Research Article

Kronecker-Pauli Operators

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In quantum mechanics, there are some classical bases which are generalized to higher-dimensional matrices. So, it is useful to express their corresponding operators using the Dirac Bra and Ket. In this paper, to express the corresponding operators, we review the Kronecker-Pauli matrices and how to construct them for an N-dimensional system, with N a prime integer, N>2. Then, we give the expression of the Kronecker-Pauli operators and show that their matrices with respect to the standard basis fulfill the conditions to form a set of Kronecker-Pauli matrices.

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1. Introduction

In quantum mechanics, some classical bases are generalized to higher-dimensional matrices. For example, the Gell-Mann matrices are generalized to any dimension without knowing the corresponding operators in the standard basis (see, for example[1]). Operators are easier to use than higher-dimensional matrices, which has led to the construction of the Gell-Mann operators by using the Dirac bra and ket (see, for example[2][3]). So, we study, in this paper, what operators whose matrices in the standard basis are the matrices in a set of Kronecker-Pauli matrices. Kronecker-Pauli matrices (KPMs) are extensions of the Pauli strings, studied in[4].

Knowing that the Kronecker product of sets of KPMs is a set of KPMs[5], we study only the case of N-dimensional, where N is a prime integer, $N \geq 3$.

By examining the sets of KPMs, we will construct the Kronecker-Pauli operators. Then, as an example, we will construct a 5×5- KPMs from the Kronecker-Pauli operators. However, we know that there are at least two sets of 5×5- KPMs. That will engage a discussion.

The paper is organized as follows. In the first section, we review the sets of KPMs and how to construct them. In the second section, we give the expression of the Kronecker-Pauli operators. We finish the paper with a discussion and conclusion.

2. Sets of Kronecker-Pauli matrices

Definition 1

For n integer $n > 1$, let us define a set of $n \times n$- KPMs as a family $(\prod_k)_{0 \le k \le n^2-1}$ of n^2 matrices which satisfy the following properties[5]:

i. $\mathbf{S}_{n \otimes n} = \frac{1}{n} \sum_{k=0}^{n^2-1} \prod_k \otimes \prod_k$ is the $n \otimes n$ swap operator;
ii. ${\prod_k}^{\dagger} = \prod_k$, for $\left(0 \le k \le n^2 - 1\right)$, (hermiticity);
iii. ${\prod_k}^{2} = I_n$, for $\left(0 \le k \le n^2 - 1\right)$, (square root of the unit);
iv. $\mathrm{Tr}\left({\prod_k}^{\dagger} \prod_j\right) = n\delta_{kj}$, for $\left(0 \le k, \ \ j \le n^2 - 1\right)$, (orthogonality).

where δ_{kj} is the Kronecker symbol.

To construct such a family, the concept of the inverse-symmetric matrix is useful[5].

Definition 2

Let us call an inverse-symmetric matrix an invertible complex matrix $\mathbf{A} = \left(A_j^i\right)$ such that $A_i^j = \frac{1}{A_j^i}$ if $A_j^i \neq 0$.

Proposition 1

For any $n \times n$ inverse-symmetric matrix $\mathbf{A}$, with only n non-zero elements, $\mathbf{A}^2 = \mathbf{I}_n$ is the unit matrix.

Consider the case of N-dimensional matrices where N is a prime integer. According to this proposition 1, the choice of inverse-symmetric matrices ensures property iii) of definition 1. To ensure property ii) of hermiticity, consider a family of inverse-symmetric matrices whose elements are the N-th roots of the unit. The following proposition[4] ensures properties i) and iv).

In the following proposition, the matrix of an operator is its matrix in the standard basis $(|\,0\rangle\,, \ \ |\,1\rangle\,, \ \ |\,2\rangle\,, \ \ldots, \ |\,N-1\rangle \ )$.

Proposition 2

Let $\mathbf{P}_1$, $\mathbf{P}_2$, ..., $\mathbf{P}_N$ be operators whose matrices are symmetric permutation matrices with only one unit in the diagonal.

$\prod_0 = \mathbf{P}_1$ and $\prod_1, \prod_2, \dots, \prod_{N-1}$ are operators whose matrices are obtained by replacing the "ones" in $\prod_0 = \mathbf{P}_1$ by the N-th roots of unity while keeping that they are inverse-symmetrics. We do the same to the operators $\mathbf{P}_2, \dots, \mathbf{P}_N$ in order to have the operators

$\prod_N = \mathbf{P}_2$ and $\prod_{N+1}, \prod_{N+2}, \dots, \prod_{2N-1}$

....

$\prod_{N^2-N} = \mathbf{P}_N$ and $\prod_{N^2-N+1}, \prod_{N^2-N+2}, \dots, \prod_{N^2-1}$

whose matrices are inverse-symmetrics.

If

1. The sum $\mathbf{P}_1 + \mathbf{P}_2 + \ldots + \mathbf{P}_N$ is equal to the operator whose matrices is the $N \times N$ ones matrix;
2. For any $l \in \{0, 1, 2, \ldots, N-1\}$, for any $k, \; j \in \{lN+1, lN+2, \ldots, lN+N-1\}$, for any two places in a $N \times N$-matrix, non-symmetrics with respect to the diagonal where the elements of $\prod_k$ are $e^{\frac{2i\pi p_k}{N}}$ and $e^{\frac{2i\pi r_k}{N}}$ and the elements of $\prod_j$ are $e^{\frac{2i\pi p_j}{N}}$ and $e^{\frac{2i\pi r_j}{N}}$ such that

$$e^{\frac{2i\pi(r_k+p_k)}{N}} \neq e^{\frac{2i\pi(r_j+p_j)}{N}}$$

Then

$\mathbf{S}_{N\otimes N} = \frac{1}{n}\sum_{k=0}^{n^2-1}\prod_k \otimes \prod_k$ is the $N \otimes N$ swap operator and $\mathrm{Tr}\left(\prod_k^{\dagger}\prod_j\right) = N\delta_{kj}$.

The following example was an example from[4], constructed following the hypotheses of Proposition 2 above. The property i) of definition 1 is checked with the help of SCILAB software.

Example 1

$$\chi_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \chi_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta & 0 & 0 \\ 0 & \eta^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 0 & 0 & \eta^3 & 0 \end{pmatrix}, \chi_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta^2 & 0 & 0 \\ 0 & \eta^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^4 \\ 0 & 0 & 0 & \eta & 0 \end{pmatrix},$$

$$\chi_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta^3 & 0 & 0 \\ 0 & \eta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & \eta^4 & 0 \end{pmatrix}, \chi_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta^4 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^3 \\ 0 & 0 & 0 & \eta^2 & 0 \end{pmatrix},$$

$$\chi_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \chi_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & \eta \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta^2 & 0 \\ 0 & 0 & \eta^3 & 0 & 0 \\ \eta^4 & 0 & 0 & 0 & 0 \end{pmatrix}, \chi_8 = \begin{pmatrix} 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta^4 & 0 \\ 0 & 0 & \eta & 0 & 0 \\ \eta^3 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\chi_9 = \begin{pmatrix} 0 & 0 & 0 & 0 & \eta^3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta & 0 \\ 0 & 0 & \eta^4 & 0 & 0 \\ \eta^2 & 0 & 0 & 0 & 0 \end{pmatrix}, \chi_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 & \eta^4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta^3 & 0 \\ 0 & 0 & \eta^2 & 0 & 0 \\ \eta & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\chi_{11} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \chi_{12} = \begin{pmatrix} 0 & 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 0 & 1 & 0 & 0 \\ \eta^4 & 0 & 0 & 0 & 0 \\ 0 & \eta^3 & 0 & 0 & 0 \end{pmatrix}, \chi_{13} = \begin{pmatrix} 0 & 0 & 0 & \eta^2 & 0 \\ 0 & 0 & 0 & 0 & \eta^4 \\ 0 & 0 & 1 & 0 & 0 \\ \eta^3 & 0 & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \end{pmatrix},$$

$$\chi_{14} = \begin{pmatrix} 0 & 0 & 0 & \eta^3 & 0 \\ 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 1 & 0 & 0 \\ \eta^2 & 0 & 0 & 0 & 0 \\ 0 & \eta^4 & 0 & 0 & 0 \end{pmatrix}, \chi_{15} = \begin{pmatrix} 0 & 0 & 0 & \eta^4 & 0 \\ 0 & 0 & 0 & 0 & \eta^3 \\ 0 & 0 & 1 & 0 & 0 \\ \eta & 0 & 0 & 0 & 0 \\ 0 & \eta^2 & 0 & 0 & 0 \end{pmatrix},$$

$$\chi_{16} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \chi_{17} = \begin{pmatrix} 0 & \eta & 0 & 0 & 0 \\ \eta^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \eta^3 & 0 & 0 \end{pmatrix}, \chi_{18} = \begin{pmatrix} 0 & \eta^2 & 0 & 0 & 0 \\ \eta^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \eta & 0 & 0 \end{pmatrix},$$

$$\chi_{19} = \begin{pmatrix} 0 & \eta^3 & 0 & 0 & 0 \\ \eta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \eta^4 & 0 & 0 \end{pmatrix}, \chi_{20} = \begin{pmatrix} 0 & \eta^4 & 0 & 0 & 0 \\ \eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \eta^2 & 0 & 0 \end{pmatrix}.$$

$$\chi_{21} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \chi_{22} = \begin{pmatrix} 0 & 0 & \eta & 0 & 0 \\ 0 & 0 & 0 & \eta^2 & 0 \\ \eta^4 & 0 & 0 & 0 & 0 \\ 0 & \eta^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \chi_{23} = \begin{pmatrix} 0 & 0 & \eta^2 & 0 & 0 \\ 0 & 0 & 0 & \eta^4 & 0 \\ \eta^3 & 0 & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\chi_{24} = \begin{pmatrix} 0 & 0 & \eta^3 & 0 & 0 \\ 0 & 0 & 0 & \eta & 0 \\ \eta^2 & 0 & 0 & 0 & 0 \\ 0 & \eta^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \chi_{25} = \begin{pmatrix} 0 & 0 & \eta^4 & 0 & 0 \\ 0 & 0 & 0 & \eta^3 & 0 \\ \eta & 0 & 0 & 0 & 0 \\ 0 & \eta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

with $\eta = e^{\frac{2i\pi}{5}}$.

3. Kronecker-Pauli Operators

The form of a Kronecker-Pauli matrix in a set of KPMs suggests the following definition of a Kronecker-Pauli operator. For a prime integer N, our goal is to construct a system of operators whose matrices in the standard basis form a set of $N \times N$- KPMs.

Definition 2

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Let us define a Kronecker-Pauli operator as the operator

$$\prod_{\sigma,l,n,N} = \sum_{k=0}^{N-1} e^{\frac{2i\pi}{N}(k-l)n} |\sigma(\ k)\rangle \langle\ k|$$

where N is a prime integer, $n = 0, 1, 2, \ldots,\ N-1,\ \ l \in \{0, 1, 2, \ldots,\ N-1\}$, σ is a symmetric permutation on the set $\{0, 1, 2, \ldots,\ N-1\}$,

$$0 \leq \sigma(k) = -k + 2l[N] < N$$

It is straightforward to check that σ is a permutation on $\{0, 1, 2, \ldots,\ N-1\}$ such that $\sigma(\sigma(k)) = k$, for any $k \in \{0, 1, 2, \ldots,\ N-1\}$, $\sigma(l) = l$ and $\sigma(k) \neq k$, if $k \neq l$.

The operator can also be written as the following

$$\prod_{\sigma,l,n,N} = \sum_{k=0}^{N-1} e^{\frac{2i\pi}{N}(k-l)n} |\ k\rangle \langle\sigma(\ k)| \quad = \sum_{k=0}^{N-1} e^{\frac{2i\pi}{N}(k-l)n} |\ k\rangle \langle\ -k + 2l[N]|$$

In order for the matrices of the operators in this system of operators to form a set of KPMs, we have to show that these matrices satisfy the hypotheses of Proposition 2 above. Let us write it as a proposition.

Proposition 3

The N^2 matrices, in the standard basis, of the system of Kronecker-Pauli operators $\left(\prod_{\sigma,l,n,N}\right)_{0\leq l,n\leq N-1}$ is a set of $N \times N$- KPMs.

Proof

For $n = 0$, $P_{l+1} = \prod_{\sigma,l,0,N}$ is a symmetric permutation matrix with only one unit, at l-th row l-th column, on the diagonal. Let us show that $P_1 + P_2 + \ldots + P_N$ is the $N \times N$ ones matrix. To do so, let us take $j,\ m \in \{0, 1, 2, \ldots,\ N-1\}$, with $j \neq m$, and show that there is $l \in \{0, 1, 2, \ldots,\ N-1\}$, $|\ j\rangle \langle\ m|$ is a term of the operator $P_{l+1} = \prod_{\sigma,l,0,N}$.

To finish, show that if $|\ j\rangle \langle\ m|$ is a term of an operator $P_{l'+1} = \prod_{\sigma,l',0,N}$, then $l = l'$. If $|\ j\rangle \langle\ m|$ is both a term of $P_{l+1} = \prod_{\sigma,l,0,N}$ and $P_{l'+1} = \prod_{\sigma,l',0,N}$, then $j = -m + 2l[N]$ and $j = -m + 2l[N]$. Thus, N divides $l - l'$ and that implies that $l = l'$.

If $j + m$ is even, there is $l \in \{0, 1, 2, \ldots,\ N-1\}$, $j + m = 2l$.

If $j + m$ is odd, for the case $0 \leq \frac{j+m+N}{2} < N$, let $l = \frac{j+m+N}{2}$, $j + m = 2l - N$. For the case $N \leq \frac{j+m+N}{2}$, $0 < j + m - N < N$, let $l = \frac{j+m-N}{2}$, $j + m = 2l + N$.

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We have seen that for any case m is of the form $m = -j + 2l\ [N]$, that is $|\ j\rangle\langle\ m|$ is a term of the operator $P_{l+1} = \prod_{\sigma,l,0,N}$.

Now, let us prove that for $m,\ n\ \in \{1,\ 2,\ \ldots,\ N-1\}$, $m < n$, for $j,\ k \in \{0,\ 1,\ 2,\ \ldots,\ N-1\}$, with $j \neq k$, $\sigma(j) \neq k$, $e^{\frac{2i\pi}{N}[(j-l)+(k-l)]n} \neq e^{\frac{2i\pi}{N}[(j-l)+(k-l)]m}$. To do so, let us suppose the contrary, that is suppose that there are $m,\ n\ \in \{1,\ 2,\ \ldots,\ N-1\}$, $m < n$, for $j,\ k \in \{0,\ 1,\ 2,\ \ldots,\ N-1\}$, with $j \neq k$, $\sigma(j) \neq k$, $e^{\frac{2i\pi}{N}[(j-l)+(k-l)]n} = e^{\frac{2i\pi}{N}[(j-l)+(k-l)]m}$. Then, $(j + k - 2k)(n - m) = 0[N]$. As N is a prime integer, thus after the Euclid lemma, N divides $(j + k - 2k)$, that is $k = \sigma(j)$. It is a contradiction.

The proposition is proved.

Example 2

For $N = 3, n = 1, \sigma(0) = 2, \sigma(1) = 1, \sigma(2) = 0$

$$\prod_{\sigma,1,1,3} = e^{-\frac{2i\pi}{3}}\ |\ 2\rangle\langle\ 0| + |\ 1\rangle\langle\ 1|\ \ +\ \ e^{\frac{2i\pi}{3}}\ |\ 0\rangle\langle\ 2|$$

whose matrix in the standard basis $(|\ 0\rangle,\ \ |1\ \rangle,\ \ |2\ \rangle\ )$ is

$$\prod_{\sigma,1,1,3} = \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix}$$

with $\omega = e^{\frac{2i\pi}{3}}$.

Example 3

For $N = 5$, $\sigma(0) = 0$, $\sigma(1) = 4$, $\sigma(2) = 3$, $\sigma(3) = 2$, $\sigma(4) = 1$, for $n = 0,\ 1,\ 2,\ 3,\ 4$ we have respectively the following five matrices which are, in the standard basis $(|\ 0\rangle,\ \ |1\ \rangle,\ \ |2\ \rangle,\ \ |3\ \rangle,\ \ |4\ \rangle\ )$, matrices of the corresponding Kronecker-Pauli operators,

$$\prod_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \prod_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^4 \\ 0 & 0 & 0 & \eta^3 & 0 \\ 0 & 0 & \eta^2 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \end{pmatrix}, \prod_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^3 \\ 0 & 0 & 0 & \eta & 0 \\ 0 & 0 & \eta^4 & 0 & 0 \\ 0 & \eta^2 & 0 & 0 & 0 \end{pmatrix},$$

$$\prod_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 0 & 0 & \eta^4 & 0 \\ 0 & 0 & \eta & 0 & 0 \\ 0 & \eta^3 & 0 & 0 & 0 \end{pmatrix}, \prod_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & \eta^2 & 0 \\ 0 & 0 & \eta^3 & 0 & 0 \\ 0 & \eta^4 & 0 & 0 & 0 \end{pmatrix}.$$

For $\sigma(0) = 2$, $\sigma(1) = 1$, $\sigma(2) = 0$, $\sigma(3) = 4$, $\sigma(4) = 3$, for $n = 0,\ 1,\ 2,\ 3,\ 4$ we have respectively

$$\prod_6 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \prod_7 = \begin{pmatrix} 0 & 0 & \eta & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \eta^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^3 \\ 0 & 0 & 0 & \eta^2 & 0 \end{pmatrix}, \prod_8 = \begin{pmatrix} 0 & 0 & \eta^2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \eta^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & \eta^4 & 0 \end{pmatrix},$$

$$\prod_9 = \begin{pmatrix} 0 & 0 & \eta^3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \eta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^4 \\ 0 & 0 & 0 & \eta & 0 \end{pmatrix}, \prod_{10} = \begin{pmatrix} 0 & 0 & \eta^4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 0 & 0 & \eta^3 & 0 \end{pmatrix}.$$

For $\sigma(0) = 4, \sigma(1) = 3, \sigma(2) = 2, \sigma(3) = 1, \sigma(4) = 0$, for $n = 0,\ 1,\ 2,\ 3,\ 4$ we have respectively

$$\prod_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \prod_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 & u^2 \\ 0 & 0 & 0 & \eta & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \eta^4 & 0 & 0 & 0 \\ \eta^3 & 0 & 0 & 0 & 0 \end{pmatrix}, \prod_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 & \eta^4 \\ 0 & 0 & 0 & \eta^2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \eta^3 & 0 & 0 & 0 \\ \eta & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\prod_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & \eta^3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \eta^2 & 0 & 0 & 0 \\ \eta^4 & 0 & 0 & 0 & 0 \end{pmatrix}, \prod_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 & \eta^3 \\ 0 & 0 & 0 & \eta^4 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \\ \eta^2 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For $\sigma(0) = 1, \sigma(1) = 0, \sigma(2) = 4, \sigma(3) = 3, \sigma(4) = 2$, for $n = 0,\ 1,\ 2,\ 3,\ 4$ we have respectively

$$\prod_{16} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \prod_{17} = \begin{pmatrix} 0 & \eta^3 & 0 & 0 & 0 \\ \eta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \eta^4 & 0 & 0 \end{pmatrix},$$

$$\prod_{18} = \begin{pmatrix} 0 & \eta & 0 & 0 & 0 \\ \eta^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \eta^3 & 0 & 0 \end{pmatrix},$$

$$\prod_{19} = \begin{pmatrix} 0 & \eta^4 & 0 & 0 & 0 \\ \eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \eta^2 & 0 & 0 \end{pmatrix}, \prod_{20} = \begin{pmatrix} 0 & \eta^2 & 0 & 0 & 0 \\ \eta^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \eta & 0 & 0 \end{pmatrix}.$$

For $\sigma(0) = 3, \sigma(1) = 2, \sigma(2) = 1, \sigma(3) = 0, \sigma(4) = 4$, for $n = 0,\ 1,\ 2,\ 3,\ 4$ we have respectively

$$\prod_{21} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \prod_{22} = \begin{pmatrix} 0 & 0 & 0 & \eta^4 & 0 \\ 0 & 0 & \eta^3 & 0 & 0 \\ 0 & \eta^2 & 0 & 0 & 0 \\ \eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \prod_{23} = \begin{pmatrix} 0 & 0 & 0 & \eta^3 & 0 \\ 0 & 0 & \eta & 0 & 0 \\ 0 & \eta^4 & 0 & 0 & 0 \\ \eta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

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$$\prod_{24} = \begin{pmatrix} 0 & 0 & 0 & \eta^2 & 0 \\ 0 & 0 & \eta^4 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \\ \eta^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \prod_{25} = \begin{pmatrix} 0 & 0 & 0 & \eta & 0 \\ 0 & 0 & \eta^2 & 0 & 0 \\ 0 & \eta^3 & 0 & 0 & 0 \\ \eta^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

3. Discussion and Conclusion

Giving N a prime integer, $N > 2$, a set of $N \times N$- KPMs is well defined by N symmetric permutation matrices with on the diagonal only one entry is the unit, on other is the zero and inverse-symmetric matrices obtained in replacing the units on the N symmetric permutation matrices by N-th roots of the unit, but in respecting that the hypotheses of the Proposition 2 are satisfied. To construct the system of operators whose matrices in the standard basis form a set of $N \times N$- KPMs we have at first defined for each $l \in \{0,\ 1,\ 2,\ \ldots,\ N-1\}$ a symmetric permutation on $\{0,\ 1,\ 2,\ \ldots,\ N-1\}$. That gives only one set of $N \times N$- KPMs. However, for $N = 5$ there are at least two sets of $N \times N$- KPMs. That is due to the definition of the symmetric permutation for constructing the system of Kronecker-Pauli operators, only one set of $N \times N$- KPMs is obtained as the matrices in the standard basis. Let us call such a family of matrices a $N \times N$-Kronecker-Pauli basis.

Perhaps the other sets of $N \times N$- KPMs would be formed by the matrices of the constructed system of Kronecker-Pauli operators in other bases than the standard basis.

But the essential is obtaining a system of operators whose matrices in the standard basis fulfill the conditions to be as an $N \times N$-Kronecker-Pauli basis.

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References

1. ^*Rakotonirina C. Expression of a tensor commutation matrix in terms of the generalized Gell-Mann matric es. Int J Math Math Sci. 2007. doi:10.1155/2007/20672.*
2. ^*Kimura G. The Bloch Vector for N-Level Systems. Phys Lett A. 2003;314:339.*
3. ^*Bertlmann RA, Krammer P. Bloch vectors for qudits. J Phys A: Math Theor. 2008;41(23). doi:10.1088/1751-81 13/41/23/235303. arXiv:0806.1174 [quant-ph].*

4. [a], [b], [c] *Rakotonirina C. Constructing a Set of Kronecker-Pauli Matrices. Qeios. 2024. doi:10.32388/F5DGRH.2.*

5. [a], [b], [c] *Rakotonirina C, Rasamizafy S. 3x3-Kronecker-Pauli Matrices. Bull Math Stat Res. 2016;4(3):127-134.*

Declarations

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