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ON A NEW TWO POINT TAYLOR EXPANSION WITH APPLICATIONS

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Abstract- A new two point Taylor series expansion is proposed. The expansion is slightly different than the classical definition. The coefficients are calculated as recursive relations in a general form. The two point Taylor expansion is applied to expressing two different functions, one of which has a finite interval of convergence and the other infinite interval of convergence. The conditions for convergence are derived and the results are compared with the results of single point Taylor expansions as well as two point Taylor expansions reported in the literature. It is found that for a finite radius of convergence, two point Taylor expansions can have a single convergence interval as well as two separate convergence intervals. Results of the new expansion are compared with the single point Taylor expansions as well as the classical two point Taylor expansion. Generally speaking two point Taylor expansions better represent the real function when the series is truncated. The new two point expansion and the classical two point expansion produced identical results for the problems treated. An application of the series to solution of a variable coefficient differential equation is also treated.

Keywords-Two Point Taylor Series, Single Point Taylor Series, Convergence Interval, Ordinary Differential Equations

MSC No-41A58, 40A30, 34A25

1. INTRODUCTION

Taylor and MacLaurin series are one of the fundamental topics in mathematics. A continuous and infinitely differentiable function can always be expressed in terms of a polynomial series, the coefficient of which is determined by the derivatives at a given point. The series expansion may be convergent over the whole domain or may have a limited convergence interval which is

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determined by the radius of convergence. In order to better approximate the functions, instead of the single point Taylor expansions, two point Taylor expansions were also proposed in the literature. Rotational symmetric lens profiles were described by the two-point Taylor polynomials. It is shown that the two point Taylor expansions better approximates the mapping functions [1]. Two point Taylor expansions were employed in the area of finance to determine density and option price expansions [2]. Such expansions were considered in the complex domain also. Singular one dimensional boundary value problems were treated [3]. Another complex domain treatment of two point Taylor expansions is presented in [4]. The two point series solutions were applied to nonlinear partial differential equations [5]. Finally, elliptic boundary value problems were also analyzed [6]. For Taylor expansions and their link to perturbation solutions, see [7]. A nonlinear curve equation with constant acceleration components were examined by numerical, single point Taylor expansion and perturbation methods [8].

In this work, a slightly different new version of the two point Taylor expansions is proposed for the first time. The new version is compared with the single point Taylor expansion as well as the classical two point Taylor expansion. Two functions, one with a finite radius of convergence and the other with an infinite radius of convergence are treated. The new expansion and the classical two point expansion produced identical results. For the two point expansions, the convergence interval is widened compared to a Taylor series with respect to the single point expansion of the lower reference point whereas it is narrower with respect to the higher reference point of the single point expansion. If the two reference points are sufficiently far away from each other, then the convergence intervals double with appearance of a divergent intermediate interval. To the best of the author's knowledge, this feature has not been exploited before for such expansions. The criterion for convergence intervals is derived as well as for the doubling of convergence intervals for a specific problem. Finally, the new series solutions are applied to a variable coefficient ordinary differential equation also.

2. TWO POINT TAYLOR SERIES EXPANSIONS

The new proposed two point Taylor expression is given first.

Theorem 1.

Given an analytical function $f(x)$ and the convergent polynomial approximation defined with respect to two reference points $x = x_0$ and $x = x_1$

$$f(x) = \sum_{m=0}^{\infty} a_{2m} (x-x_0)^m (x-x_1)^m + a_{2m+1}(x-x_0)^{m+1}(x-x_1)^m \ , \quad (1)$$

the coefficients a_{2m} and a_{2m+1} are uniquely determined by the equations

$$f^{(k)}(x_0) = \frac{d^k}{dx^k}\{\sum_{m=0}^{\infty} a_{2m} (x-x_0)^m (x-x_1)^m + a_{2m+1}(x-x_0)^{m+1}(x-x_1)^m\}_{x=x_0} \quad (2)$$

$$f^{(k)}(x_1) = \frac{d^k}{dx^k}\{\sum_{m=0}^{\infty} a_{2m} (x-x_0)^m (x-x_1)^m + a_{2m+1}(x-x_0)^{m+1}(x-x_1)^m\}_{x=x_1} \ , \quad (3)$$

$k = 0,1,2,\ldots$, where $f^{(k)} = \frac{d^k f}{dx^k}$ with the coefficients being calculated from the recursive relations

$$a_{2m} = \Bigg[f^{(m)}(x_0) - \sum_{i=1}^{Int\left(\frac{m}{2}\right)} \binom{m}{i} (m-i)! \left(\prod_{k=m-2i+1}^{m-i} k\right) (x_0-x_1)^{m-2i} a_{2m-2i}$$

$$- \sum_{i=1}^{Int\left(\frac{m-1}{2}\right)} \binom{m}{i} (m-i)! \left(\prod_{k=m-2i}^{m-i-1} k\right) (x_0-x_1)^{m-2i-1} a_{2m-2i-1} - m!\,(x_0-x_1)^{m-1} a_{2m-1}\Bigg]$$

$$/m!\,(x_0-x_1)^m \quad (4)$$

$$a_{2m+1} = \Bigg[f^{(m)}(x_1) - \sum_{i=1}^{Int\left(\frac{m}{2}\right)} \binom{m}{i} (m-i)! \left(\prod_{k=m-2i+1}^{m-i} k\right) (x_1-x_0)^{m-2i} a_{2m-2i}$$

$$- \sum_{i=1}^{Int\left(\frac{m+1}{2}\right)} \binom{m}{m-i} (m-i)! \left(\prod_{k=m-2i+2}^{m-i+1} k\right) (x_1-x_0)^{m-2i+1} a_{2m-2i+1} - m!\,(x_1-x_0)^m a_{2m}\Bigg]$$

$$/m!\,(x_1-x_0)^{m+1} \ . \quad \square \quad (5)$$

Proof

A straightforward calculation of the *m*'th derivatives at points $x = x_0$ and $x = x_1$ yield

$$f^{(m)}(x_0) = a_{2m} \binom{m}{0} m!\ (x_0-x_1)^m + a_{2m-2} \binom{m}{1} (m-1)!\ (m-1)(x_0-x_1)^{m-2} + \cdots$$

$$+ a_{2m-2i} \binom{m}{i} (m-i)!\ (m-i)(m-i-1)\ldots(m-2i+1)(x_0-x_1)^{m-2i}$$

$$+ a_{2m-1} \binom{m}{0} m!\ (x_0-x_1)^{m-1} + a_{2m-3} \binom{m}{1} (m-1)!\ (m-2)(x_0-x_1)^{m-3} +$$

$$\cdots + a_{2m-2i-1} \binom{m}{i} (m-i)!\ (m-i-1)\ldots(m-2i)(x_0-x_1)^{m-2i-1} \quad (6)$$

$$f^{(m)}(x_1) = a_{2m}\binom{m}{m} m!\ (x_1 - x_0)^m + a_{2m-2}\binom{m}{m-1}(m-1)!\ (m-1)(x_1 - x_0)^{m-2} + \cdots$$
$$+a_{2m-2i}\binom{m}{i}(m-i)!\ (m-i)(m-i-1)\dots(m-2i+1)(x_1 - x_0)^{m-2i}$$
$$+a_{2m+1}\binom{m}{m} m!\ (x_1 - x_0)^{m+1} + a_{2m-1}\binom{m}{m-1} m(m-1)!\ (x_1 - x_0)^{m-1} + \cdots$$
$$+a_{2m-2i+1}\binom{m}{m-i}(m-i)!\ (m-i+1)(m-i)\dots(m-2i+2)(x_1 - x_0)^{m-2i+1} \quad . \tag{7}$$

Solving a_{2m} from (6) and a_{2m+1} from (7), and using the summation and multiplication signs, the recursive relations (4) and (5) are obtained. The first twelve coefficients in explicit form are

$$a_0 = f(x_0) \tag{8}$$
$$a_1 = \frac{f(x_1)-a_0}{x_1-x_0} \tag{9}$$
$$a_2 = \frac{f'(x_0)-a_1}{x_0-x_1} \tag{10}$$
$$a_3 = \frac{f'(x_1)-(x_1-x_0)a_2-a_1}{(x_1-x_0)^2} \tag{11}$$
$$a_4 = \frac{f''(x_0)-2(x_0-x_1)a_3-2a_2}{2(x_0-x_1)^2} \tag{12}$$
$$a_5 = \frac{f''(x_1)-2(x_1-x_0)^2a_4-4(x_1-x_0)a_3-2a_2}{2(x_1-x_0)^3} \tag{13}$$
$$a_6 = \frac{f'''(x_0)-6(x_0-x_1)^2a_5-12(x_0-x_1)a_4-6a_3}{6(x_0-x_1)^3} \tag{14}$$
$$a_7 = \frac{f'''(x_1)-6(x_1-x_0)^3a_6-18(x_1-x_0)^2a_5-12(x_1-x_0)a_4-6a_3}{6(x_1-x_0)^4} \tag{15}$$
$$a_8 = \frac{f^{(iv)}(x_0)-24(x_0-x_1)^3a_7-72(x_0-x_1)^2a_6-48(x_0-x_1)a_5-24a_4}{24(x_0-x_1)^4} \tag{16}$$
$$a_9 = \frac{f^{(iv)}(x_1)-24(x_1-x_0)^4a_8-96(x_1-x_0)^3a_7-72(x_1-x_0)^2a_6-72(x_1-x_0)a_5-24a_4}{24(x_1-x_0)^5} \tag{17}$$
$$a_{10} = \frac{f^{(v)}(x_0)-120(x_0-x_1)^4a_9-480(x_0-x_1)^3a_8-360(x_0-x_1)^2a_7-360(x_0-x_1)a_6-120a_5}{120(x_0-x_1)^5} \tag{18}$$
$$a_{11} = \frac{f^{(v)}(x_1)-120(x_1-x_0)^5a_{10}-600(x_1-x_0)^4a_9-480(x_1-x_0)^3a_8-720(x_1-x_0)^2a_7-360(x_1-x_0)a_6-120a_5}{120(x_1-x_0)^6}. \tag{19}$$

To prove the uniqueness of the coefficients, assume that there are some other b_{2m} and b_{2m+1} coefficients to express the same function

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)^2(x - x_1) + \cdots \tag{20}$$
$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)^2(x - x_1) + \cdots \quad . \tag{21}$$

Subtracting (20) from (21)

$$a_0 - b_0 + (a_1 - b_1)(x - x_0) + (a_2 - b_2)(x - x_0)(x - x_1) + (a_3 - b_3)(x - x_0)^2(x - x_1) + \cdots = 0 \quad . \tag{22}$$

But the Wronskian of $(x - x_0)^m(x - x_1)^m$ and $(x - x_0)^n(x - x_1)^n$ is

$$W(x) = \begin{vmatrix} (x - x_0)^m(x - x_1)^m & (x - x_0)^n(x - x_1)^n \\ \frac{d}{dx}[(x - x_0)^m(x - x_1)^m] & \frac{d}{dx}[(x - x_0)^n(x - x_1)^n] \end{vmatrix}$$

$$= (n - m)(2x - a - b)(x - x_0)^{m+n-1}(x - x_1)^{m+n-1} \neq 0 \text{ for } m \neq n, \tag{23}$$

which dictates that the different powers are linearly independent functions of each other. Therefore, in order (22) to identically equal to zero for each x, the coefficients should vanish leading to $a_i = b_i, i = 0,1,2 \ldots$ which proves the uniqueness of the coefficients □

The classical version of the two point Taylor expansion used in the literature is slightly different from the expansion (1)

$$f(x) = \sum_{m=0}^{\infty}[a_m (x - x_0) + b_m(x - x_1)][(x - x_0)(x - x_1)]^m \quad , \tag{24}$$

leading to different coefficients

$$a_0 = \frac{f(x_1)}{x_1 - x_0} \tag{25}$$

$$b_0 = \frac{f(x_0)}{x_0 - x_1} \tag{26}$$

$$a_1 = \frac{f'(x_1) - a_0 - b_0}{(x_1 - x_0)^2} \tag{27}$$

$$b_1 = \frac{f'(x_0) - a_0 - b_0}{(x_0 - x_1)^2} \tag{28}$$

$$a_2 = \frac{f''(x_1) - 2(2a_1 + b_1)(x_1 - x_0)}{2(x_1 - x_0)^3} \tag{29}$$

$$b_2 = \frac{f''(x_0) - 2(a_1 + 2b_1)(x_0 - x_1)}{2(x_0 - x_1)^3} \tag{30}$$

$$a_3 = \frac{f'''(x_1) - 6(3a_2 + b_2)(x_1 - x_0)^2 - 6(a_1 + b_1)}{6(x_1 - x_0)^4} \tag{31}$$

$$b_3 = \frac{f'''(x_0) - 6(a_2 + 3b_2)(x_0 - x_1)^2 - 6(a_1 + b_1)}{6(x_0 - x_1)^4} \tag{32}$$

$$a_4 = \frac{f^{(iv)}(x_1) - 24(4a_3 + b_3)(x_1 - x_0)^3 - 24(3a_2 + 2b_2)(x_1 - x_0)}{24(x_1 - x_0)^5} \tag{33}$$

$$b_4 = \frac{f^{(iv)}(x_0) - 24(a_3 + 4b_3)(x_0 - x_1)^3 - 24(2a_2 + 3b_2)(x_0 - x_1)}{24(x_0 - x_1)^5} \tag{34}$$

$$a_5 = \frac{f^{(v)}(x_1) - 120(5a_4 + b_4)(x_1 - x_0)^4 - 360(2a_3 + b_3)(x_1 - x_0)^2 - 120(a_2 + b_2)}{120(x_1 - x_0)^6} \tag{35}$$

$$b_5 = \frac{f^{(v)}(x_0) - 120(a_4 + 5b_4)(x_0 - x_1)^4 - 360(a_3 + 2b_3)(x_0 - x_1)^2 - 120(a_2 + b_2)}{120(x_0 - x_1)^6} \tag{36}$$

It can be proven that this representation is a unique way of expressing an analytical convergent function, that is there exist only unique polynomial coefficients a_m and b_m for a given function.

It is well known that the single point Taylor expression in the vicinity of $x = x_0$ is

$$f(x) = \sum_{m=0}^{\infty} c_m (x - x_0)^m \tag{37}$$

where

$$c_m = \frac{f^{(m)}(x_0)}{m!} \tag{38}$$

Numerical comparisons of the two point Taylor expansions and the single point expansion will be given in the next section.

3. FUNCTIONAL APPROXIMATIONS

In this section, two functions will be approximately expressed in terms of two point Taylor series. The two point Taylor series and the single point Taylor series will be compared with the exact solution.

3.1. The function y=1/(1+x)

The two point Taylor expression of

$$f(x) = \sum_{m=0}^{\infty} a_{2m} (x - x_0)^m (x - x_1)^m + a_{2m+1}(x - x_0)^{m+1}(x - x_1)^m \;, \tag{39}$$

is derived for arbitrary two points x_0 and x_1 ($x_0 < x_1$). A straightforward calculation of (4) and (5) yields

$$a_{2m} = \frac{1}{(1+x_0)^{m+1}(1+x_1)^m}\,, \qquad a_{2m+1} = -\frac{1}{(1+x_0)^{m+1}(1+x_1)^{m+1}}\,. \tag{40}$$

For the classical two point Taylor expression

$$f(x) = \sum_{m=0}^{\infty}[a_m (x - x_0) + b_m(x - x_1)][(x - x_0)(x - x_1)]^m \tag{41}$$

the coefficients are

$$a_m = \frac{1}{(1+x_0)^m(1+x_1)^{m+1}(x_1 - x_0)}\,, \quad b_m = -\frac{1}{(1+x_0)^{m+1}(1+x_1)^m(x_1 - x_0)}\,. \tag{42}$$

Finally the single point Taylor expression about x_2 is

$$f(x) = \sum_{m=0}^{\infty} c_m (x - x_2)^m \tag{43}$$

where

$$c_m = \frac{(-1)^m}{(1+x_2)^{m+1}} \tag{44}$$

Using the ratio test for convergence of series, i.e. $\lim_{n\to\infty} \frac{p_{n+1}}{p_n} < 1$, p_n being the n'th term of the polynomial expression, the convergence criterion for both of the two point Taylor expansions turns out to be

$$|(x - x_0)(x - x_1)| < (1 + x_0)(1 + x_1)\ , \tag{45}$$

and for the single point expansion

$$|(x - x_2)| < 1 + x_2\ . \tag{46}$$

All expansions cease to be valid at the singular point of the function $x = -1$, no matter what the values of x_{0-2} is.

Condition (45) may lead to a single convergence interval as well as double convergence intervals. To the best of the author's knowledge, double convergence intervals were not discussed previously in the literature. Using the properties of the quadratic functions, the criterion for a single convergence interval turns out to be

$$x_1 < 3x_0 + 2 + 2\sqrt{2}(x_0 + 1)\ \ , \qquad (x_1 > x_0) \tag{47}$$

The convergence intervals are given in Table 1 for a number of specific numerical values of the reference points.

Table 1- Convergence intervals for some reference points

x_0	x_1	Convergence Interval	Criterion (47)
0	2	(-1,3)	Satisfied
0	6	(-1,1.58) and (4.41,7)	Not Satisfied
1	4	(-1,6)	Satisfied
1	9	(-1,11)	Satisfied
1	19	(-1,3.59) and (16.40,21)	Not satisfied
3	4	(-1,8)	Satisfied
18	19	(-1,38)	Satisfied
-2	2	(-$\sqrt{7}$,-1) and (1, $\sqrt{7}$)	Not satisfied

In Figure 1, the two point Taylor expansions are contrasted with the exact solution for the case of $x_0 = 0$ and $x_1 = 6$. Table 1 predicts two convergence regions for this case. As the number of terms increase, the intervals which predict the exact function closely widen converging to the intervals predicted by Table 1. Outside the convergence regions however, the predictions are worse as the number of terms increase.

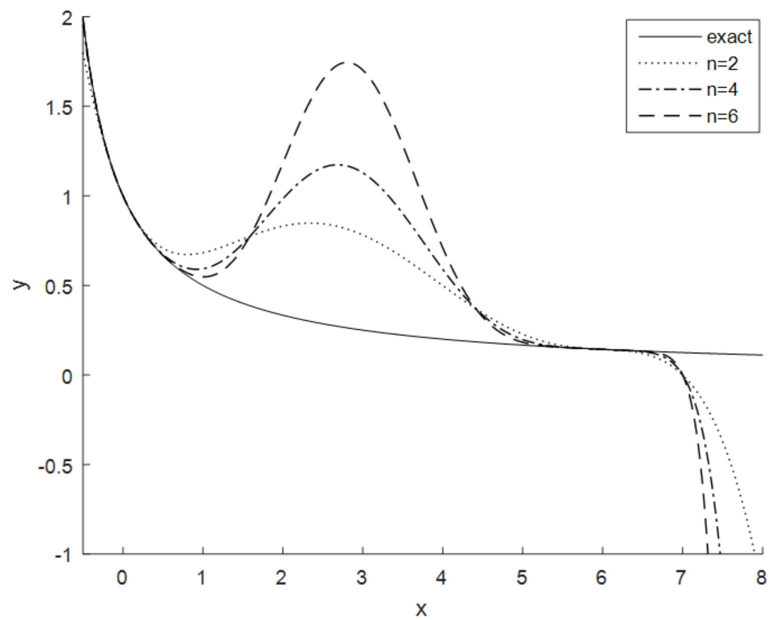

Figure 1- Two point Taylor approximations of function y=1/(1+x) for

$x_0 = 0, x_1 = 6$

Note that, although the formulations and coefficients are somewhat different, both two-point Taylor expansions produced identical results even for the finite truncations of the series.

In the case of single point expansions, the convergence interval is (-1,1) for $x_2 = 0$, and (-1,13) for $x_2 = 6$. That is, the two point Taylor expansion has a wider convergence region compared to the single point expansion in the vicinity of lower reference value and possesses a narrower convergence region compared to the single point expansion in the vicinity of higher reference value.

Another case in which there is only one convergence interval is shown in Figure 2. Four terms are taken in all expansions. It is obvious that two point Taylor expansion has a wider convergence region compared to the single point expansion in the vicinity of $x_2 = 1$, and a narrower convergence region compared to the single point expansion about $x_2 = 4$ as predicted by the theory. Note that for finite number of truncations, the two point Taylor expansion ($x_0 = 1$, $x_1 = 4$) is better at the left than the single point expansion about $x_2 = 4$ and better at the right than the single point expansion about $x_2 = 1$.

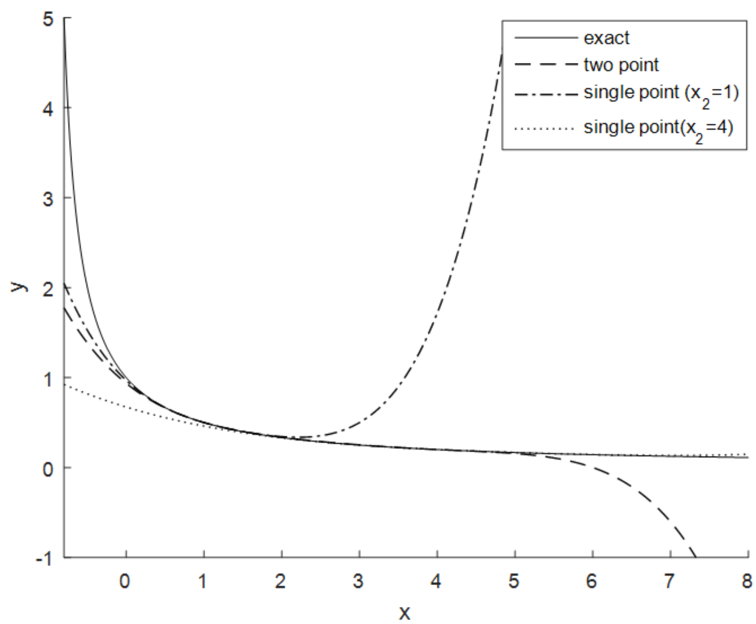

Figure 2- Comparison of Two point Taylor approximations ($x_0 = 1$, $x_1 = 4$) and Single Point approximations ($x_2 = 1$, $x_2 = 4$) of function y=1/(1+x)

One of the advantages of the two point Taylor expansions is that they can produce solutions at the right and left hand sides of the singular point, i.e. $x_s = -1$ for this specific case. Single point Taylor expansion solutions cannot cross the singularity points because they have only one convergence interval and the function ceases to be analytic at the singular point. Figure 3 is such an example in which the two reference points are $x_0 = -2$, $x_1 = 2$. Table 1 predicts two convergence intervals i.e., ($-\sqrt{7}$,-1) and (1, $\sqrt{7}$) which can be visualized from Figure 3 with 22 terms taken in the expansion.

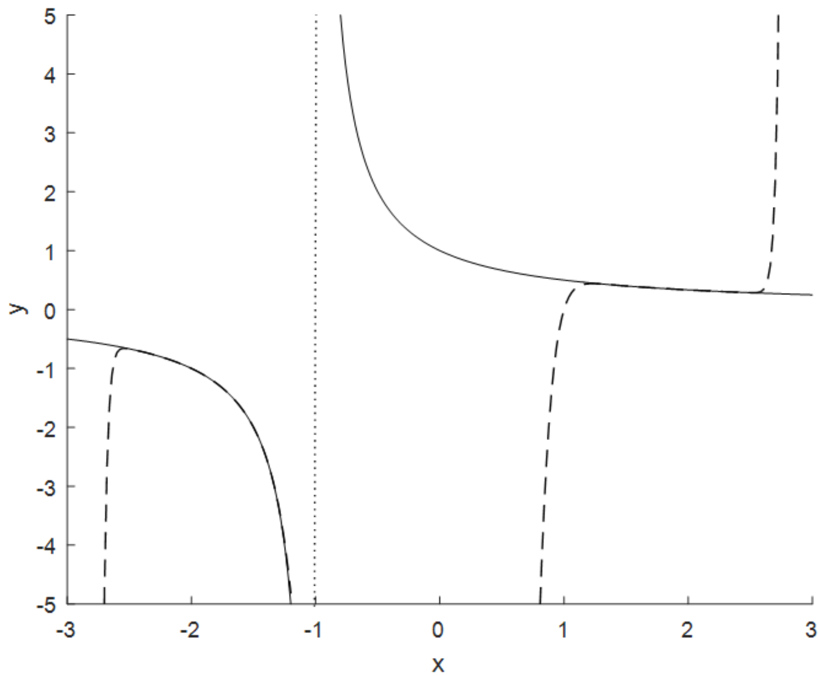

Figure 3- Comparison of the two point Taylor approximation ($x_0 = -2, x_1 = 2$) (dashed) and the exact function y=1/(1+x) (solid) about the singular point

3.2. The function y=exp(x)

This is a characteristic example where the convergence interval is infinity. The first twelve terms are taken in all the expansions. The performance of the two point Taylor expansion about $x_0 = -2$ and $x_1 = 2$ is slightly better than the performance of the single expansion about $x_2 = 0$ and much better than the single expansion about $x_2 = 2$ (Figure 4). As the number of terms increases, all the solutions will converge to the real solution. However, there are differences between the performances of the extensions for finite mode truncations.

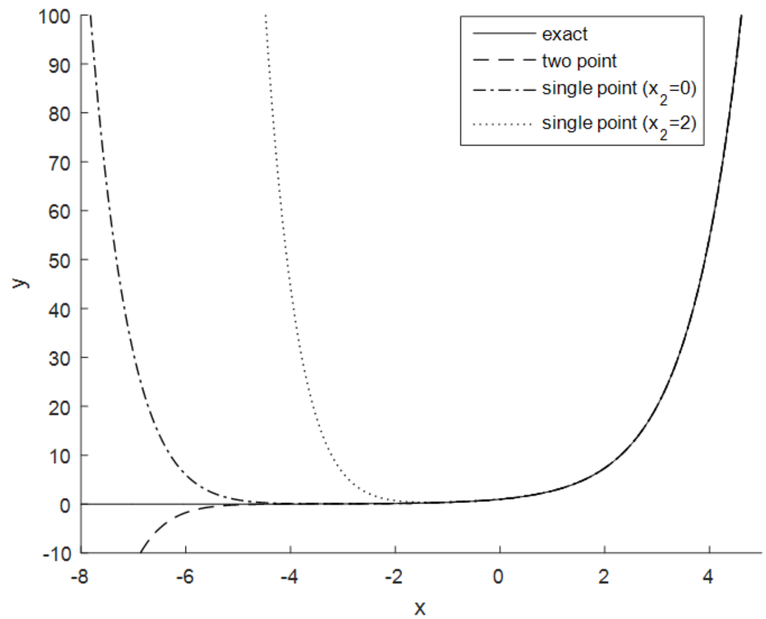

Figure 4- Comparison of the two point Taylor approximation, single point Taylor approximations and the exact function y=exp(x)

4. DIFFERENTIAL EQUATIONS

One of the common analytical solution methods of differential equations is the Taylor series solutions. Two point Taylor expansions can also be employed in search of approximate solutions. Consider the first order differential equation

$$y' + (1-2x)y = 0\,, \qquad y(0) = 1, \tag{48}$$

which has an exact solution

$$y_e = \exp[x(x-1)] \quad . \tag{49}$$

Assuming a two point series solution of the form

$$y(x) = \sum_{m=0}^{\infty} a_m\, x^m (x-1)^m + b_m x^{m+1} (x-1)^m \;, \tag{50}$$

Substituting into (48) and grouping the terms, one finally has the recursive relations

$$a_{m+1} = \frac{a_m}{m+1}\,, \qquad b_m = 0 \tag{51}$$

The general form of the coefficients is then

$$a_m = \frac{1}{m!} \quad . \tag{52}$$

The approximate two point series solution is

$$y(x) = \sum_{m=0}^{\infty} \frac{1}{m!} x^m (x-1)^m \quad . \tag{53}$$

The exact and approximate series solutions are contrasted in Figure 5. As the number of terms in the series solution increases, the approximate solution converges to the real solution.

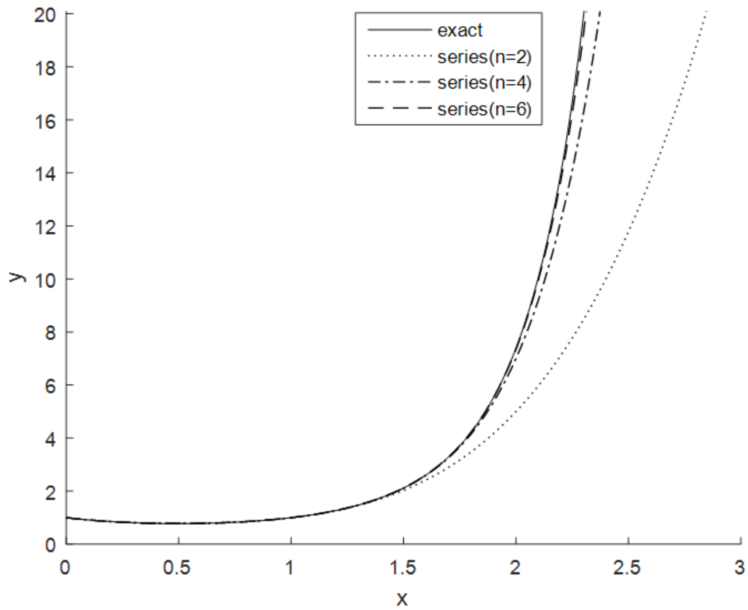

Figure 5- Comparison of the two point series approximation with the exact solution of the differential equation

5. CONCLUDING REMARKS

A new version of the two point Taylor expansion is given. The new version produces identical results with the classical version reported in the literature. For problems with finite radius of convergence, two point Taylor expansions may possess two different convergence intervals or a single convergence interval. When the selected two points are distant to each other, the single convergence interval may separate into two. A worked example is treated in detail. For problems of infinite radius of convergence, there is no separation of the convergence intervals. The two point and single point expansions are compared with each other. For functions with finite radius of convergences, the two point expansion is definitely advantageous compared to the single point expansion about the lower reference point. However, the single point expansion about the higher reference point may possess a wider convergence interval depending on the problem investigated. Despite the narrowing of the convergence interval, finite number of truncations of the series may produce better results compared to the single point expansions. The two point Taylor series can approximate the function at opposite sides of a singular point with two convergence intervals lying at the left and right of the singular point whereas single point expansions cannot be valid at both sides. The proposed two point expansion may be applied to solve approximately the differential equations. A variable coefficient linear differential equation with an exact solution is treated to demonstrate the application of the method.

Availability of Supporting Data- There is no additional data associated with the paper
Competing Interests- Author declares no competing interests

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