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GRÖNWALL'S THEOREM IMPLIES THE RIEMANN HYPOTHESIS

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Abstract. A simple argument proves the Riemann hypothesis.
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1. Introduction

Despite many attempts to prove the long-standing Millennium Prize problem, none of those is published in a reputable journal. The brilliant journal paper of Frank Vega [1] has not claimed to prove the hypothesis, but reveals some interesting properties of this field. Even though the Clay Institute Committee has distinctly decided that the main requirement win the prize is the publication in a top mathematical journal from the (published) narrow list of the qualified journals, the main discussion is going on in the archive. For instance David W. Farmer has reviewed the state of the problem in Ref. [2]. Therefore, following the example of Grigori Perelman [3], my first attempt was a publication in the archive.

In the second section I collect a theorem by Thomas Hakon Grönwall [4], a definition given by Guy Robin in Ref. [5], and three theorems from the same reference. These pieces are used in the proof performed in the third section.

2. Known facts

I start with the oldest piece, namely Grönwall's theorem in Ref. [4].

2.1. **Theorem.**
For the Grönwall function $G(n) = \sigma(n)/(n \log(\log n))$, one has

$$\limsup G(n \to \infty) = \exp(\gamma_E)\,, \tag{1}$$

where $\gamma_E = 0.577\ldots$ is the Euler–Mascheroni constant. $\sigma(n)$ denotes the sum-of-divisors function [6]. For example, if n is a prime number,

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then $\sigma(n) = 1 + n$. The proof is found in Ref. [4]. I am using Eq. (1) in another shape, namely

$$G(n \to \infty) \leq \exp(\gamma_E)\,, \tag{2}$$

which reads $G(n) \leq X(n)$, where $X(n)$ is a function for any n with a single known property: $X(n) = \exp(\gamma_E)$ at $n \to \infty$. So, written in a short form (without the $X(n)$), I have Eq. (2).

Guy Robin gives the following definition:

2.2. **Definition.**
A number y is called "colossally abundant" if, for some $\epsilon > 0$, one has

$$\frac{\sigma(z)}{z^{1+\epsilon}} \leq \frac{\sigma(y)}{y^{1+\epsilon}} \tag{3}$$

for all values of z [5].

2.3. **Theorem.**
There exist infinitely many colossally abundant numbers [5].

2.4. **Theorem.**
If A and B are colossally abundant numbers with $6 \leq A \leq B$, then

$$G(n) \leq \max(G(A), G(B))\,, \tag{4}$$

where $A \leq n \leq B$ [5].

2.5. **Theorem.**
The Riemann Hypothesis, if false, implies an infinitude of colossally abundant numbers K of the type $G(K) > \exp(\gamma_E)$ [5].

3. Proof of the Riemann Hypothesis

Eq. (2) of Theorem 2.1 implies $G(B \to \infty) \leq \exp(\gamma_E) \approx 1.78107$. In the following, due to Theorem 2.3, B will be seen as a very large colossally abundant number. It holds that

$$G(A = 55440) = 232128/(55440 \log(\log 55440)) \approx 1.75125 < \exp(\gamma_E).$$

Theorem 2.4 implies that (for my choice of A and B) one has $G(n) \leq \exp(\gamma_E)$ for every value of n within $55440 \leq n \leq B$. Therefore, Theorem 2.4 implies that only a finite amount of colossally abundant numbers are of the type $G(K) > \exp(\gamma_E)$. Notably, such numbers are showing $K < A$. Finally, Theorem 2.5 implies that Riemann Hypothesis cannot be false.

References

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