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Two Intrinsic Formulae Generated by the Jones Polynomial

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Abstract. This paper notes and derives two simple, intrinsic mathematical relations generated by the Jones polynomial.

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"Out of clutter, find simplicity." - John A. Wheeler (on Albert Einstein)[1]

1. Introduction

This paper notes two nice and simple intrinsic relations generated by the Jones polynomial for knots that, though the relation's *terms* can be generated by the Skein relations and Kaufmann bracket [2-3] (once the relation is noticed), the relations themselves must be actually perceived and discovered and first noticed in order to then realize that the Skein relations and Kaufmann bracket generate their terms.

This author in studying knots for the purpose of modeling the elementary particles of physics [5-8], has not found through due diligence these specific relations recognized or even alluded to in any journal article or book on knots or otherwise by any author. Perhaps they are too trivial to note? Or perhaps they just simply weren't noticed, until now. They are thus noted herein and derived, but in novel way.

I contemplate the polynomials:

$$t^{-2} - t^{-1} + t^0 - t + t^2 \tag{1}$$

$$t^2 - t^3 + 2t^4 - 2t^5 + 3t^6 - 2t^7 + t^8 - t^9$$

$$t^3 + t^5 - t^8$$

$$-t^{-11} + 3t^{-10} - 5t^{-9} + 6t^{-8} - 8t^{-7} + 8t^{-6} - 6t^{-5} + 5t^{-4} - 2t^{-3} + t^{-2}$$

$$-t^{-4} + t^{-3} + t^{-2}$$

$$t^0 - 2t + 4t^2 - 5t^3 + 6t^4 - 6t^5 + 6t^6 - 4t^7 + 2t^8 - t^9$$

[1] Cosmic Search Magazine, Vol. 1 No. 4, Forum: John A. Wheeler, From the Big Bang to the Big Crunch, http://www.bigear.org/vol1no4/wheeler.htm

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$$-t^{1/2} - t^{5/2}$$

$$t^{-5} - 2t^{-4} + 3t^{-3} - 3t^{-2} + 3t^{-1} - 3t^{0} - 2t - t^{2}$$

...Clutter. But staring long enough an utterly simple (and so perhaps profound?) pattern emerges. Two of these polynomials are not like the others; in a very certain specific sense they are not the same.

One of the polynomials is a Jones polynomial which signals a link with two components, whereas the other polynomials (excepting one other polynomial among them) are Jones polynomials which signal only one-component links. And one of the polynomials isn't a Jones polynomial at all.

We know the distinguished two by simply reading the next sentence. These are the respective Jones polynomials for the 4_1, the 7_3, the 8_{19}, the 9_{23}, the trefoil (3_1), the 9_{36}, and the Hopf (2_1^2) links [1-3], along with the last polynomial which, we will see, is not even a Jones polynomial.

But what if one wasn't given that information? Can one, by scanning the polynomial and doing a simple operation determine whether or not it can be a Jones polynomial in the first instance? And if so then with the same operation can one determine how many components its signaled link must have? What about the following list:

$$f_1 = t^{-9} - t^{-8} + 6t^{-7} - 6t^{-6} + 16t^{-5} - 15t^{-4} + 21t^{-3} - 15t^{-2} + 15t^{-1} - 10t^{0} + 5t^{1} - t^{2} \tag{2}$$

$$f_2 = -t^{-9} + 3t^{-8} - 5t^{-7} + 6t^{-6} - 7t^{-5} + 7t^{-4} - 6t^{-3} + 4t^{-2} - 2t^{-1} + t$$

$$f_3 = -t^{-7/2} - t^{-3/2} - 2t^{-1/2} - 2t^{1/2} - t^{3/2} - t^{7/2}$$

$$f_4 = t^{9/2} - 2t^{13/2} + 3t^{15/2} - 3t^{17/2} + t^{21/2}$$

$$f_5 = -t^{7/2} - t^{11/2} + t^{13/2} - t^{15/2} + t^{17/2} - t^{19/2} + t^{21/2} - t^{23/2}$$

$$f_6 = -t^{-9/2} + 3t^{-7/2} - 5t^{-5/2} + 5t^{-3/2} - 6t^{-1/2} + 5t^{1/2} - 4t^{3/2} + 2t^{5/2} - t^{7/2}?$$

The answer is that f_4 is not a Jones polynomial. All the rest potentially are (and happen to be). How can one tell?

2. The Link Class Invariant

Designate the class containing all links L with the same number n of components as $\mathcal{L}_n$. We include in $\mathcal{L}_n$ both the link L and its Jones polynomial J. Thus, for example, $\mathcal{L}_2$ is the class of all Jones polynomials and their referent links having 2 and only 2 components. For instance:

$$7_3 \in \mathcal{L}_1 \qquad 2^2_1 \in \mathcal{L}_2 \qquad 8^4_3 \in \mathcal{L}_4, \tag{3}$$

with the link L implicitly carrying its Jones polynomial J into the class with it.

Let $\{J\}_{L_n}$ be the sequence of coefficients of the Jones polynomial of any link L with n components. For example, for the Hopf link 2^2_1 we have [1-2]: $\{J\}_{2^2_1} = \{-1, 0, -1\}$. For the 8^2_1 link we have [2]: $\{J\}_{8^2_1} = \{-1, 0, -1, 1, -1, 1, -1, 1, -1\}$. Call $|J|_{L_n}$ the sum of the coefficients of any link L with n components - a member of $\mathcal{L}_n$:

$$|J|_{L_n} = \sum \{J\}_{L_n} : L_n \in \mathcal{L}_n. \tag{4}$$

The link class conjecture is:

$$|J|_{L_n} = X_n \quad n \in \mathbb{N}, \tag{5}$$

where $X_n \in \mathbb{Z}$ and is completely determined and calculable for all n, to be shown infra. Therefore one can state that any two Jones polynomials J and J' belong to the same $\mathcal{L}$-class iff $|J| = |J'|$. One can also state that given any polynomial p, if $|p| \neq X_n$ for some $n \in \mathbb{N}$ then p is not a Jones polynomial.

3. The Link Class Iteration

One can sit down and go through all the links with 1 component [1-2,4], and calculate $|J|_{L_1}$ for each. One will find:

$$\sum \{J\}_{L_1} = 1 \quad \forall L \in \mathcal{L}_1. \tag{6}$$

Do this for those links with 2, 3 and 4 components and one will find:

$$\sum \{J\}_{L_2} = -2 \quad \forall L \in \mathcal{L}_2 \tag{7}$$

$$\sum \{J\}_{L_3} = 4 \quad \forall L \in \mathcal{L}_3$$

$$\sum \{J\}_{L_4} = -8 \quad \forall L \in \mathcal{L}_4.$$

That is to say, we have for the Jones polynomial link class:

$$\sum \{J\}_L = (-2)^{n-1} \quad \forall L \in \mathcal{L}_n, \tag{8}$$

or what is the same:

$$|J|_{L_n} = (-2)^{n-1}. \tag{9}$$

If the sum of the coefficients of a purported Jones polynomial does not equal one of the numbers given by (9), then it's not a Jones polynomial. Looking back to (2), one finds that $|f_4| = 0$, and therefore f_4 is not a Jones polynomial. The others are, and are given by [4]:

$$L_{f_1} = L11a548 \tag{10}$$

$$L_{f_2} = 9_{20} \qquad L_{f_3} = 8_3^4$$

$$L_{f_5} = 8_1^2 \qquad L_{f_6} = 8_{10}^2.$$

Likewise looking back to (1) we see that the last polynomial p has $|p| = -4$, and therefore it cannot be a Jones polynomial.

Equation (9) applies as well if the orientation of the component(s) are changed, applies to alternating and non-alternating L, and applies to unoriented L. For an example, reversing the orientation of one component of the 2-component Hopf link gives the following Jones polynomial and coefficient sum:

$$-t^{-5/2} - t^{-1/2} \qquad |J|_L = -2. \tag{11}$$

Consider the Jones polynomial for unoriented (u) 7_1^3 [9]:

$$t^{-5/2} - t^{-3/2} + 4t^{-1/2} - 3t^{1/2} + 4t^{3/2} - 3t^{5/2} + 3t^{7/2} - t^{9/2}, \tag{12}$$

which is strikingly different from the Jones polynomial for oriented 7_1^3 [2]:

$$-t^{-3} + 3t^{-2} - 3t^{-1} + 4t^0 - 3t^1 + 4t^2 - t^3 + t^4. \tag{13}$$

Nevertheless we have $|J|_{7_1^3(u)} = 4 = |J|_{7_1^3}$.

Considering the pattern in (6)-(7) it is straightforward to deduce that the sum of the coefficients for any $n{:}J \in \mathcal{L}_n$ is determinable and can be calculated by utilizing the following iteration:

$$\sum \{J\}_{L_{n+1}} - \sum \{J\}_{L_n} = (-1)^n \left(2^n + 2^{n-1}\right) \tag{14}$$

$$n \in \mathbb{N}, \qquad \sum \{J\}_{L_1} = 1.$$

4. Conclusion

Number (14) is a nice, simple intrinsic relation generated by the Jones polynomial, just as (8) is. Their application? One never knows.

Statements and Declarations

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