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Quantum distinguishability and symplectic topology

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The distinguishability between pairs of quantum blobs, as measured by quantum fidelity, is defined on complex phase space. Fidelity is physically interpreted as the probability that the pair are mistaken for each other upon a measurement. The mathematical representation is based on the concept of symplectic capacity in symplectic topology. The fidelity is the absolute square of the complex-valued overlap between the symplectic capacities of the pair of states. The symplectic capacity for a given state, onto any conjugate plane of degrees of freedom, is postulated to be bounded from below by the Gromov width $h/2$. This generalizes the Liouville theorem in classical mechanics, which states that the volume of a region of phase space is invariant under the Hamiltonian flow of the system, by constraining the shape of the flow. It is shown that for closed Hamiltonian systems, the Schrödinger equation is the mathematical representation for the conservation of fidelity.

Keywords: Non-squeezing theorem, Indeterminacy relation, Quantum blobs, Quantum fidelity, Schrödinger equation

Introduction

What are the key character traits of quantum mechanics which distinguish it from classical mechanics? Several features could be mentioned, including e.g. the quantum indeterminacy relation, the superposition of complex-valued probability amplitudes, quantum entanglement, and non-locality, to mention some. The point of departure taken in this article to address this question has to do with the ability of the observer to distinguish between the pairs of states in classical and quantum mechanics. In the former, it is straightforward. Either the pair is identical or they are distinct. There are no alternatives to these two extremes. In quantum mechanics, the situation is quite different due to the indeterminacy relation. In essence, the problem is to mathematically describe how close the pairs of states are on the space of states. There are well-established measures for this distance. Two such measures are trace distance and fidelity. In this article, we will consider fidelity. The fidelity is physically interpreted as the probability that the pair are mistaken for each other upon measurement by an observer.

In the canonical approach to non-relativistic quantum mechanics [1–6], the space of states is the Hilbert space of complex-valued state vectors where Hermitian operators, representing observables, act on this space. The properties of state vectors and Hermitian operators are summarized in a set of axioms which set the foundation for the mathematical representation of canonical quantum mechanics. The dynamical evolution of the state vector is then encoded in the postulate that it should be unitary, represented in differential form by the Schrödinger equation. The connection between the state vector and experimental reality is then given by postulating the Born rule, which interprets the state vector as a complex-valued probability amplitude whose squared modulus gives the probability for the system to occupy the specific state. From this mathematical structure, and the physical postulates, distance measures on Hilbert space, such as fidelity, can be clearly defined and their properties investigated [7]. An equivalent formulation, geometric quantum mechanics [8–21], is obtained by considering the projective Hilbert space, whose elements are the complex-valued rays, as the space of states, where observables are real-valued functions. This space is endowed with a metric, the Fubini-Study metric, which is symplectic, complex-valued, and Riemannian, i.e. it is Kählerian. It is worth noting that the projective Hilbert space, unlike Hilbert space, is non-linear and has a symplectic structure, thus sharing the same key features as the classical phase space, but with the key difference that the metric has two additional compatible structures associated with it. The geometry of the projective Hilbert space, as described by the Fubini-Study metric, then allows for a clear discussion on the distance between pairs of states for a given quantum system [21].

To the best of the author's knowledge, there is no well-known definition of fidelity in the other most commonly used mathematical representations of non-relativistic quantum mechanics, e.g. the phase-space formulation [22–25] or the spacetime approach by Feynman [26]. In the former, the state of a quantum system is given by a quasiprobability distribution, the most well-known being the Wigner probability function, on the phase space. Real-valued functions on the phase space represent observables, with the star product between observables replacing the operator multiplication in the Hilbert space formulation. In Feynman's path-integral formulation, alternative paths in which an event could unfold are defined by a complex-valued probability amplitude, with the probability for the event to occur being given by the squared modulus of the sum of amplitudes.

Given the existence of well-defined mathematical representations of the concept of quantum distinguishability,

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given e.g. by fidelity, one might ask what the purpose of the present article may be. The article intends to initiate a study on the possibility that the branch of symplectic topology might act as yet another mathematically equivalent framework for non-relativistic quantum mechanics. This seems to be an interesting and worthwhile study considering the role played by symplectic geometry in the Hamiltonian formulation of classical mechanics. There, symplectic geometry describes the local geometry on the phase space for the Hamiltonian flow. Perhaps, symplectic topology can play an important role in the topological description of the quantum flow of systems on the phase space. Possible hints that this might be the situation is the discovery of the non-squeezing theorem [27] and the concept of symplectic capacity [28] and their relation with the quantum indeterminacy relations [29, 30]. It is worthwhile to emphasize that it is not the ambition of the article to develop new physical insights or results to specific problems in the foundations of quantum mechanics. For the moment, the attempt is to see whether the concept of quantum probability and the Schrödinger equation can be properly defined within the language of symplectic topology. It is thus important to clearly state that we do not begin the present study within any of the familiar mathematical representations, e.g. the Hilbert-space or the phase-space formulations. The task is to begin from scratch with the key concept in symplectic topology, i.e. symplectic capacity, and that it is bounded from below as described by the Gromov non-squeezing theorem, and from this build up the concept of probability. It is a later task to try to check whether this construction is mathematically equivalent to the well-known formulations of quantum mechanics.

The space of states considered in the article is the phase space of generalized position q and momentum p, where the momentum is extended into the complex-valued domain. This extension is made to obtain the Schrödinger equation for the overlap between pairs of states. In the classical limit, i.e. $h \rightarrow 0$, where h is Planck's constant, the complex-valued phase space should reduce to the real-valued classical phase space. Therefore, we define the complex-valued momentum as

$$p \rightarrow p \cdot e^{ih\alpha}. \tag{1}$$

The idea to extend the phase space to become complex-valued is not as strange as it may seem. The space of states in geometric quantum mechanics, i.e. the projective Hilbert space, is also symplectic and complex-valued, in addition to being Riemannian.

For simplicity, we will consider only the squeezed coherent states, whose phase-space representation, in the language of symplectic topology, is given by the recently developed concept of quantum blobs [31]. The quantum blobs are the smallest units of phase space that are compatible with the Robertson-Schrödinger indeterminacy relation and invariant under general symplectic transformations.

Limited distinguishability

In classical mechanics, it is assumed that the state of the system can be specified with infinite precision. There is no uncertainty in the state. An observer is infinitely able to specify the physical degrees of freedom for the state. This is seen by the fact that the classical state for the system, e.g. in the Hamiltonian formulation, is given by an infinitesimal point on the phase space. Consider the situation where a single system at some time $t = 0$ is prepared in the state ψ. This defines the initial condition for the system. When the system evolves in time, the initial state ψ will be continuously updated in time, according to the Hamilton equations of motion. A smooth curve will thus be traced out on the phase space. If the same system is prepared in a distinctly different initial condition ϕ, the Liouville theorem states that the initial distinctions between the pair of states ψ and ϕ are conserved in time [32]. Therefore, the phase-space paths traced out by the system as it evolves in time from the pair of distinctly different initial conditions ψ and ϕ are not allowed to cross each other anywhere on the phase space, see Fig.1. For this reason, any given pair of states on the phase space can either be identified by an observer to be identical, i.e. $\psi = \phi$, or, completely distinct, i.e. $\psi \neq \phi$. These are the only two possibilities.

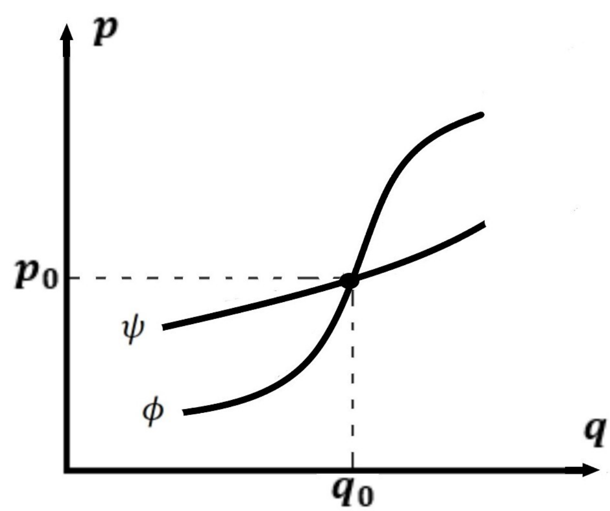

FIG. 1. The pair of possible initial conditions ψ and ϕ for a given system cross each other at the point (q_0, p_0), thus becoming indistinguishable. This violates the Liouville theorem and is not allowed in classical mechanics.

In statistical mechanics, the observer is not perfectly able to specify the state of the system. This is not due to an inherent property of the system. It is entirely due to the difficulty of the observer to keep perfect track of a large number of degrees of freedom [33]. Due to the uncertainty in the state of the system, the ability of the observer to distinguish between states decreases exponentially over time, as stated by the second law of thermodynamics, until the system has reached statistical equilibrium where all states are indistinguishable [34]. Due to the presence of statistical uncertainty, the observer is unable to predict a unique path on the phase space along which the system evolves in time.

The situation in quantum mechanics is different from statistical mechanics, primarily for two reasons. Firstly, there are states in quantum mechanics that can, depending on the set of observables that are being measured, be known with perfection. These observables are said to be compatible. Other states can be uncertain if the observables being measured are incompatible, meaning that they do not commute with each other. Secondly, the physical origin for the appearance of uncertainty is different. Specifically, consider the quantum indeterminacy relation between any given pair of non-commuting observables A and B, as given by

$$\langle(\Delta A)^2\rangle \cdot \langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A, B]\rangle|^2, \tag{2}$$

where $\langle(\Delta A)^2\rangle$ and $\langle(\Delta B)^2\rangle$ are the dispersions of the observables A and B, respectively. If the observables do not commute, i.e. if

$$[A, B] \neq 0, \tag{3}$$

then there is a lower bound on the product of the dispersions. Put differently, the dispersions cannot simultaneously be made arbitrarily small. There will always be a "fuzziness" in the observables of the system. In this situation, it is not possible to perfectly distinguish between pairs of initial quantum states ψ and ϕ as the system evolves in time. Consider e.g. the components of the spin observable S for a spin 1/2 system which satisfies the commutation relations

$$[S_i, S_j] = i\epsilon_{ijk}\frac{h}{2\pi}S_k. \tag{4}$$

The non-commutation of e.g. S_x and S_y means that they cannot in a Stern-Gerlach experiment be simultaneously determined with arbitrary precision. Another example is the historical Heisenberg uncertainty relation, i.e.

$$\Delta q_i \cdot \Delta p_j \geq \frac{h}{2\pi}\delta_{ij}, \tag{5}$$

which follows from the non-commutativity between the conjugate pair (q, p), i.e.

$$[q_i, p_j] = i\frac{h}{2\pi}\delta_{ij}. \tag{6}$$

In this article, we will consider the limited ability to distinguish between quantum states, within the language of symplectic topology. The fuzziness for non-commuting observables, such as e.g. q_i and p_i, will be interpreted to correspond to the impossibility of squeezing the projected area of the quantum blob onto the $(q_i, p_i)-$plane to a value smaller than the Gromov width $h/2$. For commuting pairs, e.g. q_i and q_j, there is no fuzziness since there is no restriction on the smallness of the projected area. The area can be made arbitrarily small.

Quantum blobs

Due to limited distinguishability, it is impossible to physically define, in the sense of observation, the notion of the state as given by an infinitesimal point on the phase space. In other words, the geometry of phase space is "pointless". To obtain a picture of the notion of state on such a phase space, consider an $N-$particle system, in d spatial dimensions, at some given time $t = 0$. Let it be assumed that the state of the system, denoted by ψ, is known, at this time, with maximum precision. Such a state is referred to as being saturated. It is further assumed that all conjugate pairs of degrees of freedom for the system, i.e. the coordinate and momenta pairs (q_k, p_k), with $k \in \{1, 2, ..., n\}$ where $n = d \cdot N$, is known to the same level of maximum precision. In this article, these symmetric states are interpreted as being the phase-space pictures of the coherent states [35–37]. The state of the system at time $t = 0$, $\psi(t = 0)$, occupy the $2n$-dimensional ball $B(\epsilon)$ defined by

$$\sum_{k=1}^{n}\left\{(q_k - a_k)^2 + (p_k - b_k)^2\right\} = \epsilon^2, \tag{7}$$

with radius $\epsilon \in \mathbb{R}$ and origin (a_k, b_k), where $a_k \in \mathbb{R}$ and $b_k \in \mathbb{C}$. This defines the initial condition of the system. Since $p \in \mathbb{C}$, the orthogonally projected area of the ball onto any given conjugate pair (q_k, p_k) is complex-valued. However, its modulus, $A^k_\psi(t = 0)$, is real-valued, see Fig.2. Due to the spherical symmetry in the initial condition, $A^k_\psi(t = 0)$ is given by

$$A^k_\psi(t = 0) = \pi\epsilon^2 \quad \forall k \in \{1, 2, ..., n\}. \tag{8}$$

The projected area $\pi\epsilon^2$ represents the maximum level of precision by which the state of the system can be known for each conjugate pair. In other words, the radius ϵ quantifies the greatest resolution available to the observer. Upon the identification of the resolution ϵ with the Planck constant h according to

$$\epsilon \equiv \sqrt{\frac{h}{2\pi}}, \tag{9}$$

the minimum uncertainty area of the projection of the ball $B(\sqrt{h/2\pi})$ onto the conjugate plane k is given by

$$A^k_\psi(t = 0) = \frac{h}{2} \quad \forall k \in \{1, 2, ..., n\}. \tag{10}$$

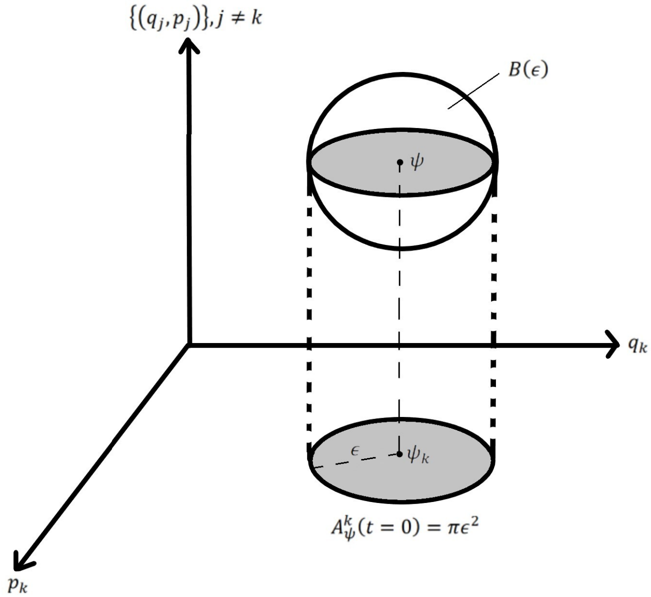

FIG. 2. The modulus of the projected area, A_ψ^k, of the phase-space ball $B(\epsilon)$ onto the conjugate pair (q_k, p_k), at time $t = 0$, is given by the minimum uncertainty area $\pi\epsilon^2$.

The ball $B(\sqrt{h/2\pi})$ is thus a representation for the phase-space coherent state ψ. More generally, the saturated initial condition ψ can have its minimum uncertainty non-symmetrically distributed between the position and momenta. In this article, these states are interpreted as the quantum blobs, which are the phase-space analogs [31] of the squeezed coherent states [37–39]. In the limit that $h \to 0$, the quantum blob ψ collapse into an infinitesimal point. This is the classical approximation.

Indeterminacy relation

The quantum blobs are the states on the complex-valued phase space which can be distinguished to the greatest resolution. Therefore, the projected area $A_\xi^k(t)$ for an arbitrary state ξ, at any given time t, onto the conjugate plane (q_k, p_k), is either equal to, or greater than $h/2$, i.e.

$$A_\xi^k(t) \geq \frac{h}{2} \quad \forall k \in \{1, 2, ..., n\}. \tag{11}$$

This is the indeterminacy relation on the phase space. It states that the shape of the state ξ cannot deform during its Hamiltonian flow in such a way that it breaches the lower bound as defined by $h/2$. In the language of symplectic topology, the projected area A_ξ^k is referred to as the symplectic capacity c_ξ^k and its minimum value, i.e. $h/2$, as the Gromov width c_G [28]. The arbitrary state ξ is thus mathematically represented by the set of symplectic capacities $\left\{c_\xi^1, ..., c_\xi^k, ..., c_\xi^n\right\}$. The indeterminacy relation thus states that the symplectic capacities of an arbitrary state ξ cannot deform during its Hamiltonian flow in such a way that its value gets smaller than the Gromov width, i.e.

$$c_\xi^k(t) \geq c_G \quad \forall k \in \{1, 2, ..., n\}. \tag{12}$$

This indeterminacy relation is related to the Robertson-Schrödinger indeterminacy relation[1] [30]. The mathematical proof of the impossibility of squeezing the state ξ into a smaller symplectic capacity than $h/2$ at any given time, as the system experiences a Hamiltonian flow, was given by Gromov in 1985 [27] and is referred to as Gromov's non-squeezing theorem.

It is important to emphasize that there is no restriction on the symplectic capacity of the state onto a non-conjugate pair of degrees of freedom, i.e. the symplectic capacities for (q_i, q_j), (p_i, p_j) or (q_i, p_j), $\forall i \neq j$, can have arbitrarily small sizes. This condition corresponds to the fact that observables, represented by operators, that commute with each other, in the standard Hilbert-space formulation, can be defined with arbitrary precision.

The key character of the quantum Hamiltonian flow, contrasting its classical approximation, is thus the constraint on the shape of the flow as encoded in the indeterminacy relation. This is in direct contradiction with the Liouville theorem. The Liouville theorem state that any initial region on the phase space can deform continuously in any conceivable way as long as its volume does not change [40]. Thus, according to the Liouville theorem, it is possible to deform the arbitrary state ξ in such a way that the symplectic capacity onto some given subset of conjugate pairs is smaller than the Gromov width $h/2$, as long as it is balanced by an increase in the symplectic capacity of another subset of conjugate pairs, keeping the volume invariant. Thus, classical mechanics, whose dynamics on the phase space are governed by the Liouville theorem, violate the indeterminacy relation.

Overlapping quantum blobs

Consider any given pair of quantum blobs, ψ and ϕ, at some given time $t = 0$. They each represent possible initial conditions for the same single system. The finite size of the pair of quantum blobs, as represented by their balls B_ψ and B_ϕ, allow for the possibility that they have a non-zero overlap Γ, see Fig.3. This implies that there exists a subset of symplectic capacities, e.g. c_ψ^k and c_ϕ^k, which have a non-zero overlap, $\Omega_k(\psi, \phi)$. In other words,

[1] The Robertson-Schrödinger indeterminacy relation [38, 41–44] generalize the Heisenberg indeterminacy relation [45] [46] due to its inclusion of the covariance between observables.

there might be a non-zero degree of indistinguishability between the pair of initial conditions ψ and ϕ if they are sufficiently close to each other. Of course, if the pair have zero overlaps, for all conjugate planes, then they are completely distinguishable. The total area of overlap,

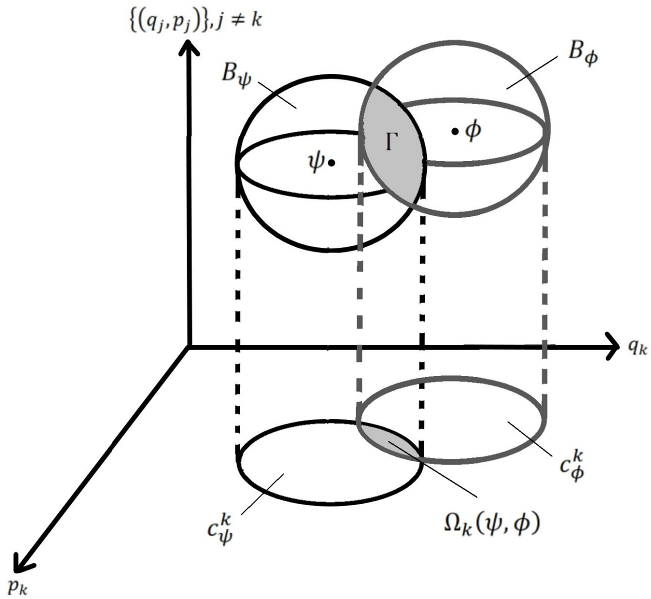

FIG. 3. The non-zero complex-valued overlap $\Omega_k(\psi, \phi)$ between the symplectic capacities c_ψ^k and c_ϕ^k, associated with the quantum blobs ψ and ϕ, represented by the balls B_ψ and B_ϕ, makes it impossible to perfectly distinguish between the pair of quantum blobs. The degree of distinguishability depends on the size of the overlap.

$\Omega(\psi, \phi)$, is given by the linear sum of the contributions Ω_k, for all $k \in \{1, 2, ..., n\}$, i.e.

$$\Omega(\psi, \phi) = \sum_{k=1}^{n} \Omega_k(\psi, \phi). \tag{13}$$

The summation is linear since the $n-$dimensional set of conjugate planes is linearly independent.

Fidelity and mistaken identity

Due to the complex-valuedness of the state overlap, it cannot serve as a physical measure for the degree of distinguishability between arbitrary pairs of quantum blobs ψ and ϕ. To construct a useful physical measure, the function $F(\Omega)$ is introduced, and required to satisfy the following set of conditions:

i It is real-valued.

ii It is non-negative, i.e. $F(\Omega) \geq 0$.

iii It is unitless.

iv $F(\Omega) = 0$ if $\Omega = 0$. The pair ψ and ϕ are completely distinguishable.

v $F(\Omega)$=1 if $\Omega_k = c_\psi^k = c_\phi^k = h/2$ for all $k \in \{1, 2, ..., n\}$. The pair ψ and ϕ are completely indistinguishable.

The conditions (ii) and (v) correspond to the first and second, respectively, Kolmogorov axioms of a probability measure [47]. The physical interpretation of $F(\Omega)$ is thus that it gives the probability that the pair of quantum blobs, ψ and ϕ, are mistaken for each other by the observer upon a measurement at the given time t. It is a quantitative measure of the belief of the observer about the state of the system, rather than a description of the state of the system itself. This point of view on the character of probability originates from the works of Cox [48] [49] and, when applied to statistical mechanics, Jaynes [50] [51]. In this article, the probability presented here is interpreted as the symplectic topological representation of the quantity known in quantum information theory as the quantum fidelity between pairs of pure states [7, 21, 52].

The Born rule [53] [54], in the symplectic representation, give the most obvious candidate for fidelity, satisfying all the imposed conditions, i.e.

$$F(\Omega) = |\Omega(\psi, \phi)|^2. \tag{14}$$

Conservation of probability

Considering that the Liouville theorem is a statement on the conservation of distinguishability between pairs of classical states, its generalization to the "pointless" geometry of the complex-valued phase space presented in this article is proposed to be given by the following statement:

The distinguishability between a pair of quantum blobs, as measured by quantum fidelity, is conserved in time.

Thus, the fidelities evaluated at arbitrarily different times, e.g. t_0 and t, are equal, i.e.

$$F(\Omega)|_t = F(\Omega)|_{t_0}, \tag{15}$$

or, alternatively,

$$\frac{F(\Omega)|_t}{F(\Omega)|_{t_0}} = 1. \tag{16}$$

Due to the Born rule, the conservation of fidelity can equivalently be stated in terms of the overlaps as

$$\frac{\Omega^*\Omega|_t}{\Omega^*\Omega|_{t_0}} = 1. \tag{17}$$

The infinitesimal flow of the overlap, from the initial time t_0 to the final time $t = t_0 + \delta t$, to the first order in the infinitesimal time step δt, is given by

$$\Omega|_{t_0} \to \Omega|_t = \Omega|_{t_0} - \delta\Omega|_{t_0,t} \cdot \Omega|_{t_0}, \tag{18}$$

or, alternatively,

$$\frac{\Omega|_t}{\Omega|_{t_0}} = 1 - \delta\Omega|_{t_0,t}, \tag{19}$$

where $\delta\Omega|_{t_0,t}$ represent the infinitesimal change in the overlap, during the time δt, relative to the initial overlap $\Omega|_{t_0}$. The flow of the complex-conjugated overlap is given by

$$\frac{\Omega^*|_t}{\Omega^*|_{t_0}} = 1 - \delta\Omega^*|_{t_0,t} \tag{20}$$

which thus gives that

$$\begin{aligned}\frac{\Omega^*\Omega|_t}{\Omega^*\Omega|_{t_0}} &= (1 - \delta\Omega^*|_{t_0,t})\,(1 - \delta\Omega|_{t_0,t}) \\ &= 1 - \delta\Omega^*|_{t_0,t} - \delta\Omega|_{t_0,t} + \delta\Omega^*|_{t_0,t} \cdot \delta\Omega|_{t_0,t} \\ &\approx 1 - \delta\Omega^*|_{t_0,t} - \delta\Omega|_{t_0,t},\end{aligned} \tag{21}$$

where the second-order term has been dropped. If quantum fidelity is conserved, then it must be that

$$\delta\Omega^*|_{t_0,t} + \delta\Omega|_{t_0,t} = 0. \tag{22}$$

This is only possible if $\delta\Omega|_{t_0,t}$ is imaginary-valued. Furthermore, if the system is assumed to be closed, it has no explicit dependence on time, i.e.

$$\delta\Omega|_{t_0,t} \sim i\delta t \cdot \mathcal{H}, \tag{23}$$

where the phase-space function $\mathcal{H}$ is explicitly time-independent. For it to be real-valued, it cannot contain any odd powers of momentum. It has units of energy and is identified with the Hamiltonian for the system. Thus, the Hamiltonian generates the flow in time of the overlap between pairs of quantum blobs. Furthermore, due to the indeterminacy relation, the Hamiltonian cannot quantify changes in the overlap with arbitrary precision. The Hamiltonian can therefore only be defined in units of the greatest possible resolution $\epsilon = \sqrt{h/2\pi}$. However, since the infinitesimal change in the overlap must be unitless, the measure of resolution that enter into its definition must be $\epsilon^2 = h/2\pi$. Thus, in conclusion, the infinitesimal flow of the overlap is given by

$$\frac{\Omega|_t}{\Omega|_{t_0}} = 1 - i\frac{\delta t \cdot \mathcal{H}}{h/2\pi}. \tag{24}$$

Extending over arbitrarily many time-steps m, such that $m \cdot \delta t = t - t_0$, the flow in time of the overlap is determined by

$$\frac{\Omega|_t}{\Omega|_{t_0}} = \lim_{m\to\infty}\left(1 - i\frac{(t-t_0)}{m}\frac{\mathcal{H}}{h/2\pi}\right)^m \tag{25}$$

$$= e^{2\pi i\mathcal{H}\cdot(t-t_0)/h}. \tag{26}$$

The relation between overlaps at different times is commonly denoted by $U(t, t_0)$, i.e.

$$U(t,t_0) \equiv \frac{\Omega|_t}{\Omega|_{t_0}} = e^{2\pi i\mathcal{H}\cdot(t-t_0)/h}, \tag{27}$$

and referred to as the time-evolution operator. It is unitary, i.e.

$$U^*U = 1. \tag{28}$$

The notion of unitarity is thus just a restatement, by the application of the Born rule, of the conservation of quantum fidelity.

The Schrödinger equation

Eq. 24 can be rewritten as a differential equation, i.e.

$$i\frac{h}{2\pi}\frac{\Omega|_t - \Omega|_{t_0}}{\delta t} = \mathcal{H}\Omega|_{t_0}, \tag{29}$$

which becomes

$$i\frac{h}{2\pi}\frac{\partial\Omega(t)}{\partial t} = \mathcal{H}\Omega(t). \tag{30}$$

This is the Schrödinger equation for the overlap. It is a direct consequence of the conservation of quantum fidelity. This is the analog of the situation in classical mechanics, where the Hamilton equations are the direct consequences of the Liouville theorem. Thus, the Schrödinger equation is a representation of the quantum generalization of the Liouville theorem.

The Schrödinger equation predicts exactly the value of the overlap at some given time if the initial condition on the overlap is known. This displays its key difference with classical mechanics regarding the notion of determinism. In classical mechanics, the exact state of the system is predictable at any given time, given the initial condition. For the present situation, it is only the overlap between the symplectic capacities of pairs of quantum blobs that is exactly predictable, given the initial overlap.

Conclusion

To summarize, we have attempted to define the concept of quantum fidelity on complex phase space with the aid of symplectic topology. The following set of postulates is taken as foundational:

i The quantum blob of a system is represented by its set of symplectic capacities on the complex-valued phase space.

ii The symplectic capacity of a quantum blob is constrained from below by the Gromov width $c_G = h/2$.

iii The probability F that the pair of quantum blobs ψ and ϕ are mistaken for each other upon measurement is given by the Born rule,

$$F = |\Omega(\psi, \phi)|^2 \quad (31)$$

where $\Omega(\psi, \phi)$ is the overlap between the symplectic capacities of the pair of quantum blobs.

iv For a closed Hamiltonian system, the probability is conserved in time.

In conclusion, the classical notion of distinguishability between pairs of states has been generalized. In classical mechanics, the pair can either be completely distinct or identical. In the present situation, the two classical possibilities are the extremum values of the quantum fidelity, i.e. $F = 0$ when they are completely distinct and $F = 1$ when identical. For values $0 < F < 1$, they are neither completely distinct nor identical. The larger value for F the more similar the pairs of states are. However, there is a continuous transition from being completely distinct to identical. This is impossible in classical mechanics.

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