Dynamics of coherent hydrodynamic systems and geometry on the Monge manifolds

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Abstract. The concept of a metric on the manifold of Monge potentials is introduced for hydrodynamic vortex system. The concept of coherence of continuous vortex structures is formulated in terms of the deviation of geodesic lines on a given manifold. Criteria of decay and collapse of a vortex continual system are established.

Keywords : Onsager vortices, geodesic deviation, Monge manifolds, hydrodynamic invariants.

1. Introduction

The possibility of introducing a metric on manifolds associated with hydrodynamic flows of various types have been repeatedly discussed in the many articles [1]–[5], however the results available to date are either overly general (which makes it difficult development of techniques suitable for describing flows of specific types) [6]–[7], or require the introduction of significant restrictions and solutions-oriented problems with a priori strictly specified setting additional conditions, the physical content of which requires special analysis [8]–[10]. If we consider a hydrodynamic flow as a statistical system in a state close to equilibrium, then its geometric properties can be studied using the general Amari–Weinhold [11]–[12] technique, using the possibility of introducing Riemannian topology on the Gibbs manifolds, determined by the relations between the thermodynamic potentials $p = p(\rho, T)$ and $s = s(u, \rho)$. In particular, if we turn to models of a real flow in the form of a set of Onsager point vortices, one can obtain meaningful differential–functional relations that are expressions for the heat capacities of the vortex ensemble and geodesic equations connecting the states levels of system equilibrium.

In this paper we consider the methods of investigations the geometric representation of a Hamiltonian dynamical system, the corresponding set of hydrodynamic conservation laws, by using the Lagrangian and Hamiltonian geometry on phase manifolds.

2. Hamiltonian formalism for a hydrodynamic system in terms of Monge potentials

Locally, the state of a hydrodynamic system can be described density, speed and entropy density (in the general case of a compressible medium) at a given point $(\mathbf{x}, t) \in K_{N+1} \subset \mathbb{R}^{N+1}$ ($\mathbf{x} \in \mathbb{R}^{N\leq 3}$, $t \in \mathbb{R}^1_+$), that is totality quantities $\{\rho(\mathbf{x}, t); \mathbf{v}(\mathbf{x}, t); s(\mathbf{x}, t)\}$. Hydrodynamic equations describing their change over time have form:

$$\rho_t + (\rho\mathbf{v})_\mathbf{x} = 0, \qquad (\rho\mathbf{v})_t + (\rho\mathbf{v}^2)_\mathbf{x} = -p_\mathbf{x}, \qquad (\rho s)_t + (\rho\mathbf{v}s)_\mathbf{x} = 0, \tag{1}$$

where $p = p(\mathbf{x}, t)$ is a scalar field pressure Using the representation velocity fields via Monge potentials $\{\big({}^{[\alpha]}M\big)\}|_{\alpha=\overline{1,m}} \in Y_m$, highlighting in the expansion of the velocity field gradient term

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and a set of quasi-solenoidal ones:

$$\mathbf{v} = -({}^{[4]}M)\cdot({}^{[1]}M)_{\mathbf{x}} - s\cdot({}^{[2]}M)_{\mathbf{x}} - ({}^{[3]}M)_{\mathbf{x}} \quad \text{at } m=4,$$

Where ${}^{[\alpha]}M(\mathbf{x},t)$ — some scalar fields. For flows c $s=\text{const}$ ${}^{[3]}M + s\cdot{}^{[2]}M \to {}^{[2]}M$, ${}^{[4]}M \to {}^{[3]}M$ (in this case $m=3$).

To ensure unambiguous selection of these fields IV additional conditions must be specified. Let us take the following as conditions I, II, III :

$$I)\ \widehat{D}{}^{[4]}M \equiv 0, \quad II)\ \widehat{D}{}^{[1]}M \equiv 0, \quad III)\ \widehat{D}{}^{[2]}M \equiv T,$$

where $\widehat{D}(...) \equiv (...)_t + \mathbf{v}\cdot(...)_{\mathbf{x}}$ — total time derivative operator, $T=T(\mathbf{x},t)$ — thermodynamic field of temperature. Let us take the fourth additional condition to be the relation for specific enthalpy of flow:

$$w(\mathbf{x},t) = ({}^{[4]}M)\cdot({}^{[1]}M)_t + s\cdot({}^{[2]}M)_t + ({}^{[3]}M)_t - \mathbf{v}^2/2.$$

Euler's equations (1) with conditions I–IV are equivalent to the consequences Whitham's variational principle $\delta\int\int p\,d\mathbf{x}dt = 0$, and where is the Lagrangian density (pressure) based on the thermodynamic relation $u = wp/\rho$:

$$p\big(\{{}^{[\alpha]}M\};\{{}^{[\alpha]}M_{\mathbf{x}}\},\{{}^{[\alpha]}M_t\}\big) \;=\; \rho(wu) \;=\; \rho\big(({}^{[4]}M)\cdot({}^{[1]}M)_t + \qquad (2)$$

$$+s\cdot({}^{[2]}M)_t + ({}^{[3]}M)_t - \frac{1}{2}\big(-({}^{[4]}M)\cdot({}^{[1]}M)_{\mathbf{x}} - s\cdot({}^{[2]}M)_{\mathbf{x}} - ({}^{[3]}M)_{\mathbf{x}}\big)^2 - u(\rho,s)\big).$$

Here $u(\rho,s)$ is the specific internal energy of the flow (the caloric equation of state of the medium is assumed to be known).

Let's move on to the Hamiltonian representation , for which we define the canonically conjugate variables $\{{}^{[\alpha]}M\}$ "momentums of the Monge representation" ${}_{[\alpha]}P$:

$${}_{[1]}P = \frac{\partial p}{\partial({}^{[1]}M_t)} = \rho{}^{[4]}M, \quad {}_{[2]}P = \frac{\partial p}{\partial({}^{[2]}M_t)} = \rho s, \quad {}_{[3]}P = \frac{\partial p}{\partial({}^{[3]}M_t)} = \rho \quad (m=4);$$

$${}_{[1]}P = \frac{\partial \pi}{\partial({}^{[1]}M_t)} = \rho{}^{[3]}M, \qquad {}_{[2]}P = \frac{\partial \pi}{\partial({}^{[2]}M_t)} = \rho \quad (m=3).$$

Let's introduce Hamilton function $H\big(\{{}^{[\varsigma]}M\},\{{}_{[\alpha]}P\}\big)$ by conversion Legendre Lagrangian density:

$$H \equiv \sum_{\alpha=1}^{m-1} {}_{[\alpha]}P\cdot{}^{[\alpha]}M_t - p = \frac{1}{2{}_{[\beta_0]}P}\big(\sum_{\alpha=1}^{m-1}{}_{[\alpha]}P\cdot{}^{[\alpha]}M_{\mathbf{x}}\big)^2 + {}_{[\beta_0]}Pu({}_{[\beta_1]}P,{}_{[\beta_0]}P),$$

where : $\beta_0(m) = m-1$, $\beta_1(m) = 2$ for $m=4$, $\beta_1(m)=0$ for $m=3$.

We will consider the Hamiltonian space $W^m = (Y_m, H)$, $H : T^*Y_m \to \mathbb{R}^1$ having a fundamental tensor $g^{\alpha\beta}\big(\{{}^{[\mu]}M\},\{{}_{[\eta]}P\}\big) = \frac{1}{2}\partial^2 H/\partial_{[\alpha']}P\partial_{[\beta]}P$; corresponding Riemannian element of the interval $d\sigma_W^2 = \sum_{\alpha,\beta} g^{\alpha\beta}d^{[\alpha]}M\otimes d^{[\beta]}M$. As an example, we give the explicit form of the metric coefficients for simplest case $m=3$, $N=2$:

$$g^{11} = \frac{{}^{[1]}M_{x_1}^2 + {}^{[1]}M_{x_2}^2}{{}_{[2]}P}, \qquad g^{12} = g^{21} = -{}_{[1]}P\frac{{}^{[1]}M_{x_1}^2 + {}^{[1]}M_{x_2}^2}{{}_{[2]}P^2},$$

$$g^{22} = {}_{[1]}P^2 \frac{{}^{[1]}M_{x_1}^2 + {}^{[1]}M_{x_2}^2}{{}_{[2]}P^3} + 2\frac{du({}_{[2]}P)}{d_{[2]}P} + {}_{[2]}P\frac{d^2u({}_{[2]}P)}{d_{[2]}P^2},$$

and at the same time the determinant $\det\bigl(g^{\alpha\beta}\bigr) = |{}^{[1]}M_{\mathbf{x}}|^2{}_{[2]}P^{-1}(2u' + {}_{[2]}Pu'') \not\equiv 0$. It should be noted that the above values $g^{\alpha\beta}$ (as well as their analogues for $m = 4$) do not depend directly on the Monge potentials (new "config variables"). This significantly simplifies the further consideration of geometrodynamic properties vortex motion of a hydrodynamic medium.

3. Canonical connections of Hamiltonian space and geodesic equations

An N-line connection on T^*Y_m is characterized by d-tensor fields

$$D\Gamma(N) = \bigl(H^\alpha_{\beta\gamma}, C_\alpha^{\beta\gamma}\bigr),$$

that is, a system of generalized Christoffel coefficients (H, C), which in the general case are functions of Monge potentials ${}^{[\mu]}M$ and canonically conjugate to them pseudo-impulses ${}_{[\mu]}P$.

Pay attention to the specific dependency structure only from the momenta of the components of the fundamental tensor $g^{\alpha\beta} = g^{\alpha\beta}({}_{[\mu]}P)$.

This will result in cancellation coefficients

$$H^\alpha_{\beta\gamma} \equiv \frac{1}{2}g^{\alpha\eta}(\delta_\zeta g_{\eta\gamma} + \delta_\gamma g_{\beta\zeta} - \delta_\zeta g_{\beta\gamma}), \qquad \delta_\mu = \partial/\partial^{[\mu]}M + N_{\mu\nu}\partial/\partial_{[\mu]}P,$$

$$N_{\mu\nu} = \frac{1}{4}\{g_{\mu\nu}, H\} - \frac{1}{4}\bigl(g_{\mu\eta}\partial^2 H/\partial^{[\nu]}M\partial_{[\eta]}P + g_{\nu\eta}\partial^2 H/\partial^{[\mu]}M\partial_{[\eta]}P\bigr)$$

– are the nonlinear connection coefficients of the Hamiltonian space W^m.

Thus, horizontal trajectories of N-linear connection D are described by a system of differential equations

$$d^{2\,[\alpha]}M/dt^2 = d_{[\alpha]}P/dt - N_{\mu\alpha}d^{[\mu]}M/dt = 0.$$

From a physical point of view, more interesting is the analysis of vertical trajectories (including ${}^{[\alpha]}M_0 \in Y_m$) with respect to the considered N-linear connection D, characterized by a system of differential equations that are analogues of the Euler–Lagrange equations:

$$\frac{d^2{}_{[\alpha]}P}{dt^2} - C_\alpha^{\beta\gamma}\bigl({}^{[\alpha]}M, \{{}_{[\eta]}P\}\bigr)\Big|_{{}^{[\alpha]}M={}^{[\alpha]}M_0} \frac{d_{[\beta]}P}{dt}\frac{d_{[\gamma]}P}{dt} = 0, \tag{3}$$

$$C_\alpha^{\beta\gamma} = -\frac{1}{2}g_{\alpha\zeta}\left(\frac{\partial g_{\zeta\gamma}}{\partial_{[\beta]}P} + \frac{\partial g_{\beta\eta}}{\partial_{[\gamma]}P} - \frac{\partial g_{\eta\gamma}}{\partial_{[\eta]}P}\right).$$

Let us present the values of some generalized Christoffel coefficients $C_\alpha^{\beta\gamma}$:

$$C_1^{11} = -\frac{{}_{[1]}P|{}^{[1]}M_{\mathbf{x}}|^2}{{}_{[2]}P^4} - \frac{{}_{[1]}P^2|{}^{[1]}M_{\mathbf{x}}|^2}{{}_{[2]}P^5}, \qquad C_1^{21} = \frac{|{}^{[1]}M_{\mathbf{x}}|^2}{{}_{[2]}P^3} + \frac{{}_{[1]}P^2|{}^{[1]}M_{\mathbf{x}}|^2}{{}_{[2]}P^5},$$

$$C_1^{22} = -\frac{2{}_{[1]}P|{}^{[1]}M_{\mathbf{x}}|^2}{{}_{[2]}P^4} + \frac{{}_{[1]}P^{[1]}M_{\mathbf{x}}}{2{}_{[2]}P^2}\left(\frac{3^{[2]}M_{\mathbf{x}}}{{}_{[2]}P^2} - \frac{3{}_{[2]}P^{2\,[2]}M_{\mathbf{x}} + 3{}_{[1]}P^{2\,[1]}M_{\mathbf{x}}}{{}_{[2]}P^4} + \right.$$

$$+3u''({}_{[2]}P)+{}_{[2]}Pu'''({}_{[2]}P)\Big)-\frac{{}^{[1]}M^2_{\mathbf{x}}}{2{}_{[2]}P^3},$$

$$C^{12}_1=\frac{{}^{[1]}M^2_{\mathbf{x}}}{2{}_{[2]}P^3}+\frac{2{}_{[1]}P^{2[1]}M^2_{\mathbf{x}}}{{}_{[2]}P^5}-\frac{{}_{[1]}P^{[1]}M_{\mathbf{x}}}{{}_{[2]}P^2}\Big(\frac{3^{[2]}M_{\mathbf{x}}}{{}_{[2]}P^2}-$$

$$-\frac{3{}_{[2]}P^{2[2]}M_{\mathbf{x}}+3{}_{[1]}P^{2[1]}M_{\mathbf{x}}}{{}_{[2]}P^4}+3u''({}_{[2]}P)+{}_{[2]}Pu'''({}_{[2]}P)\Big).$$

The above equations (3) can be considered as the basic equations when studying the stability of the dynamics of a vortex fluid flow, initially described by the system of Euler equations. The main information contained in their decisions relates to the shape of the geodetic trajectory in "momentum" space (in fact, on the manifold of densities of scalar flow characteristics). Of particular interest are closed trajectories that are naturally associated with periodic hydrodynamic structures of various scales.

4. Deviation of geodesics on Monge manifolds and its connection with the evolution of coherent hydrodynamic systems

The question naturally arises about the stability of periodic orbits in momentum space. For research deviations from the geodetic motion described by the equation (3), we represent

$$_{[\alpha]}P={}_{[\alpha]}P_0+\epsilon_{[\alpha]}\Pi+O(\epsilon^2), \tag{4}$$

where ${}_{[\alpha]}P_0$ — solution equation (3), ϵ — small parameter, ${}_{[\alpha']}\Pi(t)$ — magnitude of deviation from the exact solution ${}_{[\alpha]}P_0$. Let's substitute expression (4) into UEL (3):

$$0=\frac{d^2{}_{[\alpha]}P_0}{dt^2}-C^{\beta\gamma}_{\alpha}\big({}^{[\alpha]}M,\{{}_{[\eta]}P_0\}\big)\Big|_{{}^{[\alpha]}M={}^{[\alpha]}M_0}\frac{d_{[\beta]}P_0}{dt}\frac{d_{[\gamma]}P_0}{dt}+$$

$$+\epsilon\bigg(\frac{d^2{}_{[\alpha]}\Pi}{dt^2}-2C^{\beta\gamma}_{\alpha}\big({}^{[\alpha]}M_0,\{{}_{[\eta]}P_0\}\big)\frac{d_{[\beta]}P_0}{dt}\frac{d_{[\gamma]}\Pi}{dt}-$$

$$-C^{\beta\gamma}_{\alpha,\eta}\big({}^{[\alpha]}M_0,\{{}_{[\eta]}P_0\}\big)\frac{d_{[\beta]}P_0}{dt}\frac{d_{[\gamma]}P_0}{dt}{}_{[\eta]}\Pi(t)\bigg)+O(\epsilon^2).$$

Transforming the expression in brackets with the factor ϵ, we obtain an analogue of the Jacobi equation (*Jacobi-Cartan equation*) for the deviation vector with components $\{{}_{[\alpha]}\Pi\}$:

$$\frac{D^2_p({}_{[\alpha]}\Pi)}{dt^2}+\left(\frac{d_{[\beta]}P_0}{dt}\right)\left(\frac{d_{[\gamma]}P_0}{dt}\right)({}_{[\eta]}\Pi)S^{\beta\gamma\eta}_{\alpha}=0, \tag{5}$$

where : $D_{p[\alpha]}Z/dt\equiv d_{[\alpha]}Z/dt-C^{\beta\gamma}_{\alpha}\big({}^{[\alpha]}M_0,\{{}_{[\eta]}P\}\big)_{[\beta]}Z(d_\gamma P_0/dt)$, $S^{\beta\gamma\eta}_{\alpha}$ — d– tensor curvature trajectories :

$$S^{\beta\gamma\eta}_{\alpha}=\frac{\partial C^{\beta\gamma}_{\alpha}}{\partial_{[\eta]}P}-\frac{\partial C^{\beta\eta}_{\alpha}}{\partial_{[\gamma]}P}+C^{\mu\beta}_{\alpha}C^{\beta\eta}_{\mu}-C^{\mu\eta}_{\alpha}C^{\beta\gamma}_{\mu}.$$

Equation (5) describes the evolution of the vector of deviations from the geodesic motion and when considering the (λ,ϵ)-congruence (λ is an affine parameter along the streamline, proportional to time t) closed trajectories, one can trace the change in density characteristics

$(\rho, \rho_{[3]}M, \rho s)$ hydrodynamic structure, containing this congruence. At the same time, it is not assumed that the system is strictly limited in the spatial sense, that is, this system has non-local correlation properties (which is typical for coherent structures of various genesis). If the solution of the Jacobi–Cartan system has stable limit cycles, then the system has a set of certain (quasi)stationary states (associated with given cycles)

Consider a simple special case of the Jacobi–Cartan equation (5), when it is assumed that the quantities ${}^{[1,2]}M_{\mathbf{x}} = {}^{[1,2]}\widetilde{M}$ are permanent. In this case, the connectivity coefficients $C_\alpha^{\beta\gamma}$ and the components of the curvature tensor $S_\alpha^{\beta\gamma\eta}$ depend only on the variables ${}_{[\alpha]}P_0$ (for $m = 3$):

$$C_1^{11} = -\frac{{}_{[1]}P_0|{}^{[1]}\widetilde{M}|^2}{{}_{[2]}P_0^4} - \frac{{}_{[1]}P_0^2|{}^{[1]}\widetilde{M}|^2}{{}_{[2]}P_0^5}, \quad \text{i t. d.}$$

In the case under consideration, (5) takes the form of a 2nd order ODE system:

$$\frac{d^2({}_{[\alpha]}\Pi)}{dt^2} + \mathcal{K}_1\big({}_{[1]}P_0, {}_{[2]}P_0\big)\frac{d({}_{[\alpha]}\Pi)}{dt} + \mathcal{K}_2\big({}_{[1]}P_0, {}_{[2]}P_0\big){}_{[\alpha]}\Pi +$$

$$+\frac{d({}_{[\beta]}P_0)}{dt}\frac{d({}_{[\gamma]}P_0)}{dt}S_\alpha^{\beta\gamma\eta}\big({}_{[1]}P_0, {}_{[2]}P_0\big) \cdot {}_{[\eta]}\Pi = 0.$$

Considering this system as an equation for the vector deviation variable ${}_{[\alpha]}\Pi$ with “frozen” coefficients, one can see an analogy with the system of equations (with dissipative terms) describing the dynamics a set of coupled oscillators (it is possible to find the conditions for the occurrence of a self-oscillatory mode). If we additionally set $P_2 = const$, then for the simplest case $m = 3$ we arrive at the case of one ODE for variable ${}_{[1]}P(\equiv \rho^{[3]}M)$; its solution is oscillations increasing/decreasing in amplitude.

5. Conclusion

The use of the geometric apparatus for analyzing hydrodynamic flows makes it possible to identify very significant aspects of their evolutions associated with the formation and decay of large-scale vortex systems (coherent structures).

Determination of local tendencies of congruence of geodetic lines (density, entropy) towards closure/intersection or divergence gives a scenario for the development of behavior of the entire observed system. Application of the formalism of Hamiltonian geometry on Monge manifolds allows you to fairly transparently and deeply analyze the situation in hydrodynamic flow.

Список литературы

[1] Arnold V.I., Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications a l’hydrodynamique des fluides parfaits // Ann. Inst. Fourier (Grenoble). V. 16, P. 319–361. 1966.

[2] Casetti L., Pettini M., Cohen E.G.D., Geometric approach to Hamiltonian dynamics and statistical mechanics // Phys. Rep. V. 337, P. 237–341. 2000.

[3] Misiołek, G., Stability of flows of ideal fluids and the geometry of the group of diffeomorphisms // Indiana Univ. Math. J. V. 42, P. 215Г±235.1993.

[4] Zeitlin V., On the structure of phase-space, Hamiltonian variables and statistical approach to the description of two-dimensional hydrodynamics and magnetohydrodynamics // J. Physics A. V. 25, P. L171–L175. 1992.

[5] Clementi C., Pettini M., A geometric interpretation of integrable motions // Celes. Mech. Dyn. Astron. V. 84, P. 263–281. 2002.

[6] Chueshov I.D., Introduction to the theory of infinite–dimensional dissipative systems. Kharkiv: ACTA Scientific Publishing House, 2002.

[7] Vasylkevych S., Marsden J.E., The Lie–Poisson structure of the Euler of an ideal fluid // Dynamics of PDE. V. 2, ПЪ 4, P. 281–300. 2005.

[8] Kambe T., Geometrical theory of two–dimensional hydrodynamics with special reference to a system of point vortices // Fluid Dyn. Res. V. 33, P. 223–249. 2003.

[9] Holm D., Marsden J.E., Ratiu T., Weinstein A., Nonlinear stability of fluid and plasma equilibria // Phys. Reports. V. 123, ПЪ 1&2, P. 1–116. 1985.

[10] Nakamura F., Hattori Y., Kambe T., Geodesics and curvature of a group of diffeomorphisms and motion of an ideal fluid // J. Physics A. V. 25, P. L45–L50. 1992.

[11] Amari S., Differential–geometrical methods in statistics. Berlin–Heidelberg–Hong Kong: Springer–Verlag. 1985. 296 p.

[12] Weinhold F., Classical and geometrical theory of chemical and phase thermodynamics. Hoboken, New Jersey: John Wiley and Sons. 2009. 490 p.

[13] van Saarloos W., Bedeaux D., Mazur P., Hydrodynamics for an ideal fluid: Hamiltonian formalism and Liouville equiation // Physica A. V. 107A. P. 109–125. 1981.

[14] Seliger R.L., Whitham G.B., Variational principles in continuum mechanics // Proc. Roy. Soc. V. A305. P. 1–25. 1968.

[15] Miron R., The geometry of higher–order Hamilton spaces. Applications to Hamiltonian mechanics. Dordrecht – Boston – London: Kluwer Academic Publishers. 2003.

[16] de Donder T., Theorie invariantive du calcul des variations. Paris: Gauthier–Villars. 1935.

[17] Weyl H., Geodesic fields in the calculus of variation for multiple integrals // Ann. Math. V. 6. P. 607–629. 1935.

[18] Neagu M., From Euler–Lagrange equations to canonical nonlinear connections // Archiv. Mathematicum. V. 42. P. 255–263. 2006.