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Qualitative Analysis of a Time-Delay Transmission Model for COVID-19 Based on Susceptible Populations With Basic Medical History

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Abstract: Based on the SEIR COVID-19 epidemic model of susceptible people with basic medical histories, this paper introduces time delay, establishes a class of COVID-19 time-delay transmission model, obtains the basic reproduction number of its transmission, and determines the existence of the equilibrium point of the model. The global stability of the equilibrium point is proved by constructing the Lyapunov function and using the LaSalle invariance principle. The theoretical results are verified by numerical simulation, and the impact of different time delays on the spread of COVID-19 is discussed.

Key words: COVID-19; Stability; Time delay; Numerical simulation

1. Introduction

In early December 2019, there was a rapid global spread of Corona Virus Disease 2019 (COVID-19), a highly contagious respiratory infection caused by Severe Acute Respiratory Syndrome Coronavirus 2 (SARS-CoV-2). Typical symptoms include fever, coughing, and shortness of breath, and in severe cases, the infection can lead to pneumonia, acute respiratory syndrome, renal failure, and death[1-2]. The outbreak of COVID-19 has had a huge impact on countries around the world, with countless lives and property losses [3].

Mathematical models and computer simulations, despite their limitations and shortcomings, remain one of the best methods for analyzing the spread of diseases and controlling their epidemics. Modeling is very important in epidemiology because, in most cases, we cannot conduct biological experiments, nor do we have pharmaceutical solutions, and mathematical modeling must be used to develop and understand epidemiological phenomena in relevant ways, as well as to quantitatively simulate the likely effects of different intervention strategies [4]. Therefore, during the disease transmission phase, It is of great significance to use appropriate mathematical models to simulate, analyze, and predict the epidemic situation of COVID-19 and put forward prevention and control suggestions.

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Since 2020, modeling and analysis of COVID-19 have become a hot issue and have received wide attention from scholars. Li Qian et al. [5] constructed a time-delayed COVID-19 non-autonomous infectious disease model driven by confirmed cases, and detailed numerical studies revealed the important impact of delayed reporting on the epidemic. Fan Ruguo et al. [6] established the SEIR dynamic model of the COVID-19 epidemic with an incubation period based on complex network theory and predicted the inflection point of the COVID-19 epidemic under three different incubation periods. Yan Yue et al. [7] proposed a time-delay dynamic model of infectious disease dynamics, introduced time-delay processes into the model to describe the virus incubation period and treatment cycle, accurately inverted the model parameters, effectively simulated the development of the epidemic situation, and predicted the epidemic situation's future trend. Based on the classical SEIR model, Zhang Liying et al. [8] established a discrete-time multi-stage dynamic time-delay model based on comprehensive consideration of epidemic development characteristics, intervention impact, medical conditions, experience transmission, and other factors and divided the transmission cycle of the virus into six stages. Zhai Yijiang et al. [9] considered that the infection probability of the group with basic disease was higher than that of the group without basic disease, they modified the classic SEIR model, divided the susceptible group into the group with basic disease and the group without basic disease, and established the SEIR model with the susceptible group of basic disease. Yu Zhenhua et al. [10] proposed a new nonlinear dynamic model of COVID-19 transmission, the SLEIR model, taking into account the low-risk population that took protective measures during the epidemic, and analyzed the model to reveal the transmission mechanism of COVID-19. Jin Wei et al. [11] established a SIR Model for the parallel transmission of two novel coronaviruses with time delay, carried out dynamic analysis of the model, and verified the conclusion by numerical simulation. Based on the fractional order model, Wei Qingdong et al. [12] proposed a fractional order population-delayed COVID-19 transmission model by introducing time delay parameters and proved the guiding significance of this model for epidemic prevention and control and the importance of the index of basic regeneration through simulation.

Although the epidemic has become normalized, the use of mathematical models to describe the transmission law of the epidemic can still accumulate relevant experience for future public health management [12]. According to the transmission characteristics of COVID-19, in order to be more in line with the actual situation, this paper proposes a COVID-19 transmission model with a time delay. Based on the established model, the basic reproduction number of the model is calculated, the disease-free equilibrium point and endemic equilibrium point of the model are analyzed, and the stability of the model equilibrium point under the condition of considering time delay is analyzed. The correctness of the theory is verified by simulation, and the influence of time delay on the spread of the epidemic is analyzed. Choosing the appropriate incubation period can predict the development of the epidemic more accurately, and shortening the incubation period of the virus can effectively control the spread of the epidemic.

2. Model with basic reproduction number

In 2021, Zhai Yijiang et al. [9] proposed the following SEIR (susceptible-exposure-infectious-recovered) COVID-19 transmission model, including susceptible infection of underlying diseases:

$$
\begin{cases}
S_1' = A - \beta_1 S_1 I - \theta S_1 - dS_1, \\
S_2' = \theta S_1 - \beta_2 S_2 I - dS_2, \\
E' = \beta_1 S_1 I + \beta_2 S_2 I - (d + \sigma) E, \\
I' = \sigma E - (\gamma + d + \alpha) I, \\
R' = \gamma I - dR,
\end{cases}
\tag{1}
$$

On the basis of model (1), this paper introduces the latent state to the incubation period of infectious force to study the following SEIR COVID-19 model with time delay, and the corresponding transmission mechanism is shown in Figure 1:

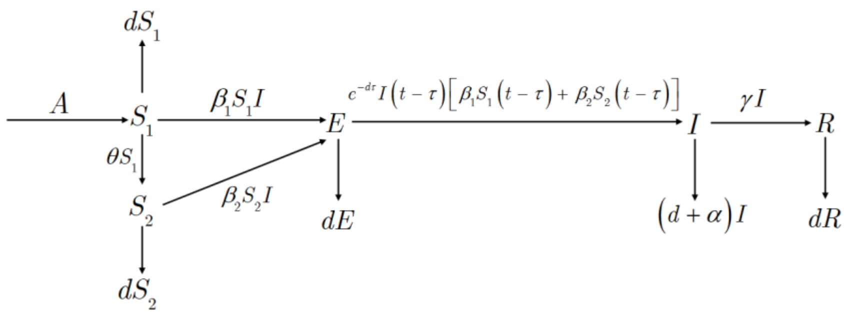

Figure 1. Mechanism of transmission

$$\begin{cases} S_1' = A - \beta_1 S_1 I - \theta S_1 - dS_1, \\ S_2' = \theta S_1 - \beta_2 S_2 I - dS_2, \\ E' = \beta_1 S_1 I + \beta_2 S_2 I - e^{-d\tau} I(t-\tau)\left[\beta_1 S_1(t-\tau) + \beta_2 S_2(t-\tau)\right] - dE, \\ I' = e^{-d\tau} I(t-\tau)\left[\beta_1 S_1(t-\tau) + \beta_2 S_2(t-\tau)\right] - (\gamma + d + \alpha) I, \\ R' = \gamma I - dR, \end{cases} \tag{2}$$

Where $S_1 = S_1(t)$ 、 $S_2 = S_2(t)$ 、 $E = E(t)$ 、 $I = I(t)$ and $R = R(t)$ represent the number of persons susceptible to no underlying disease, susceptible to underlying disease, exposure, infected and recovered at time *t*, respectively, A is the input rate, and β_1 is the transmission rate coefficient of susceptible persons without underlying diseases after effective contact with infected persons, β_2 is the transmission rate coefficient after effective contact between susceptible persons and infected persons; θ is the rate coefficient of healthy individuals to acquire the underlying disease; d is the natural mortality coefficient; α is the mortality coefficient of infected persons due to the disease; γ is the recovery rate coefficient of infected individuals; and τ is the incubation period, that is, the time from exposure to the onset of the disease.

Obviously, in model (2), the states of E and R are not included in equations 1, 2, and 4, so for convenience, only the following subsystems are considered below:

$$\begin{cases} S_1' = A - \beta_1 S_1 I - \theta S_1 - dS_1, \\ S_2' = \theta S_1 - \beta_2 S_2 I - dS_2, \\ I' = e^{-d\tau} I(t-\tau)\left[\beta_1 S_1(t-\tau) + \beta_2 S_2(t-\tau)\right] - mI, \end{cases} \tag{3}$$

$m = \gamma + d + \alpha.$

Considering the biological significance, let the initial conditions of system (3) be as follows: $\left(S_1(t), S_2(t), I(t)\right) = \left(\phi_1(t), \phi_2(t), \phi_3(t)\right) \in C\left([-\tau, 0], R_+^{\ 3}\right), \phi_1(t) > 0, \phi_2(t) \ge 0, \phi_3(t) \ge 0.$

Lemma 2.1 The solution $\left(S_1(t), S_2(t), I(t)\right)$ of system (3) satisfying the above initial conditions is positive for all $t \ge 0$.

Let $\Omega = \left\{\left(S_1, S_2, I\right) : S_1 \ge 0, S_2 \ge 0, I \ge 0, S_1 + S_2 + I \le \frac{A}{d}\right\}$. Obviously Ω is the positive invariant set of system (3).

Easy to know system (3) always has a disease-free equilibrium point

$$E_0\left(\frac{A}{\theta+d},\frac{\theta A}{d(\theta+d)},0\right).$$

Let $x=(I,S_1,S_2)^T$, then the original system can be rewritten as follows:

$$\frac{dx}{dt}=\mathrm{F}\ (x)-\mathrm{V}(x),$$

here

$$\mathrm{F}\ (x)=\begin{pmatrix} e^{-d\tau}I(t-\tau)\left[\beta_1S_1(t-\tau)+\beta_2S_2(t-\tau)\right] \\ 0 \\ 0 \end{pmatrix},\ \mathrm{V}(x)=\begin{pmatrix} mI \\ \beta_1S_1I+\theta S_1+dS_1-A \\ \beta_2S_2I+dS_2-\theta S_1 \end{pmatrix},$$

After linearization at the disease-free equilibrium point $E_0\left(\frac{A}{\theta+d},\frac{\theta A}{d(\theta+d)},0\right)$:

$$F=\left(e^{-d\tau}\left(\frac{\beta_1A}{d+\theta}+\frac{\beta_2\theta A}{d(d+\theta)}\right)\right),\ V=(m),$$

the calculated regeneration matrix is:

$$FV^{-1}=\left(\frac{1}{m}e^{-d\tau}\left(\frac{\beta_1A}{d+\theta}+\frac{\beta_2\theta A}{d(d+\theta)}\right)\right)=\frac{Ae^{-d\tau}(d\beta_1+\theta\beta_2)}{md(d+\theta)},$$

and its spectral radius is as follows:

$$\rho(FV^{-1})=\frac{Ae^{-d\tau}(d\beta_1+\theta\beta_2)}{md(d+\theta)}.$$

Therefore, according to Theorem 2 of reference [13], the basic reproduction number of system (3) is:

$$R_0=\frac{Ae^{-d\tau}(d\beta_1+\theta\beta_2)}{md(d+\theta)}.$$

3. Existence of equilibrium point

When $R_0\le 1$, the system has a disease-free equilibrium point $E_0\left(\frac{A}{\theta+d},\frac{\theta A}{d(\theta+d)},0\right)$; When $R_0>1$, the system has an endemic equilibrium $E^*\left(S_1^*,S_2^*,I^*\right)$ in addition to E_0, which is a positive solution of the following system:

$$
\begin{cases}
A-\beta_1 S_1 I-\theta S_1-dS_1=0,\\
\theta S_1-\beta_2 S_2 I-dS_2=0,\\
e^{-d\tau}I\left(\beta_1 S_1+\beta_2 S_2\right)-mI=0,
\end{cases}
\tag{4}
$$

from the first equation of system (4), the following can be obtained:

$$S_1=\frac{A}{\beta_1 I+\theta+d},$$

the second equation can be transformed as follows:

$$S_2=\frac{\theta A}{\left(\beta_1 I+\theta+d\right)\left(\beta_2 I+d\right)},$$

and the third equation can be transformed into:

$$\left[e^{-d\tau}\left(\beta_1 S_1+\beta_2 S_2\right)-m\right]I=0,$$

Substitute S_1 and S_2 into the above equation:

$$\frac{\beta_1 A}{\beta_1 I+\theta+d}+\frac{\beta_2\theta A}{\left(\beta_1 I+\theta+d\right)\left(\beta_2 I+d\right)}-me^{d\tau}=0,$$

Put $f\left(I\right)=\frac{\beta_1}{\beta_1 I+\theta+d}+\frac{\beta_2\theta}{\left(\beta_1 I+\theta+d\right)\left(\beta_2 I+d\right)}-\frac{me^{d\tau}}{A}$, then

$$f\left(0\right)=\frac{\beta_1}{\theta+d}+\frac{\beta_2\theta}{\left(\theta+d\right)d}-\frac{me^{d\tau}}{A}=\left(R_0-1\right)\frac{me^{d\tau}}{A}>0,$$

$$
\begin{aligned}
f\left(\frac{A}{d}\right)&=\frac{d\beta_1}{\beta_1 A+\theta d+d^2}+\frac{\beta_2\theta d^2}{\left(\beta_1 A+\theta d+d^2\right)\left(\beta_2 A+d^2\right)}-\frac{me^{d\tau}}{A}\\
&=\frac{d}{\beta_1 A+\theta d+d^2}\left(\beta_1+\frac{\beta_2\theta d}{\beta_2 A+d^2}\right)-\frac{me^{d\tau}}{A}\\
&\le\frac{d}{\beta_1 A+\theta d+d^2}\left(\beta_1+\frac{\theta d}{A}\right)-\frac{me^{d\tau}}{A}\\
&\le\frac{A}{d}-\frac{me^{d\tau}}{A}\\
&=\frac{1}{A}\left(d-me^{d\tau}\right)<0,
\end{aligned}
$$

It follows that the function $f\left(I\right)$ is monotonically decreasing with respect to I, so if and only if $R_0>1$, there exists a unique root of the equation $f\left(I\right)=0$ in the interval $\left(0,\frac{A}{d}\right)$.

By $f(I)=0$, we get:

$$\frac{\beta_1}{\beta_1 I+\theta+d}+\frac{\beta_2\theta}{(\beta_1 I+\theta+d)(\beta_2 I+d)}-\frac{me^{d\tau}}{A}=0,$$

that is

$$\frac{A(\beta_1\beta_2 I+\beta_1 d+\beta_2\theta)-me^{d\tau}(\beta_1 I+\theta+d)(\beta_2 I+d)}{A(\beta_1 I+\theta+d)(\beta_2 I+d)}=0,$$

therefore, the quadratic equation with respect to I is obtained:

$$\begin{aligned}
&A(\beta_1\beta_2 I+\beta_1 d+\beta_2\theta)-me^{d\tau}(\beta_1 I+\theta+d)(\beta_2 I+d)\\
&=-me^{d\tau}\beta_1\beta_2 I^2+\left(A\beta_1\beta_2-me^{d\tau}(\beta_1 d+\beta_2\theta+\beta_2 d)\right)I+A\beta_1 d+A\beta_2\theta-me^{d\tau}d(\theta+d)\\
&=0,
\end{aligned}$$

the equation can be deformed into:

$$me^{d\tau}\beta_1\beta_2 I^2-\left(A\beta_1\beta_2-me^{d\tau}(\beta_1 d+\beta_2\theta+\beta_2 d)\right)I-\left[A\beta_1 d+A\beta_2\theta-me^{d\tau}d(\theta+d)\right]=0,$$

the positive solution can be obtained as follows:

$$I^*=\frac{b+\sqrt{b^2+4me^{d\tau}\beta_1\beta_2\left[md(d+\theta)e^{d\tau}(R_0-1)\right]}}{2me^{d\tau}\beta_1\beta_2},$$

where $b=\left[A\beta_1\beta_2-me^{d\tau}(\beta_1 d+\beta_2\theta+\beta_2 d)\right]$.

From the previous derivation, we can see that:

$$S_1^*=\frac{A}{\beta_1 I^*+\theta+d},\ S_2^*=\frac{\theta A}{(\beta_1 I^*+\theta+d)(\beta_2 I^*+d)}.$$

In conclusion, the existence theorem of the equilibrium point of endemic diseases can be obtained.

Theorem 1 When $R_0>1$, a unique endemic equilibrium $E^*\left(S_1^*,S_2^*,I^*\right)$ exists in the system.

4 The stability of the equilibrium

4.1 $\tau=0$, the local stability of the equilibrium point

Theorem 2 When $\mathrm{R}_0<1$, the equilibrium point is locally asymptotically stable; When $\mathrm{R}_0>1$, the equilibrium point is unstable.

Proof The Jacobian of the system at $E_0\left(\frac{A}{\theta+d},\frac{\theta A}{d(\theta+d)},0\right)$:

$$J\left(E_0\right)=\begin{pmatrix} -(\theta+d) & 0 & -\beta_1 S_{10} \\ \theta & -d & -\beta_2 S_{20} \\ 0 & 0 & \beta_1 S_{10}+\beta_2 S_{20}-m \end{pmatrix},$$

clearly $-(d+\theta)$, and $-d$ are the two eigenvalues of the matrix $J\left(E_0\right)$, and the third eigenvalue of the matrix is:

$$\beta_1 S_{10}+\beta_2 S_{20}-m=\frac{\beta_1 A}{\theta+d}+\frac{\beta_2 \theta A}{d(\theta+d)}-m=R_0-1,$$

thus, when $\mathrm{R}_0<1$, the equilibrium point E_0 is locally asymptotically stable; when $\mathrm{R}_0>1$, the equilibrium point is unstable.

Theorem 3 When $\mathrm{R}_0>1$, the endemic equilibrium point $E^*\left(S_1^*, S_2^*, I^*\right)$ of the model is locally asymptotically stable.

Proof The equilibrium point E^* satisfies

$$\left(\beta_1 S_1^*+\beta_2 S_2^*\right) I^*-m I^*=0,$$

so

$$\beta_1 S_1^*+\beta_2 S_2^*=m.$$

The characteristic equation of the system at E^* is as follows.:

$$\begin{vmatrix} \lambda+\beta_1 I^*+\theta+d & 0 & \beta_1 S_1^* \\ -\theta & \lambda+\beta_2 I^*+d & \beta_2 S_2^* \\ -\beta_1 I^* & -\beta_2 I^* & \lambda \end{vmatrix}=\lambda^3+c_1 \lambda^2+c_2 \lambda+c_3=0.$$

where

$c_1=I^*\left(\beta_1+\beta_2\right)+2 d+\theta,$

$c_2=\beta_1 \beta_2 I^{*2}+\left(\beta_1^2 S_1^*+\beta_2^2 S_2^*+\left(\beta_1+\beta_2\right) d+\beta_2 \theta\right) I+d^2+d \theta,$

$c_3=I^*\left[\left(\beta_1 I^*+d+\theta\right) \beta_2^2 S_2^*+\left(\beta_1 I^*+\theta\right) \beta_1 \beta_2 S_1^*+\beta_1^2 d S_1^*\right],$

The calculation shows that

$$c_1>0, \quad c_1 \cdot c_2-c_3>0, \quad c_3>0.$$

Therefore, according to the Rough-Hurwitz criterion, the endemic equilibrium E^* of the model is locally asymptotically stable when $\mathrm{R}_0>1$.

4.2 $\tau > 0$, the stability of the equilibrium

In order to discuss the stability of the equilibrium point, the system is linearized at the equilibrium point $E^*\left(S_1^*, S_2^*, I^*\right)$ as follows:

Let

$$U = S_1 - S_1^*, V = S_2 - S_2^*, W = I - I^*,$$

then:

$$U' = A - \beta_1\left(UW + I^*U + S_1^*W + S_1^*I^*\right) - (\theta+d)U - (\theta+d)S_1^*,$$

$$V' = \theta\left(S_1^* + U\right) - \beta_2\left(V + S_2^*\right)\left(W + I^*\right) - d\left(V + S_2^*\right),$$

$$W' = e^{-d\tau}\left(W(t-\tau) + I^*\right)\left[\beta_1\left(U(t-\tau) + S_1^*\right) + \beta_2\left(V(t-\tau) + S_2^*\right)\right] - m\left(W + I^*\right),$$

Take the linear term:

$$\begin{cases} U' = -\left(\beta_1 I^* + \theta + d\right)U - \beta_1 S_1^* W, \\ V' = \theta U - \left(\beta_2 I^* + d\right)V - \beta_2 S_2^* W, \\ W' = e^{-d\tau}\beta_1 I^* U(t-\tau) + e^{-d\tau}\beta_2 I^* V(t-\tau) + \left[e^{-d\tau}\left(\beta_1 S_1^* + \beta_2 S_2^*\right)\right]W(t-\tau) - mW, \end{cases}$$

Substituting $\begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} e^{\lambda\tau}$ and eliminating c_i leads to the following characteristic matrix:

$$\begin{pmatrix} \lambda + \left(\beta_1 I + \theta + d\right) & 0 & \beta_1 S_1 \\ -\theta & \lambda + \beta_2 I + d & \beta_2 S_2 \\ -e^{-d\tau-\lambda\tau}\beta_1 I & -e^{-d\tau-\lambda\tau}\beta_2 I & \lambda - e^{-d\tau-\lambda\tau}\left(\beta_1 S_1 + \beta_2 S_2\right) + m \end{pmatrix}$$

Theorem 4 When $\mathrm{R}_0 \le 1$, the equilibrium point $E_0\left(\dfrac{A}{\theta+d}, \dfrac{\theta A}{d(\theta+d)}, 0\right)$ is locally asymptotically stable.

Proof $\tau > 0$, the characteristic equation of the system at is as follows.

$$\begin{vmatrix} \lambda+(\theta+d) & 0 & \dfrac{\beta_1 A}{\theta+d} \\ -\theta & \lambda+d & \dfrac{\beta_2\theta A}{d(\theta+d)} \\ 0 & 0 & \lambda-e^{-d\tau-\lambda\tau}\left(\dfrac{\beta_1 A}{\theta+d}+\dfrac{\beta_2\theta A}{d(\theta+d)}\right)+m \end{vmatrix}$$

$$=(\lambda+d)(\lambda+\theta+d)\left(\lambda-e^{-d\tau-\lambda\tau}\left(\frac{\beta_1 A}{\theta+d}+\frac{\beta_2\theta A}{d(\theta+d)}\right)+m\right)$$

$$=0,$$

Clearly the above equation has two negative roots $\lambda=-d$ and $\lambda=-\theta-d$, and the remaining roots are determined by the following equation:

$$\lambda+m-e^{-d\tau-\lambda\tau}\left(\frac{\beta_1 A}{\theta+d}+\frac{\beta_2\theta A}{d(\theta+d)}\right)=0,$$

Put $f(\lambda)=\lambda+m-e^{-d\tau-\lambda\tau}\left(\dfrac{\beta_1 A}{\theta+d}+\dfrac{\beta_2\theta A}{d(\theta+d)}\right)=0,$

Suppose that there exists a root $\lambda=\mu+i\omega$ $(\mu>0)$ in the above equation, substitute it in to obtain:

$$\mu+i\omega+m-e^{-d\tau-(\mu+i\omega)\tau}\left(\frac{\beta_1 A}{\theta+d}+\frac{\beta_2\theta A}{d(\theta+d)}\right)=0,$$

that is

$$\mu+i\omega+m-e^{-(d+\mu)\tau}\left(\frac{\beta_1 A}{\theta+d}+\frac{\beta_2\theta A}{d(\theta+d)}\right)(\cos\omega\tau+i\sin\omega\tau)=0,$$

comparing the real parts of both sides of the equation gives:

$$\mu+m-e^{-(d+\mu)\tau}\frac{(\beta_1 d+\beta_2\theta)A}{d(\theta+d)}\cos\omega\tau=0,$$

since $\mu>0$, $e^{-(d+\mu)\tau}$ is decreasing in $[0,+\infty]$, meanwhile $0<e^{-(d+\mu)\tau}<1$, $0<|\cos\omega\tau|\le 1$,

so

$$\begin{aligned} &\mu+m-e^{-(d+\mu)\tau}\frac{(\beta_1 d+\beta_2\theta)A}{d(\theta+d)}\cos\omega\tau \\ &\ge \mu+m-e^{-d\tau}\frac{(\beta_1 d+\beta_2\theta)A}{d(\theta+d)} \\ &=\mu+m(1-R_0), \end{aligned}$$

therefore when $R_0 \le 1$ and $\mu \ge 0$, there is $\mu + m - e^{-(d+\mu)\tau} \dfrac{(\beta_1 d + \beta_2\theta)A}{d(\theta+d)}\cos\omega\tau \ge 0$. So the roots of the characteristic equation cannot have nonnegative real parts, that is, only negative real roots.

In summary, When $\mathrm{R}_0 \le 1$, the equilibrium point E_0 is locally asymptotically stable.

Theorem 5 When $\mathrm{R}_0 < 1$, the equilibrium point $E_0 = (S_{10}, S_{20,0}) = \left(\dfrac{A}{\theta+d}, \dfrac{\theta A}{d(\theta+d)}, 0\right)$ is globally asymptotically stable.

Proof Since $A = (\theta+d)S_{10}$, the first two equations of the model can be rewritten as follows:

$$S_1' = (\theta + d + \beta_1 I)(S_{10} - S_1) - \beta_1 S_{10} I,$$

$$S_2' = \theta(S_1 - S_{10}) - (d + \beta_2 I)(S_2 - S_{20}) - \beta_2 S_{20} I,$$

then we take the Lyapunov function:

$$L_1 = \frac{1}{2}(S_1 - S_{10})^2 + \frac{1}{2}x(S_2 - S_{20})^2 + yI + ye^{-d\tau}\int_{t-\tau}^{t}\left[S_1(\theta)I(\theta) + S_2(\theta)I(\theta)\right]d\theta,$$

the direct calculation has the total derivative of the solution of the function L_1 along the system (3):

$$\begin{aligned}
L_1'\big|_{(3)} &= (S_1 - S_{10})S_1' + x(S_2 - S_{20})S_2' + yI' \\
&+ye^{-d\tau}\left[\beta_1 S_1 I + \beta_2 S_2 I - \beta_1 S_1(t-\tau)I(t-\tau) + \beta_2 S_2(t-\tau)I(t-\tau)\right] \\
&= -(\theta + d + \beta_1 I)(S_1 - S_{10})^2 - \beta_1 S_{10} I(S_1 - S_{10}) + x\theta(S_1 - S_{10})(S_2 - S_{20}) \\
&-x(d + \beta_2 I)(S_2 - S_{20})^2 - x\beta_2 S_{20} I(S_2 - S_{20}) + ye^{-d\tau}(\beta_1 S_1 I + \beta_2 S_2 I) - ymI \\
&\le -(\theta + d)(S_1 - S_{10})^2 + x\theta(S_1 - S_{10})(S_2 - S_{20}) - xd(S_2 - S_{20})^2 \\
&-x\beta_2 S_{20} I(S_2 - S_{20}) - \beta_1 S_{10} I(S_1 - S_{10}) + ye^{-d\tau}(\beta_1 S_1 I + \beta_2 S_2 I) - ymI \\
&= -(\theta + d)(S_1 - S_{10})^2 + x\theta(S_1 - S_{10})(S_2 - S_{20}) - xd(S_2 - S_{20})^2 \\
&-(S_{10} - ye^{-d\tau})\beta_1 S_1 I - (xS_{20} - ye^{-d\tau})\beta_2 S_2 I + (\beta_1 S_{10}{}^2 + x\beta_2 S_{20}{}^2 - ym)I,
\end{aligned}$$

In order to cancel out the $S_1 I$ and $S_2 I$ terms, set $x = \dfrac{S_{10}}{S_{20}} = \dfrac{d}{\theta}$ and $y = S_{10}e^{d\tau}$, so

$$\begin{aligned}
L_1'\big|_{(3)} \le &-(\theta + d)(S_1 - S_{10})^2 + d(S_1 - S_{10})(S_2 - S_{20}) - \frac{d^2}{\theta}(S_2 - S_{20})^2 \\
&+(\beta_1 S_{10}{}^2 + x\beta_2 S_{20}{}^2 - ym)I,
\end{aligned}$$

while for the quadratic polynomial in the above equation

$$-(\theta+d)(S_1-S_{10})^2+d(S_1-S_{10})(S_2-S_{20})-\frac{d^2}{\theta}(S_2-S_{20})^2,$$

because

$$\Delta=d^2-4\frac{d^2}{\theta}(d+\theta)=-3d^2-4\frac{1}{\theta}d^3<0,$$

thus this polynomial is negative definite with respect to S_{10} and S_{20}.

Also, note that when $R_0<1$, the coefficient of the I term

$$\begin{aligned}&\beta_1 S_{10}{}^2+x\beta_2 S_{20}{}^2-ym\\&=\beta_1 S_{10}{}^2+\beta_2 S_{10}S_{20}-mS_{10}e^{d\tau}\\&=S_{10}\left(\beta_1 S_{10}+\beta_2 S_{20}-me^{d\tau}\right)\\&=S_{10}\left(R_0-1\right)<0,\end{aligned}$$

so

$$\begin{aligned}L_1'\big|_{(3)}&\le-(\theta+d)(S_1-S_{10})^2+d(S_1-S_{10})(S_2-S_{20})-\frac{d^2}{\theta}(S_2-S_{20})^2\\&\quad+\left(\beta_1 S_{10}{}^2+x\beta_2 S_{20}{}^2-ym\right)I\\&\le 0,\end{aligned}$$

It is also easy to know that the largest invariant set of $L_1'\big|_{(3)}=0$ on Ω is $\{E_0\}$. Therefore, according to LaSalle's invariance principle [14], when $\mathrm{R}_0<1$, the equilibrium point $E_0=\left(S_{10},S_{20,0}\right)$ is globally asymptotically stable.

Theorem 6 When $\mathrm{R}_0>1$, the endemic equilibrium point $E^*\left(S_1^*,S_2^*,I^*\right)$ of the model is locally asymptotically stable.

Proof The characteristic equation of system (3) at e0 is as follows：

$$\begin{vmatrix}\lambda+\beta_1 I^*+\theta+d & 0 & \beta_1 S_1^*\\ -\theta & \lambda+\beta_2 I^*+d & \beta_2 S_2^*\\ -\beta_1 e^{-d\tau-\lambda\tau}I^* & -\beta_2 e^{-d\tau-\lambda\tau}I^* & \lambda-e^{-d\tau-\lambda\tau}\left(\beta_1 S_1^*+\beta_2 S_2^*\right)+m\end{vmatrix}=0,$$

Clearly the above equation is equivalent to

$$\lambda^3+a_1\lambda^2+b_1\lambda+c_1+\left(a_2\lambda^2+b_2\lambda+c_2\right)e^{-\lambda\tau}=0, \qquad (5)$$

where

$a_1 = \beta_1 I^* + \beta_2 I^* + 2d + \theta + m$,

$b_1 = \beta_1\beta_2 I^{*2} + \left[(m+d)(\beta_1+\beta_2) + \theta\beta_2\right] I^* + d(d+2m+\theta) + m\theta$,

$c_1 = m\beta_1\beta_2 I^{*2} + m\left[\beta_1 d + \beta_2 d + \beta_2\theta\right] I^* + dm(d+\theta)$,

$a_2 = -\left(\beta_1 S_1^* + \beta_2 S_2^*\right) e^{-d\tau}$,

$b_2 = -\left[\beta_1\beta_2 I^*\left(S_1^* + S_2^*\right) + \left(\beta_1 S_1^* + \beta_2 S_2^*\right)(\theta + 2d)\right] e^{-d\tau}$,

$c_2 = -d\left[\beta_1\beta_2 I^*\left(S_1^* + S_2^*\right) + \left(\beta_1 S_1^* + \beta_2 S_2^*\right)(\theta + d)\right] e^{-d\tau}$,

Let $\lambda = i\omega$ be the root of the above equation, then substitute in and obtain:

$$\begin{cases} b_2\omega\sin(\tau\omega) + \left(c_2 - a_2\omega^2\right)\cos(\tau\omega) = a_1\omega^2 - c_1, \\ b_2\omega\cos(\tau\omega) + \left(c_2 - a_2\omega^2\right)\sin(\tau\omega) = \omega^3 - b_1\omega, \end{cases} \tag{6}$$

and then

$$\omega^6 + a_3\omega^4 + b_3\omega^2 + c_3 = 0. \tag{7}$$

Let $\omega^2 = z$, then

$$h(z) = z^3 + a_3 z^2 + b_3 z + c_3 = 0. \tag{8}$$

where: $a_3 = a_1^2 - 2b_1 - a_2^2$, $b_3 = b_1^2 - b_2^2 - 2a_1c_1 + 2a_2c_2$, $c_3 = c_1^2 - c_2^2$.

If the coefficients in h(z) satisfy the Routh-Hurwitz condition, then there is no positive real root in Equation (8), that is, there may be no positive ω satisfying the transcendental equation (6). On the other hand, considering that the values of the coefficients in Equation (8) do not satisfy the Routh-Hurwitz condition, we may wish to assume condition P2: $c_3 < 0$, that is $c_1 + c_2 < 0$, $c_1 - c_2 > 0$, then there exists at least one positive real root in Equation (8), in which case there exists $\omega_0 > 0$ such that there exists a pair of pure imaginary roots $\lambda = \pm i\omega_0$ of equation (5). If condition P2 holds, that is $c_1 + c_2 < 0$, $c_1 - c_2 > 0$, at this point $c_1 - c_2 > 0$ is clearly true, by calculation

$$
\begin{aligned}
c_1+c_2 &= m\beta_1\beta_2 I^{*2}+m\left[\beta_1 d+\beta_2 d+\beta_2\theta\right]I^*+dm(\theta+d)\\
&\quad -d\left[\beta_1\beta_2 I^*(S_1^*+S_2^*)+(\beta_1 S_1^*+\beta_2 S_2^*)(\theta+d)\right]e^{-d\tau}\\
&= I^*\left[m\beta_1\beta_2 I^*+m(\beta_1 d+\beta_2 d+\beta_2\theta)-e^{-d\tau}d\beta_1\beta_2(S_1^*+S_2^*)\right]\\
&= I^*\left[e^{-d\tau}\beta_1\beta_2\left[S_1^*(\beta_1 I^*-d)+S_2^*(\beta_2 I^*-d)\right]+m(\beta_1 d+\beta_2 d+\beta_2\theta)\right]\\
&> 0.
\end{aligned}
$$

Therefore, there is no positive real root in Equation (8), so when $\tau>0$ and $\mathrm{R}_0>1$, the endemic equilibrium point E^* is locally asymptotically stable.

The global stability of the endemic disease equilibrium point is proved below:

Theorem 7 When $\mathrm{R}_0>1$, the endemic equilibrium point $E^*\left(S_1^*,S_2^*,I^*\right)$ of the model is globally asymptotically stable.

Proof Take the Lyapunov function

$$
L_{21}=S_1^* g\left(\frac{S_1}{S_1^*}\right)+S_2^* g\left(\frac{S_2}{S_2^*}\right)+xI^* g\left(\frac{I}{I^*}\right),
$$

where $g\left(u\right)=u-1-\ln u$.

Then the total derivative of the solution of function L_{21} along system (3) is:

$$
\begin{aligned}
L_{21}'\Big|_{(3)} &= \left(1-\frac{S_1^*}{S_1}\right)\left[A-\beta_1 S_1 I-\theta S_1-dS_1\right]+\left(1-\frac{S_2^*}{S_2}\right)\left[\theta S_1-\beta_2 S_2 I-dS_2\right]\\
&\quad +x\left(1-\frac{I^*}{I}\right)\left[e^{-d\tau}I\left(t-\tau\right)\left(\beta_1 S_1(t-\tau)+\beta_2 S_2(t-\tau)\right)-mI\right],
\end{aligned}
$$

let $U=\dfrac{S_1}{S_1^*}, V=\dfrac{S_2}{S_2^*}, W=\dfrac{I}{I^*}$, then

$$
\begin{aligned}
L_{21}'\Big|_{(3)} &= \left(1-\frac{1}{U}\right)\left[A-\beta_1 S_1^* I^* UW-\theta S_1^* U-dS_1^* U\right]+\left(1-\frac{1}{V}\right)\left[\theta S_1^* U-\beta_2 S_2^* I^* VW-dS_2^* V\right]\\
&\quad +x\left(1-\frac{1}{W}\right)\left[e^{-d\tau}I^* W(t-\tau)\left(\beta_1 S_1^* U(t-\tau)+\beta_2 S_2^* V(t-\tau)\right)-mI^* W\right]\\
&= A+\left(\theta+d\right)S_1^*+dS_2^*+xmI^*-dS_1^* U-\frac{A}{U}-dS_2^* V-\theta S_1^*\frac{U}{V}\\
&\quad +\left(\beta_1 S_1^*+\beta_2 S_2^*-xm\right)I^* W-\beta_1 S_1^* I^* UW-\beta_2 S_2^* I^* VW\\
&\quad +xe^{-d\tau}I^* W(t-\tau)\left[\beta_1 S_1^*\left(U(t-\tau)-\frac{U(t-\tau)}{W}\right)+\beta_2 S_2^*\left(V(t-\tau)-\frac{V(t-\tau)}{W}\right)\right].
\end{aligned}
$$

To get rid of the W term, we can take $x=\dfrac{1}{m}\left(\beta_1 S_1^*+\beta_2 S_2^*\right)$, In addition $A=\beta_1 S_1^* I^*+\theta S_1^*+dS_1^*$,

$\theta S_1^* = \beta_2 S_2^* I^* + dS_2^*$, substitute them into the above equation:

$$\begin{aligned} L_{21}'\Big|_{(3)} &= dS_1^*\left(2-U-\frac{1}{U}\right)+dS_2^*\left(3-V-\frac{1}{U}-\frac{U}{V}\right) \\ &+\beta_1 S_1^* I^*\left(2-\frac{1}{U}-UW+U(t-\tau)W(t-\tau)-\frac{U(t-\tau)W(t-\tau)}{W}\right) \\ &+\beta_2 S_2^* I^*\left(3-\frac{1}{U}-\frac{U}{V}-VW+V(t-\tau)W(t-\tau)-\frac{V(t-\tau)W(t-\tau)}{W}\right). \end{aligned}$$

take $L_{22}=\int_{t-\tau}^{t}\left[U(\theta)W(\theta)-1-\ln(U(\theta)W(\theta))\right]d\theta,\ L_{23}=\int_{t-\tau}^{t}\left[V(\theta)W(\theta)-1-\ln(V(\theta)W(\theta))\right]d\theta,$

let

$$L_2 = L_{21}+\beta_1 S_1^* I^* L_{22}+\beta_2 S_2^* I^* L_{23},$$

then the total derivative of the solution of function L_2 along system (3) is:

$$\begin{aligned} L_2'\Big|_{(3)} &= dS_1^*\left(2-U-\frac{1}{U}\right)+dS_2^*\left(3-V-\frac{1}{U}-\frac{U}{V}\right) \\ &+\beta_1 S_1^* I^*\left(2-\frac{1}{U}-\frac{U(t-\tau)W(t-\tau)}{W}+\ln\frac{U(t-\tau)W(t-\tau)}{UW}\right) \\ &+\beta_2 S_2^* I^*\left(3-\frac{1}{U}-\frac{U}{V}-\frac{V(t-\tau)W(t-\tau)}{W}+\ln\frac{V(t-\tau)W(t-\tau)}{VW}\right) \\ &\le 0. \end{aligned}$$

It is also easy to know that the largest invariant set of $L_2'\big|_{(3)}=0$ on Ω is $\{E^*\}$. Therefore, according to LaSalle's invariance principle [14], when $\mathrm{R}_0>1$, the endemic disease equilibrium point $E^*=\left(S_1^*, S_2^*, I^*\right)$ is globally asymptotically stable.

5. Numerical simulation

According to the biological significance of the system and some parameter values and initial values in the literature [9, 15-16], MATLAB software was used for numerical simulation. The following parameters and initial values that meet the conditions are taken to simulate the existence of the endemic equilibrium point and analyze the influence of time delay on the changing trend of the epidemic situation. In order to combine with reality, the initial value here will be taken to be the size equivalent to the population of a large city:

$$A=60000,\ \beta_1=2.1011\times10^{-7},\ \beta_2=7.1443\times10^{-7},\ \theta=0.1,\ d=0.045,\ \gamma=0.33029,$$

$$\alpha=1.7826\times10^{-5},\ \tau=5.3,\ S_1(0)=1000000,\ S_2(0)=100000,\ I(0)=1000.$$

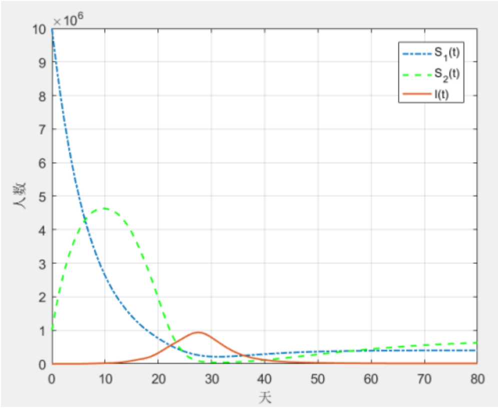

Figure 2. Epidemic spread

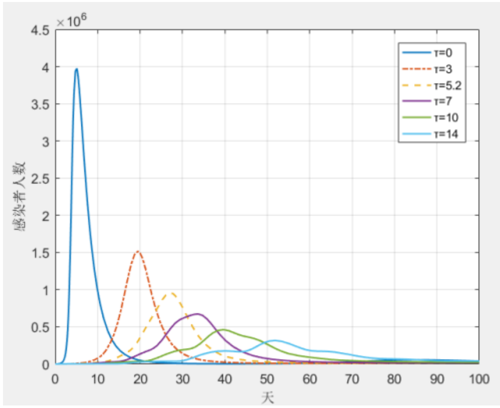

Figure 3. Effects of different time delays on epidemic transmission

The basic reproduction number $R_0 = 1.56 > 1$ was calculated, and the final simulation result is shown in Figure 2. It was observed that all variables tend to stabilize after a period of time, which proved the existence of a non-negative endemic equilibrium point. According to Theorem 7, the endemic equilibrium point is globally asymptotically stable at this point, and clearly the numerical simulation results are in agreement with the theory.

The incubation period of COVID-19 is generally 3-7 days, and the maximum is 14 days, so the other parameters are unchanged and the time delay is set to, respectively. The numerical simulation results are shown in Figure 3. We can observe that the larger the time delay, the slower the increase in the cumulative number of infected people. However, the final stability level corresponding to different time delays is not much different. In contrast, the results of considering time lag (such as an inflection point) would be more consistent with the actual situation.

The change in time delay makes the solution (such as $I(t)$) of the time delay system (3) shake, which may lead to multiple outbreaks of the epidemic. For example, when $\tau > 7$, the number of infected persons appears to have reached a second peak immediately after the first peak, a large time-delay may lead to a second outbreak of the epidemic.

6. Conclusion

In this paper, the existence and stability of the equilibrium point are discussed separately according to the constructed SEIR-based time-delay model. The basic reproduction number R_0 is calculated. First, it is proved that when $\tau = 0$, the disease-free equilibrium is locally asymptotically-stable when $R_0 < 1$, while the endemic equilibrium is locally asymptotically-stable when $R_0 > 1$. Then we mainly analyze the stability of the system when $\tau > 0$. The following conclusions are drawn: when $R_0 < 1$, the disease-free equilibrium is globally asymptotically stable; when $R_0 > 1$, the endemic equilibrium is globally asymptotically stable. Finally, the results of simulations show that the previously derived conclusions about the stability of the equilibrium point are correct. Therefore, in order to better prevent and control the epidemic, we should take measures to keep R_0 below 1. In addition, the influence of different time delays on the epidemic was observed through numerical simulation, indicating the importance of selecting appropriate time delays for the prediction of epidemic development.

Although the conclusions obtained from the model established in this paper can well reflect the changes in the spread of COVID-19, the epidemic modeling needs to consider many factors. This paper is relatively simple and has limitations. For example, only a discrete time delay is assumed in the model, and the impact of quarantine, vaccination, and other measures on the spread of the epidemic is not considered.

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