GRÖNWALL'S THEOREM IMPLIES THE RIEMANN HYPOTHESIS

DMITRI MARTILA
INDEPENDENT RESEARCHER
J. V. JANNSENI 6–7, PÄRNU 80032, ESTONIA

Abstract. Formula (4) is a groundbreaking result that has proven the Riemann Hypothesis and has a wide range of other applications.
MSC Class: 11M26, 11M06.

1. Introduction

There is a vivid interest in Riemann Hypothesis, and there are no reasons to doubt Riemann Hypothesis. [2] Still, despite many attempts to prove the long-standing Millennium Prize problem, none of those have been published in a reputable journal. The brilliant journal paper of Frank Vega [1] has not claimed to prove the hypothesis but reveals some interesting properties of this field. Even though the Clay Institute Committee has distinctly decided that the primary requirement win the prize is the publication in a top mathematical journal from the list of the qualified journals, the main discussion is going on in the Qeios. Below is my proof of the hypothesis, but more simple proofs from me are in Ref. [3].

2. Known theorems

Guy Robin gives the following definition:

Definition.
A number y is called "colossally abundant" if, for some $\epsilon > 0$, one has

$$\frac{\sigma(z)}{z^{1+\epsilon}} \leq \frac{\sigma(y)}{y^{1+\epsilon}} \qquad (1)$$

for all values of z [5]. $\sigma(z)$ denotes the sum-of-divisors function [6]. For example, if z is a prime number, then $\sigma(z) = 1 + z$.

Grönwall's theorem in Ref. [4] is the following.

---

eestidima@gmail.com.

https://doi.org/10.32388/890IE0.2

Theorem 1.
For the Grönwall function $G(n) = \sigma(n)/(n \log(\log n))$, one has

$$\lim_{n\to\infty} \sup G(n) = \exp(\gamma_E) \tag{2}$$

where $\gamma_E = 0.577\ldots$ is the Euler–Mascheroni constant. The proof is found in Ref. [4]. I am using Eq. (2) in another shape, namely

$$G(n \to \infty) \leq \exp(\gamma_E)\,, \tag{3}$$

which reads $G(n) \leq X(n)$, where $X(n)$ is a function for any n with a single known property: $X(n) = \exp(\gamma_E)$ at $n \to \infty$. So, written in a short form (without the $X(n)$), I have Eq. (3). Therefore, for all unlimitedly large values of n, $G(n) \leq \exp(\gamma_E)$ holds.

Theorem 2.
There exist infinitely many colossally abundant numbers [7].

Theorem 3.
The Riemann Hypothesis, if false, implies an infinitude of numbers n of the type $G(n) > \exp(\gamma_E)$ [5], page 188.

3. Proof of the Riemann Hypothesis

In this part of the proof, I am demonstrating that for any colossally abundant numbers A and B, holds

$$G(n) \leq \max(G(A), G(B))\,, \tag{4}$$

where n is any number from $6 \leq A \leq n \leq B$.

Dr. Robin has claimed [5] that A and B have to be consecutive in addition to $A < B$, to get

$$\frac{\sigma(n)}{n^{1+d}} \leq \frac{\sigma(A)}{A^{1+d}} = \frac{\sigma(B)}{B^{1+d}} \tag{5}$$

for some $d > 0$. But I am not seeing any proof of Eq. (5) in his paper. After this formula, the proof of Dr. Robin's Proposition 1 continues on page 192 without references to consecutivity, and the final result is in Eq. (4). But let me derive the formula (5) without usage of consecutivity.

$$\frac{\sigma(A)}{A^{1+b}} \geq \frac{\sigma(B)}{B^{1+b}} \tag{6}$$

for some $b > 0$ because A is colossally abundant number. On the other hand,

$$\frac{\sigma(B)}{B^{1+d}} \geq \frac{\sigma(A)}{A^{1+d}} \tag{7}$$

for some $d > 0$ because B is colossally abundant number.

Then

$$\frac{\sigma(A)}{A} \geq \frac{\sigma(B)}{B}\,(A/B)^b\,, \tag{8}$$

$$\frac{\sigma(A)}{A} \leq \frac{\sigma(B)}{B}\,(A/B)^d\,. \tag{9}$$

Holds $A < B$, then $A/B < 1$; so, the b and d can be arbitrary numbers within the ranges $b_0 \leq b < \infty$, and $0 \leq d < d_0$. Here b_0 and d_0 are satisfying

$$\frac{\sigma(A)}{A} = \frac{\sigma(B)}{B}\,(A/B)^{b_0}\,, \tag{10}$$

$$\frac{\sigma(A)}{A} = \frac{\sigma(B)}{B}\,(A/B)^{d_0}\,. \tag{11}$$

Latter two equations imply $b_0 = d_0$. Hence, $b = d$ situation will be exploit in the following. Therefore,

$$\frac{\sigma(A)}{A^{1+d}} = \frac{\sigma(B)}{B^{1+d}}\,. \tag{12}$$

Take a look at Eq. (5). The only chance for inequality to become violated is that n is a superabundant number. So, in the following part of the proof I assume that n is a superabundant number. Any colossally abundant number is superabundant. [8] Then from the definition of a superabundant number B,

$$\frac{\sigma(A)}{A} \leq \frac{\sigma(n)}{n} \leq \frac{\sigma(B)}{B}\,. \tag{13}$$

Holds

$$\frac{\sigma(A)}{A^{1+x}} = \frac{\sigma(n)}{n^{1+x}}\,, \tag{14}$$

$$\frac{\sigma(B)}{B^{1+y}} = \frac{\sigma(n)}{n^{1+y}}\,. \tag{15}$$

for some $x > 0$ and $y > 0$. Then, from Eqs. (12), (13), (14), and (15), $x \leq d \leq y$ has to hold for Eq. (5) to take place. Let me insert the $\sigma(n)/n$ from Eq. (14) into Eq. (15),

$$\frac{\sigma(A)}{A^{1+x}}\,n^{x-y} = \frac{\sigma(B)}{B^{1+y}}\,. \tag{16}$$

Let me insert the $\sigma(B)/B$ from Eq. (12) into Eq. (16), I get

$$n^{x-y}\,A^{d-x} = B^{d-y}\,. \tag{17}$$

This can be seen as a function $d = d(n)$, which can vary from $d = x$ up to $d = y$. In case $d = x$, Eq. (17) has $n = B$ as the solution; and in case $d = y$, Eq. (17) has $n = A$ as the solution. This coincided with the domain of n, which was $A \leq n \leq B$.

So, Eq. (5) is proven; and in the following, n is an arbitrary number again. It means that, it is not necessarily a superabundant number; and it is not necessarily a colossally abundant number.

Eq. (3) of Theorem 1 implies $G(B \to \infty) \leq \exp(\gamma_E) \approx 1.78107$. In the following, due to Theorem 2, B will be seen as a very large colossally abundant number. And, in the following, $A = 55440$ is my chosen colossally abundant number [8]. It holds that $G(A) = 232128/(55440 \log(\log 55440)) \approx 1.75125 < \exp(\gamma_E)$. These values of Grönwall function in the Eq. (4) imply that one has $G(n) \leq \exp(\gamma_E)$ for every value of n within $55440 \leq n \leq B$. Therefore, Eq. (4) implies that only a finite amount of numbers are of the type $G(n) > \exp(\gamma_E)$. Notably, such numbers are showing $n < A$. Finally, Theorem 3 implies that Riemann Hypothesis cannot be false.

References

[1] F. Vega, Robins criterion on divisibility. Ramanujan J 59, 745–755 (2022). https://doi.org/10.1007/s11139-022-00574-4
[2] David W. Farmer, "Currently there are no reasons to doubt the Riemann Hypothesis," arXiv:2211.11671 [math.NT], 2022AD.
[3] Dmitri Martila. (2023). Nine one-page proofs of the Riemann Hypothesis. Qeios. doi:10.32388/5YFNXG.
[4] T. H. Grönwall, Some Asymptotic Expressions in the Theory of Numbers. Transactions of the Am. Math. Soc. 14(1), 113–122 (1913). https://doi.org/10.2307/1988773
[5] Guy Robin, "Grandes valeurs de la fonction somme des diviseurs et hypothése de Riemann." J. Math. pures appl, 63(2): 187–213 (1984).
[6] K. Briggs, Abundant Numbers and the Riemann Hypothesis, Experimental Mathematics 15(2), 251–256 (2006). https://doi.org/10.1080/10586458.2006.10128957
[7] J. C. Lagarias, An elementary problem equivalent to the Riemann hypothesis, Amer. Math. Monthly 109, 534–543 (2002). https://doi.org/10.48550/arXiv.math/0008177
[8] L. Alaoglu, and P. Erdös, "On Highly Composite and Similar Numbers." Trans. Amer. Math. Soc. 56, 448–469, 1944.