (\langle \boldsymbol{a}_I | * |\boldsymbol{b}_J \rangle)^R \right\rangle = [(\boldsymbol{AB})^R]_{IJ}$$

Thus, a similar demonstration to the previous one is obtained. One can deduce that the reversal of a matrix product corresponds to the product of the reversed matrices present in the product.

9. Determinant of a Reversed Matrix

It is easy to deduce that the determinant of a square matrix is invariant concerning the reversal operation as defined before. That is:

$$Det|\boldsymbol{A}| = Det|\boldsymbol{A}^R|$$

The reason for this invariance is easy to understand, as a $(N \times N)$ square matrix reversal corresponds to an even number of row-column interchanges precisely $2N$ of them. Determinants change the sign for every interchange of columns or rows; thus, an even number of interchanges leaves the determinant invariant.

10. Reversal of the Inverse of a Matrix

Non-singular $(N \times N)$ matrices: $\boldsymbol{A} = \{a_{IJ} | I, J = 1, N\}$, possess a non-null determinant, that is: $Det|\boldsymbol{A}| \neq 0$, and in this case, an inverse exists: $\boldsymbol{A}^{-1} = \left\{a_{IJ}^{(-1)} | I, J = 1, N\right\}$, concerning the matrix product. Moreover, one can write:

$\boldsymbol{AA}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}_N$,

being: $\boldsymbol{I}_N = \{\delta_{IJ} | I, J = 1, N\}$ the unit matrix of dimension $(N \times N)$.

One can describe the reversal of the inverse of a matrix using the reversal behavior on matrix multiplication. Taking into account the invariance of the unit matrix upon reversal: $\boldsymbol{I}_N^R = \boldsymbol{I}_N$, one can write:

$$(\boldsymbol{A}\boldsymbol{A}^{-1})^R = (\boldsymbol{A}^{-1}\boldsymbol{A})^R = \boldsymbol{I}_N \Rightarrow \boldsymbol{A}^R\boldsymbol{A}^{-R} = \boldsymbol{A}^{-R}\boldsymbol{A}^R = \boldsymbol{I}_N,$$

This result implies that the inverse of the reverse of a matrix is the reverse of the inverse.

11. Discussion

The reversal of vectors and matrices of arbitrary dimension has been studied. As a result, a new operator can be adopted, acting similarly to the conjugation operator but reordering the final elements of the involved matrix. Thanks to such a definition of a new internal operator, matrix elements, which usually are not mentioned, become relevant. The anti-diagonal and the sub-anti-diagonals become relevant in this manner, adding more information to the study of matrix structure at the same footing as the role played by the diagonal and subdiagonals. Application of some aspects of this new point of view to number theory is underway.

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References

1. F. Ayres; "Matrices" (Spanish Version) Mc Graw Hill (México) (1970).
2. R. Carbó, LL. Domingo; "Algebra Matricial y Lineal" Schaum-Mc Graw Hill (Madrid) (1987).

3. J. H. Wilkinson; “The Algebraic Eigenvalue Problem” Oxford University Press (Oxford) (1965).
4. J. H. Wilkinson, C. Reinsch; “Handbook for Automatic Computation (II) Linear Algebra” Springer Verlag (Berlin) (1971).
5. E. Durand; “Solutions Numériques des Équations Algébriques” Masson et Cie., Editeurs (Paris) (1960).
6. L. Hogben (editor); “Handbook of Linear Algebra” Chapman & Hall/CRC (Boca Raton, FA) (2007).
7. https://en.wikipedia.org/wiki/Time_reversibility
8. https://en.wikipedia.org/wiki/T-symmetry
9. J. Chang, R. Carbó-Dorca; “Fuzzy Hypercubes and their Time-Like Evolution” J. Math. Chem. **58** (2020) 1337-1344.
10. T. Gowers (editor); “The Princeton Companion to Mathematics” Princeton University Press (Princeton) (2008).
11. R. Carbó-Dorca; “Natural Vector Spaces, (Inward Power and Minkowski Norm of a Natural Vector, Natural Boolean Hypercubes) and Fermat’s Last Theorem” J. Math. Chem. 55 (2017) 914-940.
12. R. Carbó-Dorca; “*N*-dimensional Boolean Hypercubes and the Goldbach Conjecture” J. Math. Chem. **54** (2016) 1213-1220.
13. R. Carbó-Dorca; “A study on Goldbach Conjecture” J. Math. Chem. **54** (2016) 1798-1809.
14. R. Carbó-Dorca; “Whole Perfect Vectors and Fermat’s Last Theorem” J. Appl. Math. Phys. **12** (2024) 34-42.
15. https://en.wikipedia.org/wiki/Exchange_matrix.