Qeios

A direct calculation in the newtonian gravity framework.

Alain Haraux

Sorbonne Université, Université Paris-Diderot SPC, CNRS, INRIA,
Laboratoire Jacques-Louis Lions, LJLL, F-75005, Paris, France.
e-mail: haraux@ann.jussieu.fr

https://doi.org/10.32388/CAO3VN

Abstract

We indicate a direct proof, without using the general Gauss theorem, of the fact that at any point outside a spherical massive body with spherically symmetric density function, both gravitational field and potential produced by the body are exactly the same as if all the mass were located at the center.

Key words: gravitation, Gauss theorem, triple integral, spherical coordinates.

1 Introduction

In deriving his formula for the gravitational field produced by a quasi punctual massive body, Newton was in a sense quite fortunate. Of course, a real mathematical point does not have any mass, and the objects to which his formulas were applied, either for the field or for the potential energy, were very large. In first approximation, the distances between planets and stars are so large that these objects can be considered punctual, but in fact the situation is even better and in this sense Newton was lucky. Calculations concerning planets and stars are indeed much more precise than what would follow by a mere point-like approximation. Since it was found later that when the object is spherically symmetric with respect to mass distribution, it behaves exactly as if all the mass was located at the centre. And this allows very precise astronomical calculations since planets and stars are close to spherical.
The principle which makes astronomical predictions much better with spherical objects can be formulated as follows.

Theorem 1.1. *Let us consider a spherical massive body with spherically symmetric density function. In this case, at any point outside the sphere, both gravitational field and potential produced by the body are exactly the same as if all the mass was located at the center.*

This result is classically proven for the gravitational field by using the Gauss theorem giving the total flux through a closed surface enclosing the body, applied to the concentric sphere containing the point, and exploiting the symmetry. The Gauss theorem is very powerful tool which applies to some other cases of symmetry in both gravitational and electro-static frameworks. In this short note we show that the calculation can be carried out without relying on the general Gauss theorem. This direct calculation may be useful since when the density is not exactly symmetric, we might rely on it to compute the deviation from the perfectly symmetric case.

2 Precise statements

In this section, we define precisely the framework in which Theorem 1.1 is valid. Let R be a positive number and define

$$B_R := \{x \in \mathbb{R}^3, ||x|| \leq R\}$$

We consider a nonnegative density function $\mu \in L^1(0, R)$ and we set

$$M := \int_{B_R} \mu(||x||)dx = \int_0^R 4\pi r^2 \mu(r)dr.$$

Theorem 2.1. *Let $u \in \mathbb{R}^3$ satisfy $||u|| > R$. Then we have*

$$\int_{B_R} \frac{\mu(||x||)}{||u-x||}dx = \frac{M}{||u||} \tag{2.1}$$

and

$$\int_{B_R} \frac{\mu(||x||)(u-x)}{||u-x||^3}dx = \frac{Mu}{||u||^3} \tag{2.2}$$

Remark 2.2. *Actually, the two formulas* (2.1) *and* (2.2) *are equivalent. One passes from* (2.1) *to* (2.2) *by taking the gradients of both sides. In the other direction, we observe that the field dermines the potential assuming the limit at infinity to be* 0.

3 Proof for the potential.

Proof. We set $U := ||u|| > R$ and we consider the integral

$$\int_{B_R} \frac{\mu(||x||)}{||u-x||}dx = \int_0^R \mu(r) \int_0^{2\pi} \int_0^{\pi} \frac{r^2 \sin\theta}{||u-x||} d\theta d\varphi dr$$

where we used the spherical coodinates

$$x = (r\cos\theta, r\sin\theta\cos\varphi, r\sin\theta\sin\varphi).$$

Since the integral is invariant by rotations of center 0, it is sufficient to consider the case where

$$u = (U, 0, 0),$$

in which case the integral becomes, by integrating with respect to φ first:

$$\int_0^R \mu(r) \int_0^{2\pi} \int_0^{\pi} \frac{r^2 \sin\theta}{||u-x||} d\theta d\varphi dr = \int_0^R 2\pi r^2 \mu(r) J(r) dr$$

with

$$J(r) := \int_0^{\pi} \frac{\sin\theta}{||u-x||} d\theta = \int_0^{\pi} \frac{\sin\theta}{\sqrt{U^2 + r^2 - 2Ur\cos\theta}} d\theta$$

coming from the fact that

$$||u-x||^2 = (U - r\cos\theta)^2 + r^2 \sin^2\theta \cos^2\varphi + r^2 \sin^2\theta \sin^2\varphi = U^2 + r^2 - 2Ur\cos\theta.$$

We claim that

$$\forall r \in (0, R), \quad J(r) = \frac{2}{U}$$

Indeed, the change of variable $\cos\theta := v$ gives

$$J(r) = \int_0^{\pi} \frac{\sin\theta}{\sqrt{U^2 + r^2 - 2Ur\cos\theta}} d\theta = \int_{-1}^{1} \frac{dv}{\sqrt{U^2 + r^2 - 2Urv}}$$

Now, we have

$$\frac{d}{dv}\sqrt{U^2 + r^2 - 2Urv} = -\frac{Ur}{\sqrt{U^2 + r^2 - 2Urv}}$$

from which it follows that

$$\int_{-1}^{1} \frac{dv}{\sqrt{U^2 + r^2 - 2Urv}} = -\frac{1}{Ur}[\sqrt{U^2 + r^2 - 2Ur} - \sqrt{U^2 + r^2 + 2Ur}]$$

whence the "miraculous" result:

$$J(r) = \frac{1}{Ur}[(U + r) - (U - r)] = \frac{2}{U}$$

Then we end up with

$$\int_{B_R} \frac{\mu(||x||)}{||u-x||}dx = \int_0^R 2\pi r^2 \mu(r) J(r) dr = \frac{1}{U} \int_0^R 4\pi r^2 \mu(r) dr = \frac{M}{U}$$

□

4 Proof for the field.

Proof. We use the same system of spherical coordinates to compute the vector

$$\int_{B_R} \frac{\mu(||x||)(u-x)}{||u-x||^3} dx = \int_0^R \mu(r) \int_0^{2\pi} \int_0^{\pi} \frac{r^2 \sin\theta}{||u-x||^3}(u-x) d\theta d\varphi dr$$

so that for fixed r we need to compute

$$\int_0^{2\pi} \int_0^{\pi} \frac{r^2 \sin\theta}{||u-x||^3}(u-x) d\theta d\varphi$$

First we observe that this integral is covariant with respect to rotations around the origin, so that we can choose $u = (U, 0, 0)$ as previously. Because

$$u - x = (U - r\cos\theta, -r\sin\theta\cos\varphi, -r\sin\theta\sin\varphi)$$

By integrating in φ first and using the property

$$\cos\varphi(s+\pi) + \cos\varphi(s) = \sin\varphi(s+\pi) + \sin\varphi(s) = 0$$

we obtain that the second and third components are 0, so that the integral is colinear to u as expected. For the first component we are led to compute the integral

$$\int_0^{2\pi} \int_0^{\pi} \frac{(U - r\cos\theta)\sin\theta}{||u-x||^3} d\theta d\varphi = 2\pi \int_0^{\pi} \frac{(U - r\cos\theta)\sin\theta}{(U^2 + r^2 - 2Ur\cos\theta)^{3/2}} d\theta$$

Introducing as before $v = \cos\theta$ we find

$$\int_0^{\pi} \frac{(U - r\cos\theta)\sin\theta}{(U^2 + r^2 - 2Ur\cos\theta)^{3/2}} d\theta = \int_{-1}^{1} \frac{(U - rv)dv}{(U^2 + r^2 - 2Urv)^{3/2}} := K(r)$$

In order to compute $K(r)$, we observe that

$$\frac{d}{dv}(U^2 + r^2 - 2Urv)^{-(1/2)} = \frac{Ur}{(U^2 + r^2 - 2Urv)^{3/2}}$$

from which it follows that

$$K(r) = \frac{1}{Ur} \int_{-1}^{1} (U - rv) \frac{d}{dv} \left[(U^2 + r^2 - 2Urv)^{-(1/2)} \right] dv$$

Since the product $(U - rv)[(U^2 + r^2 - 2Urv)^{-(1/2)}$ is equal to 1 at both endpoints 1 and -1, the contribution of the integrated part is 0 and we find

$$K(r) = -\frac{1}{Ur} \int_{-1}^{1} \frac{d}{dv}(U - rv) \left[(U^2 + r^2 - 2Urv)^{-(1/2)} \right] dv$$

$$= \frac{1}{U} \int_{-1}^{1} \left[(U^2 + r^2 - 2Urv)^{-(1/2)} \right] dv = \frac{2}{U^2}$$

according to the calculation made in the previous section. Finally we obtain

$$\int_0^R \mu(r) \int_0^{2\pi} \int_0^{\pi} \frac{r^2 \sin\theta}{||u-x||^3}(u-x)d\theta d\varphi dr = (\frac{2}{U^2}\int_0^R 2\pi r^2 \mu(r)dr, 0, 0)$$

The first component is

$$\frac{1}{U^2}\int_0^R 4\pi r^2 \mu(r)dr = \frac{M}{U^2}$$

and this concludes the proof. □

Remark 4.1. *Both results on the field and the potential are still valid in the limiting case $||u|| = R$ under the hypothesis $\mu \in L^1(0, R)$, since in both cases the singularity coming from the boundary point is integrable.*

Remark 4.2. *In our calculation, we did not require the density μ to be strictly positive for all values of r, we just need it to be positive on a set of positive measure in order for the total mass to be positive. So the result applies in particular to the case of thin spherical shells. A slightly more complicated proof for this case is given in Wikipedia.*

5 Conclusion.

We start this last section by a remark:

Remark 5.1. *We actually proved that whenever $||u|| > r > 0$, it holds*

$$\int_{S_r} \frac{d\sigma}{||u-x||} = \frac{|S_r|}{||u||} = \frac{4\pi r^2}{||u||}$$

with

$$S_r := \{x \in \mathbb{R}^3, ||x|| = r\}$$

and $d\sigma$ the area element along the sphere S_r.

We conclude with the observation that Newton's gravitational law is very efficient because it has a strong geometric background. Historically, the comparison with propagation of light played a basic role to formulate it with the help of Robert Hooke. Newton was also very interested in optics, so that in a sense all of this is not the result of chance. The confirmation that the 1/r potential implies the 3 Kepler's laws was the icing on the cake.

References

[1] J. Evans; Physics, the Human Adventure: From Copernicus to Einstein and Beyond, *Physics Today* 54(10), 69 (2001); doi: 10.1063/1.142055

[2] I. Newton; Philosophiæ naturalis principia mathematica, *Londini, iussu Societatis Regiae ac typis Josephi Streater,* anno MDCLXXXVII (editio princeps)

[3] C.F. Gauss; Theoretische Astronomie, Übergeordnetes Werk, Werke, Jahr 1906, Digitalisierungsdatum 2002-04-15.

[4] L. Landau, E. Lifchitz; Mécanique, *Editions MIR* (1969).