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New computational methods using seventh derivative type for the solution of first order initial value problems

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Abstract

In this research, a class of implicit block methods of a seventh derivative type are examined through interpolation and collocation techniques using finite power series as the basis function. The discrete schemes, which are implicit two-point block methods, are obtained by carefully and unevenly choose collocation points that ensure better methods' stability via test. However, these schemes require seventh derivative functions unlike other existing numerical formulae. The new methods are found, investigated and proven to be convergent and A-stable. The implementation of methods is achieved by using Newton Raphson's method. Experiments show the efficiency and accuracy of the developed formulae on different class of first-order initial value problems, including SIR, growth models and Prothero-Robinson oscillatory problem and with comparison to such existing methods. In addition, it is observed that uneven and positioning of collocation points greatly influence the efficiency and accuracy of numerical methods.

Keywords: Seventh derivative functions, implicit block methods, Algorithm, numerical stability, interpolation and collocation.

1. Introduction

Stiff differential equations have been studied over the years with a view to developing robust numerical methods that will not only be robust but adequate. It is worthy to note that [1], first examined the best approach in terms of numerical methods to solving stiff ODEs. Several scholars have different definitions to this resounding area of research. Therefore, it can be defined as equations that are ill-conditioned.To unveil the nature of stiffness of the ill-conditioning and to motivate the need to formulate efficient numerical methods for stiff

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differential equations, consider the first order initial value problems of the form:

$$y' = f(x, y), \;\; a \leq x \leq b, \quad y(0) = y_0, \tag{1}$$

where, $x_n = x_0 + nh$, h is the step size. Also, a stiff system of equations is one for which $|\lambda_{max}|$ is enormous, so that either the stability or the error bound or both can only be assured by unreasonable restrictions on h (i.e., an excessively small h requiring too many steps to solve the initial value problem). Enormous here means, enormous relative to a scale which is $\frac{1}{b}$. Thus, an equation with $|\lambda_{max}|$ small may also be viewed as stiff if we must solve it for great values of time, where $f : [x_n, x_N] \times R^m \rightarrow R^m$ in (1) is continuous and differentiable; so that, f is assumed to satisfy the existence and uniqueness theorem within the interval of $[a, b]$; while stability is clearly necessary, it is not sufficient to obtain accurate solutions to stiff systems of ordinary differential equations. A phenomenon that is commonly observed is that when applied to stiff problems, many implicit methods do not seem to achieve the order of accuracy that is expected for the method. This phenomenon is called order reduction. Certainly, order reduction occurs with Runge-Kutta methods, but not backward differentiation formula methods. In addition, explicit methods fail on solving stiff ODEs as a result of step-size being restricted to maintain the potential accuracy of the methods. This problem is overcome by using appropriate implicit methods (see [2]). However, some of the famous numerical methods, among others are the Euler method by [3], linear multistep methods in [4] and Runge-Kutta methods in [5]. In addition, the methods mentioned above cannot solve difficult problems with stiff nature that arise in many fields of science and engineering. Hence, the need to develop more viable methods for approximation. Also, [6] formulated a diagonally implicit block backward differentiation formula for stiff IVPs. In [8, 11, 12, 14, 16, 17, 19], implicit linear block multistep methods for first-order stiff and non-stiff IVPs have been derived and implemented respectively. Interestingly, [21] also developed and implemented an implicit four-point hybrid block integrator on stiff models relating to some real-life situations with method near optimal as with other existing methods. Another implicit block methods have been considered for solving stiff IVPs using Chebyshev polynomial in [22, 23]. However, their methods depend on the perturbed collocation approximation with shifted Legendre polynomials as perturbation term.

More recently, are the applications of multi-derivatives block methods to first-order stiff initial value problems [24]. However, higher derivative methods have a general disadvantage of having to provide and evaluate derivatives of function thereby resulting to more functions evaluations. Therefore, this drawback could result to round-off errors in the global iterations if numerical methods are not sufficiently stable, that is, the numerical errors are not under check by the zero stability and consistency properties.

Consequently, [25] derived and implemented fourth derivative k-point block formula on first-order stiff IVPs through interpolation and collocation techniques. Similarly, [26] proposed a third derivative trigonometrically fitted block method of a low order 2 for solving Equation (1).

Others like [27], considered a family of third derivative multi-step methods for solving (1) and a class of continuous third derivative block methods of order $(k+3)$ for direct approximation of (1) has also been derived through interpolation and collocation techniques by [28]. For second derivative methods, [29, 30, 31] solved Equation (1) respectively.
Summarily, in this research, a class of seventh derivative implicit block methods are derived. They are a collection of discrete schemes of a first order function with seventh derivative type. The objectives are to derive higher-order derivative implicit block formulae which solves (1) directly with increased stability and reduced computational time using interpolation and collocation approach. The proposed methods require seven derivative functions unlike other numerical methods. This technique makes the methods unique, though have the burden of having to provide the aforementioned derivative functions, but the efficiency and accuracy of the proposed methods prove their significance. Test on numerical examples indicate that our derived formulae are viable on stiff IVPs.
Therefore, this research is organized as follows: section two gives the derivation of the proposed methods, section three shows the analysis of the numerical properties, section four presents the implementation strategy, section five shows the numerical experiment and section six displays the real-life application of methods, section seven presents conclusion and future research.

2. Derivation of the seventh derivative methods

We consider the power series polynomial of the form:

$$y(x) = \sum_{j=0}^{k+8} a_j x^j \tag{2}$$

with its derivatives given as:

$$y'(x) = \sum_{j=0}^{k+8} j a_j x^{j-1} = f(x, y) \tag{3}$$

$$y''(x) = \sum_{j=0}^{k+8} j(j-1) a_j x^{j-2} = g(x, y) \tag{4}$$

$$y'''(x) = \sum_{j=0}^{k+8} j(j-1)(j-2) a_j x^{j-3} = u(x, y) \tag{5}$$

$$y''''(x) = \sum_{j=0}^{k+8} j(j-1)(j-2)(j-3) a_j x^{j-4} = v(x, y) \tag{6}$$

$$\vdots$$

With the seventh derivative given as:

$$y^{(7)}(x) = \sum_{j=0}^{k+8} j(j-1)(j-2)(j-3)(j-4)(j-5)(j-6)a_j x^{j-7} = q(x,y) \tag{7}$$

Here, we define:
$y'_{n+i} = f_{n+i}, i = 0(1)k, y''_{n+i} = g_{n+i}, i = 1, \dots k, y'''_{n+i} = u_{n+i}, i = k, y^{(4)}_{n+i} = v_{n+i}, i = k, y^{(5)}_{n+i} = w_{n+i}, i = k, y^{(6)}_{n+i} = m_{n+i}, i = k, y^{(7)}_{n+i} = q_{n+i}, i = k$. Where $a_{j's} \in \mathbb{R}$ in Equations (2)–(7) are found using Gaussian elimination method. Therefore, Equation (2) and Equations (3)–(7) are then interpolated and collocated at x_n and $x_{n+l}, l = 0(1)k$ (where k is the step number and $k = 2$) to give the following system of equation using Maple 18 soft environment:

$$PX = Q \tag{8}$$

Where,

$$P = \begin{pmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 & x_n^{10} \\
0 & 1 & 2\,x_{n+l} & 3\,x_{n+l}^2 & 4\,x_{n+l}^3 & 5x_{n+l}^4 & 6x_{n+l}^5 & 7x_{n+l}^6 & 8x_{n+l}^7 & 9x_{n+l}^8 & 10x_{n+l}^9 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 2 & 6x_{n+k} & 12x_{n+k}^2 & 20x_{n+k}^3 & 30x_{n+k}^4 & 42x_{n+k}^5 & 56x_{n+k}^6 & 72x_{n+k}^7 & 90x_{n+k}^8 \\
0 & 0 & 0 & 6 & 24x_{n+k} & 60x_{n+k}^2 & 120x_{n+k}^3 & 210x_{n+k}^4 & 336x_{n+k}^5 & 504x_{n+k}^6 & 720x_{n+k}^7 \\
0 & 0 & 0 & 0 & 24 & 120x_{n+k} & 360x_{n+k}^2 & 840x_{n+k}^3 & 1680x_{n+k}^4 & 3024x_{n+k}^5 & 5040x_{n+k}^6 \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+k} & 2520x_{n+k}^2 & 6720x_{n+k}^3 & 15120x_{n+k}^4 & 30240x_{n+k}^5 \\
0 & 0 & 0 & 0 & 0 & 0 & 720 & 5040x_{n+k} & 20160x_{n+k}^2 & 60480x_{n+k}^3 & 151200x_{n+k}^4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5040 & 40320\,x_{n+1} & 181440x_{n+1}^2 & 604800x_{n+1}^3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5040 & 40320\,x_{n+k} & 181440x_{n+k}^2 & 604800x_{n+k}^3
\end{pmatrix}$$

$$X = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k+8} \end{pmatrix}, \quad Q = \left(y_n, f_{n+l}, f'_{n+k}, f^{(2)}_{n+k}, f^{(3)}_{n+k}, f^{(4)}_{n+k}, f^{(5)}_{n+k}, f^{(6)}_{n+1}, f^{(6)}_{n+k} \right)^\top$$

Where, $l = 0(1)k$.
Equation (8) is then solved for $a_{j's} \in \mathbb{R}, j = 0(1)k$ and substitution made into Equation (2) gives the

Linear Multi-step Method (LMM) of the form:

$$y(x_{n+\xi}) = \alpha_0(\xi)y_n \ + \ h\sum_{j=0}^{k}\beta_j(\xi)f_{n+j} \ + \ h^2\sum_{j=3}\beta_j(\xi)g_{n+k} \ + \ h^3\sum_{j=4}\beta_j(\xi)u_{n+k}$$

$$+h^4\sum_{j=5}\beta_j(\xi)v_{n+k} \ + \ h^5\sum_{j=6}\beta_j(\xi)w_{n+k} \ + \ h^6\sum_{j=7}\beta_j(\xi)m_{n+k} \ + \ h^7\sum_{j=8}^{9}\beta_j(\xi)\sum_{i=1}^{k}q_{n+i}, \tag{9}$$

Therefore, the parameters $\alpha_0(\xi)$ and $\beta_j(\xi)$ are obtained with $\xi = x - x_n$ as:

$$\alpha_0 = 1 \tag{10}$$

$$\beta_0 = \xi - \frac{3}{2}\frac{\xi^2}{h} + \frac{2}{3}\frac{\xi^3}{h^2} + \frac{7\,\xi^4}{8\,h^3} - \frac{63\,\xi^5}{40\,h^4} + \frac{7}{6}\frac{\xi^6}{h^5} - \frac{1}{2}\frac{\xi^7}{h^6} + \frac{33\,\xi^8}{256\,h^7} - \frac{43\,\xi^9}{2304\,h^8} + \frac{3\,\xi^{10}}{2560\,h^9} \tag{11}$$

$$\beta_1 = \frac{128\,\xi^3}{3\,h^2} - 112\,\frac{\xi^4}{h^3} + \frac{672\,\xi^5}{5\,h^4} - \frac{280\,\xi^6}{3\,h^5} + 40\,\frac{\xi^7}{h^6} - \frac{21}{2}\frac{\xi^8}{h^7} + \frac{14\,\xi^9}{9\,h^8} - \frac{1}{10}\frac{\xi^{10}}{h^9} \tag{12}$$

$$\beta_2 = \frac{3}{2}\frac{\xi^2}{h} - \frac{130\,\xi^3}{3\,h^2} + \frac{889\,\xi^4}{8\,h^3} - \frac{5313\,\xi^5}{40\,h^4} + \frac{553\,\xi^6}{6\,h^5} - \frac{79\,\xi^7}{2\,h^6} + \frac{2655\,\xi^8}{256\,h^7} - \frac{3541\,\xi^9}{2304\,h^8} + \frac{253\,\xi^{10}}{2560\,h^9} \tag{13}$$

$$\beta_3 = \frac{-5}{2}\xi^2 + 44\,\frac{\xi^3}{h} - \frac{441\,\xi^4}{4\,h^2} + \frac{525\,\xi^5}{4\,h^3} - 91\,\frac{\xi^6}{h^4} + 39\,\frac{\xi^7}{h^5} - \frac{1311\,\xi^8}{128\,h^6} + \frac{583\,\xi^9}{384\,h^7} - \frac{25\,\xi^{10}}{256\,h^8} \tag{14}$$

$$\beta_4 = 2\,\xi^2 h - \frac{45\,\xi^3}{2} + \frac{217\,\xi^4}{4\,h} - \frac{1281\,\xi^5}{20\,h^2} + \frac{133\,\xi^6}{3\,h^3} - 19\,\frac{\xi^7}{h^4} + \frac{639\,\xi^8}{128\,h^5} - \frac{853\,\xi^9}{1152\,h^6} + \frac{61\,\xi^{10}}{1280\,h^7} \tag{15}$$

$$\beta_5 = -\xi^2 h^2 + \frac{23\,\xi^3 h}{3} - \frac{419\,\xi^4}{24} + \frac{203\,\xi^5}{10\,h} - 14\,\frac{\xi^6}{h^2} + 6\,\frac{\xi^7}{h^3} - \frac{101\,\xi^8}{64\,h^4} + \frac{15\,\xi^9}{64\,h^5} - \frac{29\,\xi^{10}}{1920\,h^6} \tag{16}$$

$$\beta_6 = \frac{1}{3}\xi^2 h^3 - \frac{17\,\xi^3 h^2}{9} + 4\,\xi^4 h - \frac{109\,\xi^5}{24} + \frac{28\,\xi^6}{9\,h} - \frac{4}{3}\frac{\xi^7}{h^2} + \frac{45\,\xi^8}{128\,h^3} - \frac{181\,\xi^9}{3456\,h^4} + \frac{13\,\xi^{10}}{3840\,h^5} \tag{17}$$

$$\beta_7 = \frac{-1}{15}\xi^2 h^4 + \frac{14\,\xi^3 h^3}{45} - \frac{37\,\xi^4 h^2}{60} + \frac{41\,\xi^5 h}{60} - \frac{67\,\xi^6}{144} + \frac{1}{5}\frac{\xi^7}{h} - \frac{17\,\xi^8}{320\,h^2} + \frac{23\,\xi^9}{2880\,h^3} - \frac{\xi^{10}}{1920\,h^4} \tag{18}$$

$$\beta_8 = -\frac{2\,h^5\xi^2}{315} + \frac{2\,h^4\xi^3}{105} - \frac{1}{36}h^3\xi^4 + \frac{11\,h^2\xi^5}{450} - \frac{h\xi^6}{72} + \frac{13\,\xi^7}{2520} - \frac{7\,\xi^8}{5760\,h} + \frac{\xi^9}{6048\,h^2} - \frac{\xi^{10}}{100800\,h^3} \tag{19}$$

$$\beta_9 = \frac{2\,h^5\xi^2}{315} - \frac{5\,h^4\xi^3}{189} + \frac{1}{20}h^3\xi^4 - \frac{49\,h^2\xi^5}{900} + \frac{1}{27}h\xi^6 - \frac{9\,\xi^7}{560} + \frac{5\,\xi^8}{1152\,h} - \frac{121\,\xi^9}{181440\,h^2} + \frac{\xi^{10}}{22400\,h^3} \tag{20}$$

We then evaluate Equations (11) – (20) at $\xi = 1$ and $\xi = 2$, substitute into Equation (9) to give the new formulated seventh derivative implicit block methods, acronym as "7D2PIB1 and 7D2PIB2" respectively.

$$y_{n+1} = y_n + \frac{5639}{23040}hf_n + \frac{121}{45}hf_{n+1} - \frac{44551}{23040}hf_{n+2} + \frac{1289}{768}h^2 g_{n+2} - \frac{7687}{11520}h^3 u_{n+2}$$
$$+\frac{287}{1920}h^4 v_{n+2} - \frac{583}{34560}h^5 w_{n+2} + \frac{1}{5760}h^6 m_{n+2} - \frac{257}{604800}h^7 q_{n+1} + \frac{121}{907200}h^7 q_{n+2} \tag{21}$$

$$y_{n+2} = y_n + \frac{11}{45}hf_n + \frac{128}{45}hf_{n+1} - \frac{49}{45}hf_{n+2} + \frac{4}{3}h^2g_{n+2} - \frac{26}{45}h^3u_{n+2} + \frac{2}{15}h^4v_{n+2}$$
$$-\frac{2}{135}h^5w_{n+2} - \frac{2}{4725}h^7q_{n+1} + \frac{2}{14175}h^7q_{n+2} \qquad (22)$$

Similarly, we derived the second formula 7D2PIB2 and it is presented as:

$$y_{n+1} = y_n + \frac{1663}{11520}hf_n + \frac{121}{45}hf_{n+1} - \frac{21119}{11520}hf_{n+2} + \frac{2837}{1920}h^2g_{n+2} - \frac{2687}{5760}h^3u_{n+2}$$
$$-\frac{257}{5040}h^4v_{n+1} + \frac{1343}{20160}h^4v_{n+2} - \frac{113}{120960}h^5w_{n+2} - \frac{37}{33600}h^6m_{n+2} + \frac{121}{907200}h^7q_{n+2} \qquad (23)$$

$$y_{n+2} = y_n + \frac{13}{90}hf_n + \frac{128}{45}hf_{n+1} - \frac{89}{90}hf_{n+2} + \frac{17}{15}h^2g_{n+2} - \frac{17}{45}h^3u_{n+2} - \frac{16}{315}h^4v_{n+1}$$
$$+\frac{16}{315}h^4v_{n+2} + \frac{h^5w_{n+2}}{945} - \frac{2}{1575}h^6m_{n+2} + \frac{2}{14175}h^7q_{n+2} \qquad (24)$$

3. The stability analysis of the methods

This section presents the numerical properties and theorems (without proof) in relation to the proposed numerical methods.

Theorem 3.1. *Convergence* [5]: The necessary and sufficient conditions for the linear multistep method (LMM) of Equations (21)–(24) to be convergent are that it must be consistent and zero stable.

Theorem 3.2. The necessary and sufficient condition for the method given by Equations (21)–(24) to be zero stable is that it satisfies the root condition (See [5]).

Definition 3.1. *Zero stability* [10]
The numerical methods in Equations (21)– (24) are said to be zero stable if no root of te first characteristic polynomial has a modulus greater than one and that every root with modulus one is simple.

Definition 3.2. *A-stability*: A numerical method is said to be A-stable if the whole of the left-half plane $z : \Re(z) \leq 0$ is contained in the region $z : \Re(z) \leq 1$. Where $\Re(z)$ is the stability polynomial of the proposed method. (See [5]).

Definition 3.3. A(α)-*stability*: A numerical algorithm is said to be A(α) stable for some $\alpha \in [0, \frac{\pi}{2}]$ if the wedge $S_\alpha = \{z : |Arg(-z)| < \alpha, z \neq 0\}$ is contained in its region of absolute stability. (See, [7]).

3.1. The Order of the 7D2PIB1 and 7D2PIB2

To establish the order of the derived methods, Equations (21) – (24) are rewritten in block form to give the linear operator:

$$L(y(x); h) = A^{(1)}Y_m - A^{(0)}Y_{m-1} - B^{(0)}F_{m-1} - B^{(1)}F_m - h^2C^1G_m - h^3C^2U_m$$
$$-h^4C^3V_m - h^5C^4W_m - h^6C^5M_m - h^7C^6Q_m \qquad (25)$$

Where,

$$A^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^{(0)} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad B^{(0)} = \begin{pmatrix} 0 & \frac{5639}{23040} \\ 0 & \frac{11}{45} \end{pmatrix}, \quad B^{(1)} = \begin{pmatrix} \frac{121}{45} & \frac{128}{45} \\ -\frac{44551}{23040} & -\frac{49}{45} \end{pmatrix},$$

$$C^{(1)} = \begin{pmatrix} 0 & \frac{1289}{768} \\ 0 & \frac{4}{3} \end{pmatrix}, \quad C^{(2)} = \begin{pmatrix} 0 & -\frac{7687}{11520} \\ 0 & -\frac{26}{45} \end{pmatrix}, \quad C^{(3)} = \begin{pmatrix} 0 & \frac{287}{1920} \\ 0 & \frac{2}{15} \end{pmatrix}, \quad C^{(4)} = \begin{pmatrix} 0 & -\frac{583}{34560} \\ 0 & -\frac{2}{135} \end{pmatrix},$$

$$C^{(5)} = \begin{pmatrix} 0 & -\frac{1}{5760} \\ 0 & -\frac{2}{135} \end{pmatrix} \quad C^{(6)} = \begin{pmatrix} -\frac{257}{604800} & \frac{121}{907200} \\ -\frac{2}{4725} & \frac{2}{14175} \end{pmatrix}, \quad Y_m = \begin{pmatrix} y_{n+1} \\ y_{n+2} \end{pmatrix}, \quad Y_{m-1} = \begin{pmatrix} y_{n-(k-1)} \\ y_n \end{pmatrix},$$

$$F(Y_m) = \begin{pmatrix} f_{n+1} \\ f_{n+k} \end{pmatrix}, \quad F(Y_{m-1}) = \begin{pmatrix} f_{n-(k-1)} \\ f_n \end{pmatrix}, \quad G_m = \begin{pmatrix} f'_{n+1} \\ f'_{n+k} \end{pmatrix}, \quad U_m = \begin{pmatrix} f^{(2)}_{n+1} \\ f^{(2)}_{n+k} \end{pmatrix}, \quad V_m = \begin{pmatrix} f^{(3)}_{n+1} \\ f^{(3)}_{n+k} \end{pmatrix},$$

$$W_m = \begin{pmatrix} f^{(4)}_{n+1} \\ f^{(4)}_{n+k} \end{pmatrix}, \quad M_m = \begin{pmatrix} f^{(5)}_{n+1} \\ f^{(5)}_{n+k} \end{pmatrix}, \quad Q_m = \begin{pmatrix} f^{(6)}_{n+1} \\ f^{(6)}_{n+2} \end{pmatrix}.$$

Note that $f_{n+l}, l = 0(1)k$ are the first-order derivative functions in x, y.
Equation (25) is expanded using Taylor series expansion, comparing their coefficients of powers of h to give:

$$\begin{aligned} L(y(x); h) = q_0 y(x) \ + \ q_1 h y'(x) \ + \ q_2 h^2 y''(x) \ + \ \dots \ + \ q_p h^p y^p(x) \ + \ \dots \\ + q_{p+1} h^{p+1} y^{p+1}(x) \ + \ \dots \end{aligned} \tag{26}$$

Therefore, the linear operator $L(y(x); h)$ in Equation (25) and the associated continuous linear multistep methods in Equations (21)–(24) are said to be of order p if $q_0 = q_1 = q_2 = \dots = q_p = 0$ and $q_{p+1} \neq 0$. q_{p+1} is the error constant and the local truncation error is given by:

$$t_{n+k} = q_{p+1} h^{(p+1)} y^{(p+1)}(x_n) \ + \ 0(h^{p+2}) \tag{27}$$

Therefore, using MAPLE 18, the order and error constants for “7D2PIB1 and 7D2PIB2” are investigated as:

Table 1: *Order and error constants*

Method	Order	Error constant (q_{p+1})
7D2PIB1	10	$-\frac{5881}{7185024000}D^{(11)}\,(y)\,(x)\,h^{11}\ +\ O\,(h^{12})$
	10	$-\frac{23}{28066500}D^{(11)}\,(y)\,(x)\,h^{11}\ +\ O\,(h^{12})$
7D2PIB2	10	$-\frac{3931}{12573792000}D^{(11)}\,(y)\,(x)\,h^{11}\ +\ O\,(h^{12})$
	10	$-\frac{31}{98232750}D^{(11)}\,(y)\,(x)\,h^{11}\ +\ O\,(h^{12})$

3.2. Zero Stability

The zero stability polynomial of the formulated block methods in Equations (21)–(24) can be expressed by evaluating:

$$R(t) = \left|(A^{(0)}t\ -\ A^{(1)})\right| \tag{28}$$

Therefore, Equation (28) is then equated to zero and solved for t to give the characteristic roots each for 7D2PIB1 and 7D2PIB2 as:
$t = 0, 1$. Therefore, by Definition 3.1, it follows that the methods in Equations (21)–(24) are zero stable.

3.3. Consistency

The necessary and sufficient condition that a numerical method be consistent is that its order, $p \geq 1$. (See [4]).
Thus, the new methods whose order is 10 each, are certainly consistent.

3.4. Convergency

Inline with Theorem 3.1, the new derived block methods are convergent since they are both zero stable and consistent. Let y_i and $y(x_i)$ be the approximate and exact solution of (1) respectively, then the absolute error is evaluated by using the formula:
$AbsErr = |(y_i)_t\ -\ (y(x_i))_t|, \quad 1 \leq t \leq NS,$
Where, NS is the total number of steps.

3.5. Linear stability

The absolute stability polynomials are presented below in the light of [13], using the test equations:
$y' = \lambda h, \quad y^{(i)} = \lambda^i h^i, \quad i = 2(1)k\ +\ 5,$

So that substituting the above into Equations (21)–(24) yields:

$$M(w,z) = -A_1 w \;+\; A_0 \;+\; zB_0 \;+\; zB_1 w \;+\; z^2 B_2 w \;+\; z^3 B_3 w \;+\; z^4 B_4 w \;+\; z^5 B_5 w \\ +z^6 B_6 w \;+\; z^7 B_7 w \tag{29}$$

Where $z = \lambda h$ and w is the difference equation shift operator. From which we have the following expression as the stability polynomial:

$$\pi_i(w,z) = |M(w,z)|, \quad i = 1,2. \tag{30}$$

The absolute stability regions are obtained by evaluating Equation (30) to give the following stability functions for 7D2PIB1 and 7D2PIB2 respectively:

$$\pi_1(w,z) = -\frac{w^2}{285768000}z^{14} + \frac{w^2}{13608000}z^{13} - \frac{23\,w^2}{27216000}z^{12} + \frac{w^2}{151200}z^{11} - \frac{67\,w^2}{1814400}z^{10} \\ +\frac{29\,w^2}{201600}z^9 + \left(-\frac{1291\,w^2}{3628800} - \frac{w}{3628800}\right)z^8 + \left(-\frac{127\,w^2}{604800} - \frac{w}{604800}\right)z^7 + \frac{11\,w^2}{1350}z^6 \\ -\frac{7\,w^2}{135}z^5 + \frac{19\,w^2}{90}z^4 - \frac{11\,w^2}{18}z^3 + \left(\frac{223\,w^2}{180} - \frac{7\,w}{180}\right)z^2 + \left(-\frac{8}{5}w^2 - \frac{2}{5}\,w\right)z + w^2 - w$$

$$\pi_2(w,z) = -\frac{w^2}{2381400}z^{11} + \frac{w^2}{113400}z^{10} - \frac{23\,w^2}{226800}z^9 + \frac{w^2}{1260}z^8 - \frac{367\,w^2}{75600}z^7 + \frac{1817\,w^2}{75600}z^6 \\ +\left(-\frac{2923\,w^2}{30240} - \frac{w}{30240}\right)z^5 + \left(\frac{523\,w^2}{1680} - \frac{w}{5040}\right)z^4 - \frac{7\,w^2}{9}z^3 + \left(\frac{64\,w^2}{45} - \frac{w}{45}\right)z^2 \\ +\left(-\frac{17\,w^2}{10} - \frac{3}{10}w\right)z + w^2 - w$$

From which π_1 and π_2 are then coded in a MATLAB software environment and the region of absolute stability for each derived method is as shown in Figures 1 and 2 below. Figures 1 and 2 indicate

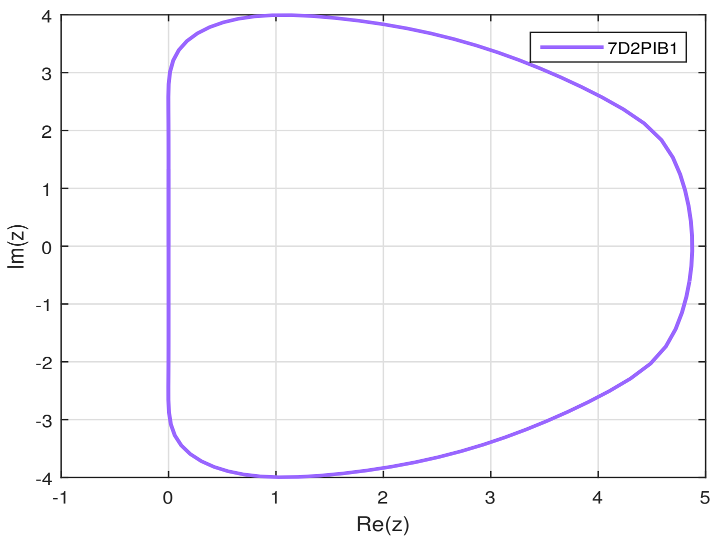

Figure 1: Absolute Stability Region of 7D2PIB1

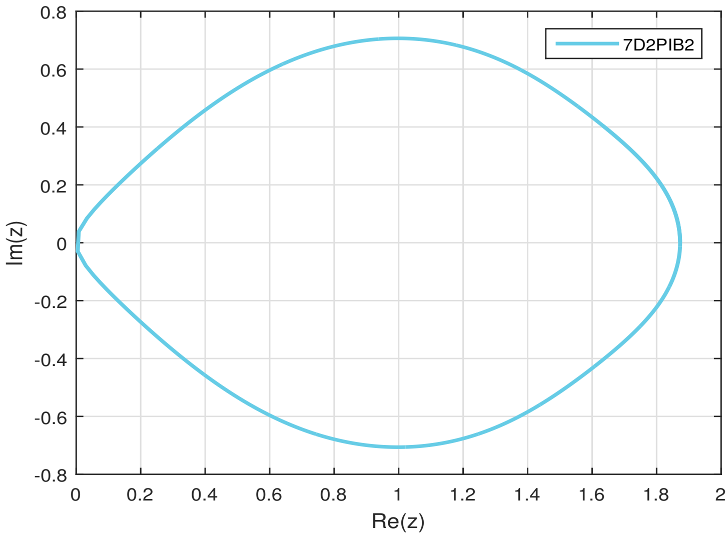

Figure 2: Absolute Stability Region of 7D2PIB2

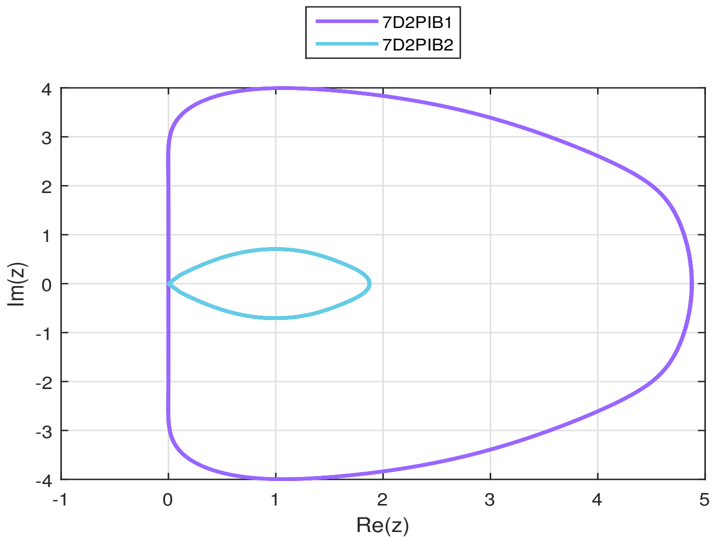

Figure 3: Compared absolute stability region of methods

the region of absolute stability of the methods. The first method, 7D2PIB1, whose unstable region is the closed region, is larger than the second method, 7D2PIB2, as precisely shown in Figure 3; implying that 7D2PIB2 has an open region of larger stability region than 7D2PIB1. However, both methods have regions of absolute stability that are left symmetric. Hence, both developed formulae are A-stable inline with Definition 3.2.

4. Implementation of the methods

The simultaneous approximation of y_{n+l} in the new methods was done using Newton Raphson's techniques on MATLAB software environment. Therefore,

$$y_{n+l}^{(i+1)} = y_{n+l}^{(i)} - \frac{f(y_{n+l}^{(i)})}{f'(y_{n+l}^{(i)})}, \quad l = 1(1)k. \tag{31}$$

So that, Equation (31) can be rewritten as:

$$y_{n+l}^{(i+1)} - y_{n+l}^{(i)} = [f(y_{n+l}^{(i)})][f'(y_{n+l}^{(i)})]^{-1} \tag{32}$$

From which we get:

$$e_{n+1}^{j+1} = [f(y_{n+l}^{(i)})][f'(y_{n+l}^{(i)})]^{-1} \tag{33}$$

where,

$$f(y_{n+l}^{(i)}) = \begin{pmatrix} y_{n+1} - y_n - \frac{1663}{11520}hf_n - \frac{121}{45}hf_{n+1} + \frac{21119}{11520}hf_{n+2} - \frac{2837}{1920}h^2g_{n+2} \\ +\frac{2687}{5760}h^3u_{n+2} + \frac{257}{5040}h^4v_{n+1} - \frac{1343}{20160}h^4v_{n+2} + \frac{113}{120960}h^5w_{n+2} \\ +\frac{37}{33600}h^6m_{n+2} - \frac{121}{907200}h^7q_{n+2} \\ y_{n+2} - y_n - \frac{29}{180}hf_n + \frac{832}{45}hf_{n+1} - \frac{3659}{180}hf_{n+2} + \frac{64}{15}h^2g_{n+1} \\ +\frac{159}{10}h^2g_{n+2} - \frac{539}{90}h^3u_{n+2} + \frac{7}{5}h^4v_{n+2} - \frac{59}{270}h^5w_{n+2} + \frac{1}{45}h^6m_{n+2} \\ -\frac{17}{14175}h^7q_{n+2} \end{pmatrix}$$

and $e_{n+1}^{j+1} = y_{n+l}^{(i+1)} - y_{n+l}^{(i)}$; while $f(y_{n+l}^{(i)})$ is a system of equations and $f'(y_{n+l}^{(i)})$ is a (2×2) Jacobian matrix, g, u, v, w, m and q are second, third, fourth, fifth, sixth and seventh derivatives respectively. Since the new block is self-starting, it does not require starting formula to incorporate all the initial values for the stiff IVPs. Therefore, approximate solutions y_{n+l} are simultaneously generated.

Algorithm 1 Proposed Methods Algorithm

Input: Define initial guess: $f(x), df(x), N, h,$ $[a, b]$, where $f(x)$ is the problem to be solved and $df(x)$ is the derivative function, e is the tolerance, N total number of iterations and h is the step-size and $[a, b]$ is the iterations interval.

Output: $y_{new} = y_{n+l}^{(i+1)}$

1: Define $y_{old} = y_{n+l}^{(i)}, [a, b], h = \frac{(b-a)}{N}$
2: **for** $j = 1 : N - 1$, **do**
3: $\quad x(j) = x_0 + jh$
4: $\quad$ **while** $|(y_{old} - y_{new})| > tol$, **do**
5: $\quad\quad y_{new} = y_{old} - \frac{f'(x,y)}{f(x,y)}$
6: $\quad\quad$ Print $y_{new} = y_{old}$
7: $\quad\quad$ **for** $j = j + 1$, **do**
8: $\quad\quad\quad$ Goto 3
9: $\quad\quad$ **end for**
10: $\quad\quad$ **if** $j \geq N$, **then**
11: $\quad\quad\quad$ Goto 4
12: $\quad\quad$ **end if**
13: $\quad$ **end while**
14: $\quad$ Goto 6
15: **end for**
16: Stop

5. Numerical Experiment

The following first order stiff initial value problems are used to test the performance of the new method and comparison, where possible, are made with some selected existing methods of close or higher orders. The test problems considered here are either mild or highly stiff first order IVPs.

Problem 1. Consider the first order system of stiff initial value problem:

$$y_1' = -8y_1 + 7y_2, \quad y_1(0) = 1, \quad h = 0.1,$$

$$y_2' = 42y_1 - 43y_2, \quad y_2(0) = 8,$$

Exact Solution:

$$y_1(t) = 2e^{-x} - e^{-50x}$$

$$y_2(t) = 2e^{-x} + 6e^{-50x}$$

Source: Skwame et al. [9]

Table 2 shows the results from solving Problem 1 with their efficiency curves shown in Figures 4 and 5. The figures show that at many grid and approximate points of iterations, proposed methods show small scale error with better accuracy in 7D2PIB2 than 7D2PIB1 and method in [9]. It is evident that with $h = 0.1$, the proposed formulae show sufficient efficiency and improved accuracy. This efficiency of the derived methods indicate that with smaller step-sizes, the methods could have smaller scale errors and therefore approximate solutions could tend to their true solutions.

Table 2: *Comparison of Absolute Error for Problem 1 with $h = 0.1$*

x	Error in [9], $p = 10$		7D2PIB1, $p = 10$		7D2PIB2, $p = 10$	
	y_1	y_2	y_1	y_2	y_1	y_2
0.1	1.32e-06	8.10e-02	4.36e-03	2.62e-02	6.31e-04	3.79e-02
0.2	1.90e-08	5.50e-04	7.78e-05	4.67e-04	8.11e-06	4.870e-05
0.3	4.00e-09	3.70e-06	1.06e-06	6.37e-06	7.82e-08	4.69e-07
0.4	4.00e-09	2.10e-08	1.31e-08	7.88e-08	6.71e-10	4.02e-09
0.5	2.00e-09	3.00e-09	1.55e-10	9.28e-10	5.39e-12	3.24e-11
0.6	3.00e-09	2.00e-09	1.73e-12	1.07e-11	3.78e-14	2.55e-13
0.7	4.50e-09	2.90e-09	2.37e-14	1.68e-13	4.33e-15	6.77e-15
0.8	4.10e-09	3.70e-09	3.88e-14	4.44e-14	4.00e-15	4.11e-15
0.9	4.60e-09	4.00e-09	3.49e-14	3.81e-14	3.55e-15	3.78e-15
1.0	4.80e-09	4.60e-09	3.24e-14	3.59e-14	2.55e-15	2.44e-15

Problem 2. Consider the first order system of stiff initial value problem:

$y_1' = -9y_1 \; + \; 95y_2, \quad y_1(0) = 1, \quad h = 0.1,$

$y_2' = -y_1 \; - \; 97y_2, \quad y_2(0) = 1,$

Exact Solution:

$y_1(t) = \frac{95}{47}e^{-2x} \; - \; \frac{48}{47}e^{-96x}$

$y_2(t) = \frac{48}{47}e^{-96x} \; - \; \frac{1}{47}e^{-2x}$

Problem 2 was solved in [9] by using an implicit block method of a uniform order 10. The same is solved using the proposed formulae. The results are presented in Table 3 with their efficiency curves shown in Figures 6 and 7. It is clear that the proposed methods show better accuracy with 7D2PIB2 outperforms 7D2PIB1 and such method in [9]. The figures show that at many grid points, the proposed methods have smaller scale absolute errors, which indicate consistency in terms of numerical properties, as errors decrease as iterations proceed.

Table 3: *Comparison of Absolute Error for Problem 2 with* $h = 0.1$

x	Error in [9], $p = 10$		7D2PIB1, $p = 10$		7D2PIB2, $p = 10$	
	y_1	y_2	y_1	y_2	y_1	y_2
0.1	1.74×10^{-4}	1.74×10^{-4}	3.70×10^{-4}	3.70×10^{-4}	9.22×10^{-4}	9.22×10^{-4}
0.2	5.40×10^{-8}	5.30×10^{-8}	1.84×10^{-7}	1.84×10^{-7}	7.07×10^{-7}	7.07×10^{-7}
0.3	1.00×10^{-9}	4.00×10^{-11}	8.01×10^{-11}	8.07×10^{-11}	5.94×10^{-10}	5.94×10^{-10}
0.4	2.30×10^{-9}	3.50×10^{-11}	4.27×10^{-13}	2.99×10^{-14}	4.46×10^{-13}	4.95×10^{-13}
0.5	2.20×10^{-9}	3.10×10^{-11}	3.69×10^{-13}	3.88×10^{-15}	3.71×10^{-14}	7.95×10^{-16}
0.6	1.80×10^{-9}	2.70×10^{-11}	2.92×10^{-13}	3.08×10^{-15}	2.81×10^{-14}	2.93×10^{-16}
0.7	1.60×10^{-9}	2.20×10^{-11}	2.34×10^{-13}	2.47×10^{-15}	1.93×10^{-14}	2.04×10^{-16}
0.8	1.40×10^{-9}	2.00×10^{-11}	1.85×10^{-13}	1.95×10^{-15}	1.44×10^{-14}	1.49×10^{-16}
0.9	1.20×10^{-9}	1.60×10^{-11}	1.45×10^{-13}	1.53×10^{-15}	9.44×10^{-15}	9.98×10^{-17}
1.0	9.10×10^{-10}	1.40×10^{-11}	1.16×10^{-13}	1.22×10^{-15}	6.50×10^{-15}	6.68×10^{-17}

Problem 3. Consider the first order stiff initial value problem:

$y' = x - y, \quad 0 \leq x \leq 1, \quad h = 0.1,$

Exact Solution:

$y(x) = x + e^{-x} - 1$

The above Problem has been considered in [32] with a uniform block order of 13. Their method was directly employed without starting values. The results of the derived formulae are presented in Table 4 with the efficiency curves shown in Figure 8. A clear comparison of our derived methods indicates that 7D2PIB2 outperformed 7D2PIB1 of the same order 10, though with minimal comparable performance in accuracy while outperformed such a method of order 13 in [32]. Figure 7 shows the competitive performance of 7D2PIB1, 7D2PIB2 and with such existing methods in [32]. It is clear that 7D2PIB1 and 7D2PIB2 show convergence at the last two grid points of the iterations.

Table 4: *Comparison of Absolute Error for Problem 3 with $h = 0.1$*

x	**Error in [32],** $p = 13$	7D2PIB1, $p = 10$	7D2PIB2, $p = 10$
0.1	1.9595×10^{-11}	3.8165×10^{-17}	3.29598×10^{-17}
0.2	3.54623×10^{-11}	4.85723×10^{-17}	3.81639×10^{-17}
0.3	4.81315×10^{-11}	4.85723×10^{-17}	6.93889×10^{-17}
0.4	5.80680×10^{-11}	1.66534×10^{-16}	1.38778×10^{-16}
0.5	6.56779×10^{-11}	2.77556×10^{-17}	2.77556×10^{-17}
0.6	7.13132×10^{-11}	1.66534×10^{-16}	1.66534×10^{-16}
0.7	7.52814×10^{-11}	1.11022×10^{-16}	1.11022×10^{-16}
0.8	7.78485×10^{-11}	5.55112×10^{-17}	5.55112×10^{-17}
0.9	7.92403×10^{-11}	0.00000×10^{-00}	0.00000×10^{-00}
1.0	7.96712×10^{-11}	0.00000×10^{-00}	0.00000×10^{-00}

Problem 4. Consider the first order stiff initial value problem:

$y' = -y, \quad 0 \leq x \leq 1, \quad h = 0.1,$

Exact Solution:

$y(x) = e^{-x}$

Problem 4 has been solved in [18] with one-step and two-step hybrid block methods of uniform order 10 and 18 respectively. The proposed methods are applied using a step-size, $h = 0.1$. Results from application, as shown in Table 5, indicate the competitive performance of our first formula, 7D2PIB1 with those in [18] particularly, 2 SHBM of order 18. Figure 9 depicts the convergence at some grid points of the iterations in 7D2PIB1 while outperforms, particularly 2 SHBM of a uniform order 18.

Table 5: *Comparison of Absolute Error for Problem 4 with* $h = 0.1$

x	1 SHBM [18], $p = 10$	2 SHBM [18], $p = 18$	7D2PIB1, $p = 10$
0.1	0.00e+00	4.10e-20	0.00e+00
0.2	1.10e-20	6.10e-20	0.00e+00
0.3	2.10e-20	8.10e-20	1.11e-16
0.4	1.10e-20	1.11e-20	0.00e+00
0.5	1.10e-20	1.21e-19	2.22e-16
0.6	2.10e-20	1.31e-19	2.22e-16
0.7	1.10e-20	1.41e-19	1.67e-16
0.8	2.10e-20	1.41e-19	0.00e+00
0.9	2.10e-20	1.51e-19	5.55e-17
1.0	3.10e-20	1.41e-20	0.00e+00

Problem 5. Consider the first order non-linear stiff initial value problem of the form:

$y' = xy, \quad 0 \leq x \leq 1, \quad h = 0.1,$

Exact Solution:

$y(x) = e^{\frac{x^2}{2}}$

Problem 5 has been solved in [32] with a uniform block method of order 13. Proposed methods are directly employed without starting values to solve the same problem. The results of the derived formulae are presented in Table 7 with the efficiency curves shown in Figure 11. Results show that proposed methods of uniform order 10 give improved accuracy with 7D2PIB2 giving potential advantage over 7D2PIB1 of the same order. A method of a uniform order 13 has been compared and our derived methods evidently show adequate accuracy over it. Observe from Figure 10 that, for this nonlinear Problem 5, as the log(x) increases from left to right, the log of absolute error also increases. This numerical results are usually common with all non-linear stiff IVPs.

Table 6: *Comparison of Absolute Error for Problem 5 with* $h = 0.1$

x	Error in [32], $p = 13$	7D2PIB1, $p = 10$	7D2PIB2, $p = 10$
0.1	2.6067×10^{-11}	1.2213×10^{-14}	4.8850×10^{-15}
0.2	8.4790×10^{-11}	3.5083×10^{-14}	1.3545×10^{-14}
0.3	1.8684×10^{-10}	7.2831×10^{-14}	2.8422×10^{-14}
0.4	3.5701×10^{-10}	1.3123×10^{-13}	5.1070×10^{-14}
0.5	6.1054×10^{-09}	2.2116×10^{-13}	8.5931×10^{-14}
0.6	1.0157×10^{-09}	3.5727×10^{-13}	1.3922×10^{-13}
0.7	1.6445×10^{-09}	5.6355×10^{-13}	2.1960×10^{-13}
0.8	2.6158×10^{-09}	8.7708×10^{-13}	3.4195×10^{-13}
0.9	4.1110×10^{-09}	1.3551×10^{-12}	5.2913×10^{-13}
1.0	6.4070×10^{-09}	2.0863×10^{-12}	8.1535×10^{-13}

6. Application problems

Problem 6.

As discussed in [15], the SIR model is an epidemiological model that computes the theoretical number of people infected with a contagious illness in a closed population over time. The name of this class of models is derived from the fact that they involve coupled equations relating the number of susceptible people S(t), number of people infected I(t) and the number of people who have recovered R(t). This is a good and simple model for many infectious diseases Including measles, mumps and rubella. It is given by the following three coupled equations:

$$\left.\begin{aligned} \frac{dS}{dt} &= \mu(1 - S) - \gamma IS \\ \frac{dI}{dt} &= -\mu I - \gamma I + \beta IS \\ \frac{dR}{dt} &= -\mu R + \gamma I \end{aligned}\right\} \tag{34}$$

Where μ, γ and β are positive parameters to be determined. Therefore, let y be given by:

$$y = S + I + R \tag{35}$$

By taking the derivative of Equation (35) and summing Equations (34) and (35) to give the SIR model of the form:

$$y' = \mu(1 - y), \quad 0 \le x \le 1, \quad h = 0.01, \tag{36}$$

Whose exact solution is:

$y(x) = 1 - 0.5e^{-0.5x}$.

Problem 6 in Equation 36 is solved using the proposed methods. The results as presented in Table 8 depict the absolute errors and time taken (seconds) at each point of iterations. The efficiency curves are also plotted using the logarithm of absolute errors against the log of time and are shown in Figure 11. Because of the stiff nature present in the modeled problem, it is clear from Figure 11 that, the scale absolute errors, particularly in 7D2PIB2 inter-nodes. Table 8 also clearly indicates the near convergence of 7D2PBI2 unlike 7D2PBI1. However, effective time cost is observed in 7D2PBI1 in comparison with 7D2PBI2. Reasonably, for this particular Problem 6 in Equation (36), 7D2PBI1 presents efficient time cost over 7D2PBI2 but improved accuracy is seen in 7D2PBI2. While time of iterations is important, accuracy of numerical methods is most significant as it shows the consistency and zero stability of methods. Finally, the proposed formulae present improved efficiency and accuracy over the compared method in [15], as shown in Table 8.

Table 7: *Comparison of Absolute Error for Problem 6 in Equation (36) (SIR Model) with* $h = 0.01$

x	Error in [15], $p = 8$	Time	7D2PIB1, $p = 10$	Time	7D2PIB2, $p = 10$	Time
0.010	1.2165824e-12	0.043527	4.4408921e-16	0.008644	0.0000000e-00	0.009575
0.020	7.0361494e-12	0.048093	1.1102230e-15	0.011411	1.1102230e-16	0.012471
0.030	1.6891821e-11	0.053913	1.5543122e-15	0.013618	1.1102230e-16	0.015296
0.040	3.0793479e-11	0.059570	1.9984014e-15	0.016258	2.2204461e-16	0.018339
0.050	5.0472182e-11	0.063933	2.5535130e-15	0.018438	1.1102230e-16	0.021397
0.060	7.1624151e-11	0.080116	2.9976022e-15	0.020523	1.1102230e-16	0.025175
0.070	1.0171974e-10	0.085281	3.5527137e-15	0.022605	2.2204461e-16	0.027427
0.080	1.2969015e-10	0.093241	4.1078252e-15	0.024689	3.3306691e-16	0.030011
0.090	1.6615576e-10	0.097912	4.5519144e-15	0.027011	3.3306691e-16	0.036050
0.100	2.0496926e-10	0.104638	5.2180482e-15	0.029718	5.5511151e-16	0.038381

Problem 7. Consider the growth model as solved in [15]:

A bacteria culture is known to grow at a rate proportional to the amount present. After one hour, 1000 strands of the bacteria are observed in the culture; and after four hours, bacteria are observed

in the culture to be 3000 strands. Find the number of strands of the bacteria present in the culture at time t, where, $0 \leq t \leq 1$.
Let $N(t)$ denote the number of bacteria strands in the culture at time t, the initial value problem modeling this problem is given by:

$$\frac{dN}{dt} = 0.366N, \quad N(0) = 694, \tag{37}$$

The exact solution is given by:

$N(t) = 694e^{0.366t}$.
Problem 7 in Equation (37), which is a population growth model, has been solved with $h = 0.01$ in [15]. The new methods are also applied for the approximations and the absolute errors are as shown in Table 9 with the efficiency curves in Figure 12. Results indicate improved accuracy with reduced computational time in 7D2PBI1 than with 7D2PBI2. Method, 7D2PIB1 performed excellently over 7D2PIB2 in terms of efficiency and accuracy as shown in Table 9. Comparison with a method in [15] showed clear performance in terms of accuracy and time of iterations in the proposed formulae.

Table 8: *Comparison of Absolute Error for Problem 7 in Equation (37) (Growth Model) with* $h = 0.01$

x	Error in [15], $p = 8$	Time	7D2PIB1, $p = 10$	Time	7D2PIB2, $p = 10$	Time
0.010	6.7871042e-11	0.022520	0.0000000e-00	0.008482	0.0000000e-00	0.008264
0.020	2.9922376e-10	0.050186	0.0000000e-00	0.010774	2.2737368e-13	0.010404
0.030	6.8837380e-10	0.070106	0.0000000e-00	0.013659	2.2737368e-13	0.029010
0.040	1.2363444e-09	0.090180	1.1368684e-13	0.015907	4.5474735e-13	0.031195
0.050	1.9656454e-09	0.110484	1.1368684e-13	0.018652	4.5474735e-13	0.033432
0.060	2.8278464e-09	0.133450	1.1368684e-13	0.021897	4.5474735e-13	0.035572
0.070	3.9101451e-09	0.152786	2.2737368e-13	0.024202	3.4106051e-13	0.037686
0.080	5.0885092e-09	0.175301	3.4106051e-13	0.026631	2.2737368e-13	0.039777
0.090	6.4850383e-09	0.210087	4.5474735e-13	0.029239	1.1368684e-13	0.041873
0.100	8.0320888e-09	0.232457	4.5474735e-13	0.031873	3.4106051e-13	0.043992

Problem 8. Consider the Prothero-Robinson oscillatory problem:

$$y' = L(y \; - \; sinx) \; + \; cosx, \quad y(0) = 0, \quad L = -1, \quad h = 0.1 \tag{38}$$

The exact solution is given by:
$y(x) = sinx$

Problem 8 in Equation (38), which is a Prothero-Robinson oscillatory problem has been solved in [15] and the proposed formulae is applied also. Results are presented in Table 9 and efficiency curves clearly shown in Figure 13. It is evident that the proposed methods show reduced computational time and improved accuracy in terms of absolute errors. However, for this Problem 8, 7D2PIB2 show improved efficiency but certainly not accuracy. Accuracy has clearly been lost to 7D2PIB1 at all points in the iterations. Therefore, each method show uniqueness in itself and 7D2PIB1 showed overall improved accuracy in terms of absolute errors, even with comparison with a method in [15].

Table 9: *Comparison of Absolute Error for Problem 8 in Equation (38) (oscillatory problem) with* $h = 0.1$

x	Error in [15], $p = 8$	Time	7D2PIB1, $p = 10$	Time	7D2PIB2, $p = 10$	Time
0.10	1.2439794e-09	0.181433	2.2689489e-12	0.004828	3.8868035e-10	0.004438
0.20	4.8347478e-09	0.380374	2.1383478e-11	0.013938	9.4229044e-10	0.010371
0.30	1.0511839e-08	0.564763	6.8256345e-11	0.017435	1.6392342e-09	0.012579
0.40	1.8015317e-08	0.751144	1.3565132e-10	0.019693	2.4580119e-09	0.014814
0.50	2.7086332e-08	0.952488	2.2087732e-10	0.022883	3.3772897e-09	0.017027
0.60	3.7467873e-08	1.129044	3.2125658e-10	0.025142	4.3759819e-09	0.026412
0.70	4.8905666e-08	1.332819	4.3413462e-10	0.028643	5.43334100e-09	0.028844
0.80	6.1149209e-08	1.620844	5.5688854e-10	0.031100	6.5290587e-09	0.031626
0.90	7.3952914e-08	1.904968	6.8693884e-10	0.036798	7.6433704e-09	0.033859
1.00	8.7077323e-08	2.125000	8.2176266e-10	0.039081	8.7571647e-09	0.036455

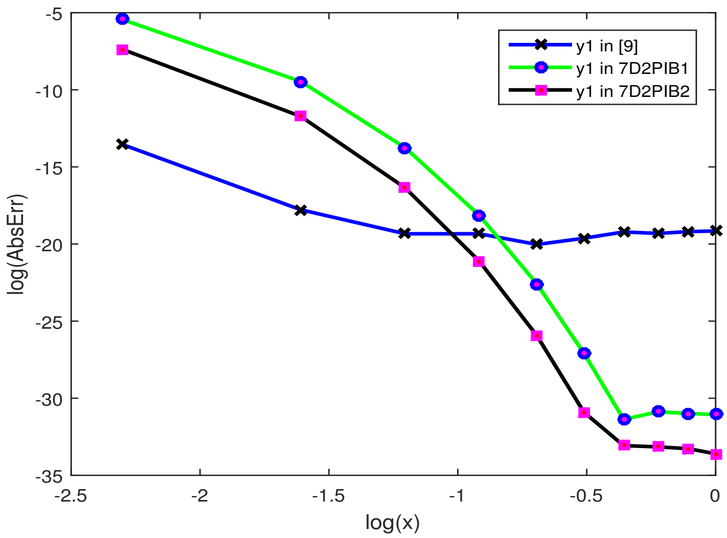

Figure 4: Efficiency curves for y_1 in Table 2

.

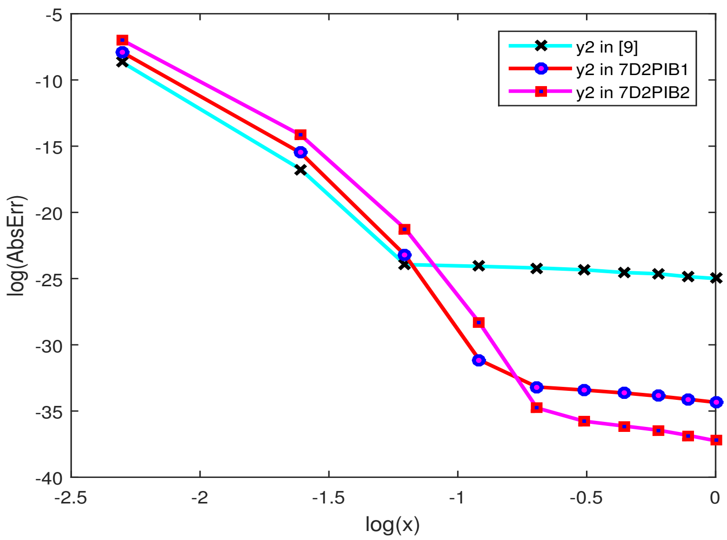

Figure 5: Efficiency curves for y_2 in Table 2

.

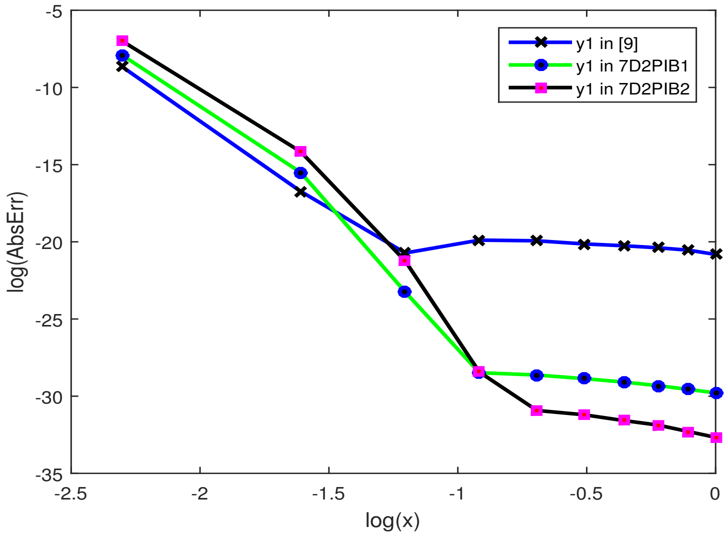

Figure 6: Efficiency curves for y_1 in Table 3

.

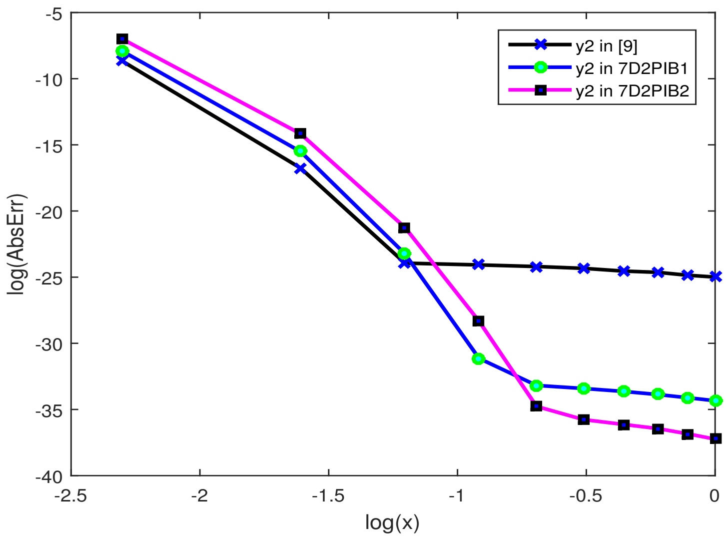

Figure 7: Efficiency curves for y_2 in Table 3

.

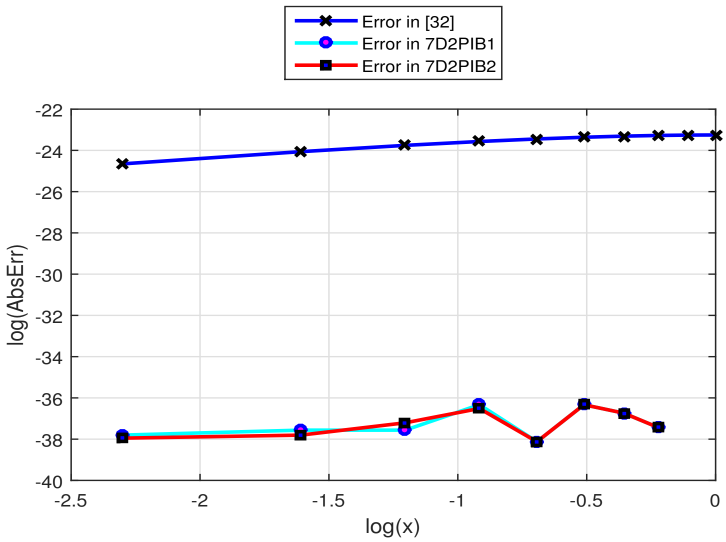

Figure 8: Efficiency curves of Table 4

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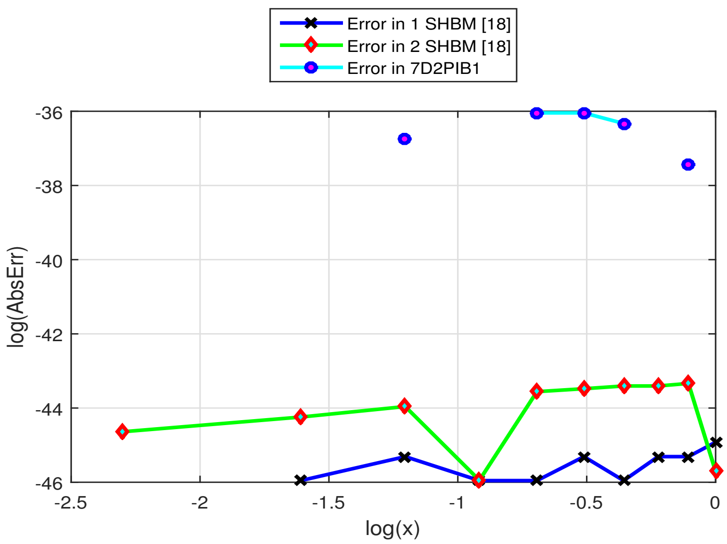

Figure 9: Efficiency curves of Table 5

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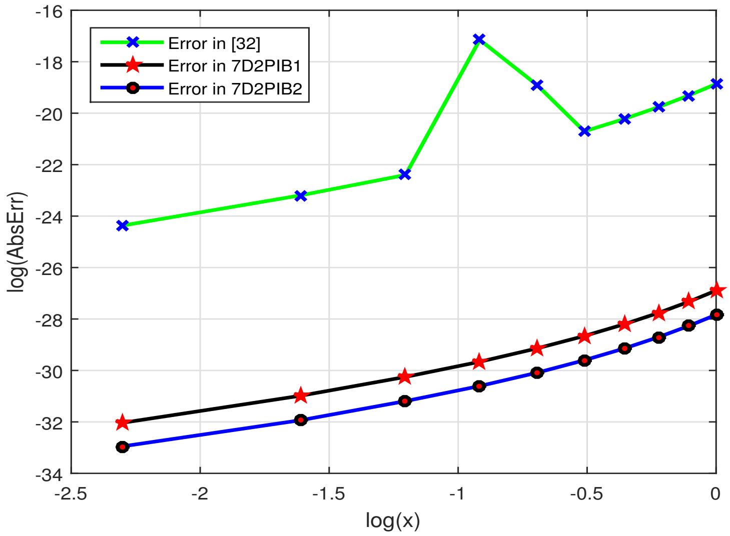

Figure 10: Efficiency curves of Table 6

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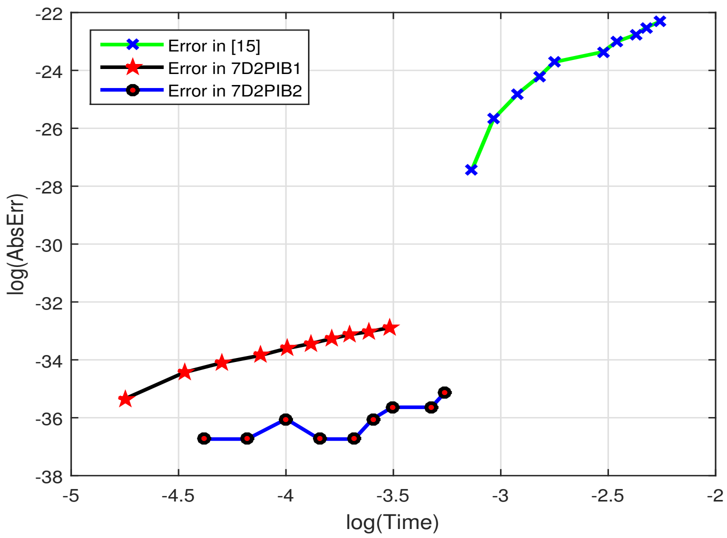

Figure 11: Efficiency curves of Table 7 (SIR Model)

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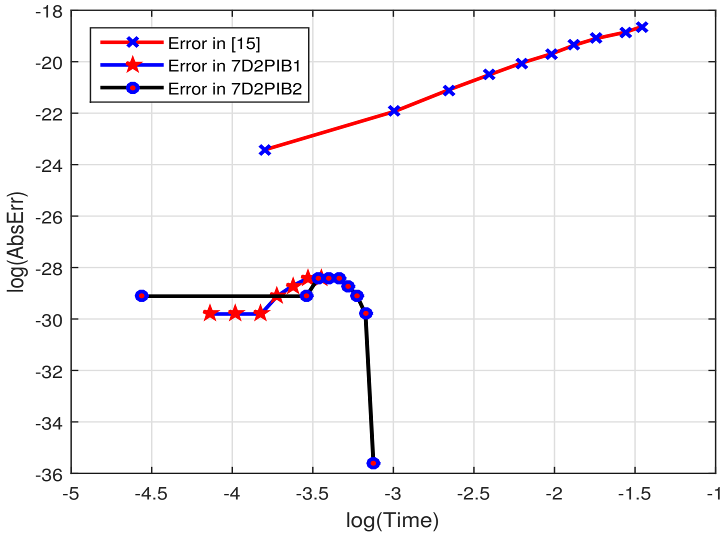

Figure 12: Efficiency curves of Table 8 (GROWTH MODEL)

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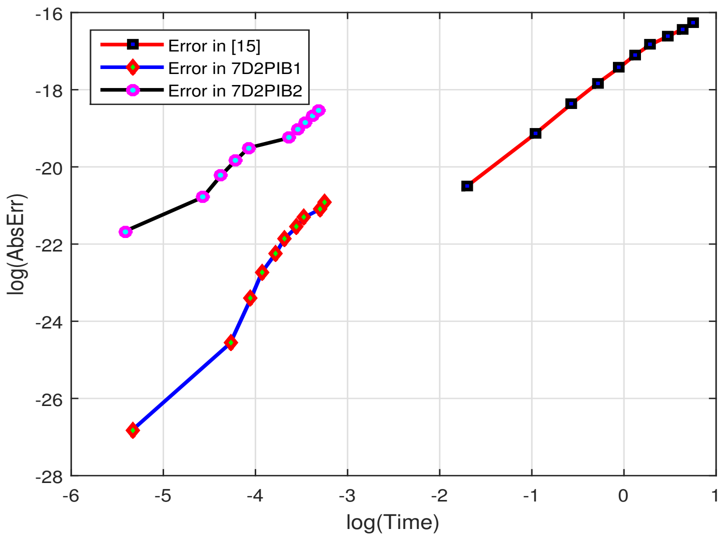

Figure 13: Efficiency curves of Table 9

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7. Conclusion and future research

A new family of computational methods, with seventh derivative type of implicit two-point block for the direct approximation of first order stiff initial value problems of uniform order 10 each have been developed. Formulae were derived through interpolation and collocation techniques. The new methods considered uneven points of collocation. They require seventh derivative type, though of a first-order function. It has been established that uneven points of collocation affect numerical schemes efficiency in terms of computational time and accuracy in terms of absolute errors. The new methods are found to be A-stable and convergent. The convergence were shown through test problems on first order stiff IVPs, including real-life problems as SIR model, growth model and oscillatory problem with comparison with some other existing methods. Results indicate that the new methods showed different numerical behaviors on different problems considered, either in terms of accuracy or efficiency while outperformed such existing methods in literature. Summarily, 7D2PIB2 displayed better accuracy and effective time cost than 7D2PIB1. This is not far-fetched as 7D2PIB2 has a larger open region of absolute stability than 7D2PIB1. In general, we have formulated numerical methods with uneven collocation points that are computationally stable with effective time cost for direct solution of (1). These methods outperformed such existing formulae in literature, as compared in this research. Our next future research will focus on developing and implementing efficient and robust numerical methods with uneven collocation points to real-life problems in chemical reaction in chemical engineering, models on drug magnetic nano-particle transport, population growth model, tumor immune interaction model, e.t.c and application to higher-order stiff IVPs may be considered also.

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Availability of data

The data used to support the results of the study are duly enclosed in the paper.

Conflict of interest

The authors declare that there is no conflict of interests.

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Author's contribution

V. O. Atabo: Conceptualization, methodology, former analysis and software. S. O. Adee : investigation, review and editing. P. O. Olatunji: validity and visualization. D. J. Yahaya: confirmation and original draft preparation. All content of manuscript were written via author's contribution, read and agreed to publish the final manuscript.

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