Exploring the Foundations of Quantum Mechanics: Bosons, Fermions, Quarks, and their q-Potentials

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Abstract: The results presented here are based on the concepts of the Cauchy continuum and, the elementary cell at the Planck scale. The symmetrization of quaternion relations and the postulate of quaternion velocity have been crucial in driving significant advancements. They allowed considering the momentum of the expanding Cauchy continuum, $\dot{u}_0(t,x)$. The momentum expansion/compression is the apparent result of the scalar potential of the expansion/compression: $\sigma_0(t,x)$. The key new results are listed below:
The vectorial $G_0(m)(\sigma_0+\hat{\phi})$, $G_0(m)\hat{\phi}$ and scalar: $G_0(m)\sigma_0$, $G_0(m)\sigma\cdot\sigma^*$, propagators are postulated and used to generate the 2nd order PDE systems for the proton, electron and neutron. The scrupulous assessment of the 2nd order PDE systems allows postulating the two 2nd order PDE systems for the *u* and *d* quarks from the *up* and *down* groups. It was verified that both the proton and the neutron obey experimental findings and are formed by three quarks. The proton and neutron are formed by *d-u-u* and *d-d-u* complexes, respectively. All particle systems comply with the Cauchy equation of motion and can be considered as stable particles. The u and d quarks do not meet the Cauchy relations. The inconsistencies of the quarks' PDE with the quaternion forms of the Cauchy equation of motion account for their lifetime and the observed Quarks Chains. That is, explain the Wilczek phenomenological paradox: "Quarks are Born Free, but Everywhere They are in Chains". Symmetrizing the variables led to the derivation of the Maxwell's equations at the macro-scale and the quarks at the Planck scale.

Keywords: vectorial potential; proton; quarks chain

1. Introduction

Already in 1936 Birkhoff and Von Neumann suggested a quaternionic quantum mechanics, QQM, where wave functions or probability amplitudes are quaternion valued [1]. But systematic work on the quaternionic extension of standard quantum mechanics has not begun. The key results relevant to the present paper are by Lanczos. His dissertation was on a quaternionic field theory of-classical electrodynamics [2,3]. In his derivation of the Dirac's equation [4], there is a doubling in the number of solutions and the concepts that still remain at the front of the fundamental theory. These articles were unnoticed by contemporaries; Lanczos abandoned quaternions and never returned to quaternionic field theory.

Almost immediately, it was demonstrated that the Cauchy-Riemann type conditions in the quaternion representation are identical in the shape to vacuum equations of electrodynamics [5] and that Dirac transition amplitudes are quaternion valued [6]. Christianto derived an original wave equation from the correspondence between Dirac equation and the Maxwell electromagnetic equations via the biquaternionic representation [7]. The Adler's schema of the quaternionizing the quantum mechanics inspired the Harari-Shupe's preon model for the composite quarks and leptons [8,9] and the substantial progress in the QQM and QFT [10]. Adler presented a major conceptual advance for the purpose of determining whether quaternionic Hilbert space is the suitable for the unification of the standard model forces with gravitation. He provided an introduction to the problem of formulating quantum field theories and concluded that the QQM may fit into the physics of unification and measurement theory issues [11].

The focus here is on quaternion quantum mechanics and quaternionic field theory, QFT. The QQM presented here is ontological in a sense that it starts with being, that is the Cauchy ideal elastic continuum at the macro-scale ($>10^{-20}$ m) and the "Planck unit cell" at the microscale ($\sim10^{-35}$ m) [14,12]. The basic categories of being and their relations are governed by the quaternion algebra [14]. The operator algebras as well as special operators are not used here.

The evolution of the P-KC model and the development of the QQM are shown in succeeding papers [13,14,15]. In this article we present the QQM in the completed, refined form:

- Without heuristically trying to find analogies with the classical operator quantum mechanics and the field equations, we use the ontology-based formalism which is constructed on the Planck-Kleinert crystal concept [12] and the quaternion algebra introduced by Hamilton, section 1.1.
- The widely used Helmholtz decomposition is used in the general form in $\mathbb{R}^4$, section 2.
- The all vectors are in the $\mathbb{R}^4$ representation, e.g., the four-velocity is the "new" variable that allows for the symmetrization of the Hamiltonian [16] and the 1st and 2nd order wave equations.
- The Standard Model of elementary particles lack adequate description for the mechanism of quark charges and their force fields. It is showed here that the quark particle waves do exist and two their PDE systems are presented.
- The further studies in order to verify or refute those propositions are suggested.

The Cauchy model of the elastic continuum is presented in Section 1.2. We construct a Lagrangian with the use of the Cauchy–Riemann operator and introduce the key new concept, the quaternion valued velocity, section 2. Abbreviations used in the text are presented in Appendix A**.**

1.1. Quaternions

The elements of the quaternion algebra used in the QQM and QFT were presented in previous papers [13-15]. The basic definitions and formulas of the quaternions and functions are limited to those already used [17,18].

In Hamilton's own words, he created the $\mathbb{R}^4$ analog of complex numbers as the equivalent of the time-space continuum [19]:

"Time is said to have only one dimension, and space to have three dimensions. The mathematical quaternion partakes of both these elements; in technical language it may be said to be 'time plus space'…': and in this sense it has, or at least involves a reference to, four dimensions."

We demonstrate here that, the Hamilton's *'time plus space'* is consistent with the Cauchy model of ideal elastic continuum in the quaternionic representation.

The algebra of quaternions, $\mathbb{Q}$, owns all laws of algebra with unique properties:

(1) the multiplication of quaternions is noncommutative;

(2) the quaternionic deformation potential, i.e., the deformation four-potential or q-potential, which is a relativistic function from which the displacement field can be derived. It combines both a compression scalar potential (pressure) and a torsion vector potential (twist) into a single quaternion (four-vector);

(3) the quaternionic deformation potential is Lorentz invariant.

The quaternion is regarded as the sum of a real (scalar q_0) and an imaginary (vector $\hat{q}$) parts: $q = q_0\mathbf{1} + \hat{q} = [q_0, \hat{q}] \in \mathbb{Q}$. The following algebraical notation is useful: $e_0 = 1, e_1 = i, e_2 = j, e_3 = k$. Thus, an arbitrary quaternion *q*, i.e., $q \in \mathbb{Q}$, can be written in terms of its basis components,

$$q = (q_0, q_1, q_2, q_3) = q_0\mathbf{1} + q_1 i + q_2 j + q_3 k = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{Q} \qquad (1)$$

where the unit vector **1** can be ignored as a factor, the unit vectors, i, j, k, are called the imaginary units.

The component-wise addition and component-wise scalar multiplication are the conventional operations. Multiplication is the fundamental operation that is defined by the multiplication of the unit vectors:

- The real quaternion **1** is the identity element;
- The real quaternions commute with all other quaternions, that is $a \cdot q = q \cdot a$, for every quaternion *q* and every real quaternion *a*;
- The Hamilton product is not commutative, $p \cdot q \neq q \cdot p$, but it is associative, $p \cdot (q \cdot r) = (p \cdot q) \cdot r$. Thus, the quaternions form an associative algebra over the real numbers;
- Every nonzero quaternion has an inverse with respect to the Hamilton product.

The quaternions form division algebra. The non-commutativity of multiplication is the *equation reference goes here* only property that makes quaternions different from a real and complex numbers. The unit vectors obey the following relations:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \qquad ki = -ik = j. \quad (2)$$

The multiplication is associative but not commutative. Instead of the simple commutative law, $p \cdot q = q \cdot p$, in quaternion algebra we have the following law:

$$p \cdot q = \left(p_0 \, q_0 - \hat{p} \circ \hat{q} \right) e_0 + \hat{p} \times \hat{q} + p_0 \, \hat{q} + q_0 \, \hat{p} \, . \tag{3}$$

From the multiplication law (3) follows the convenient formula $\left(p \cdot q \right)^* = q^* \cdot p^*$. A conjugate quaternion is defined as follows:

$$q^* = q_0 - \hat{q} = q_0 - q_1 i - q_2 j - q_3 k \; , \tag{4}$$

where the asterisk means the following: one goes over to the "conjugate" of the quaternion, that is to say, one gives the imaginary units the opposite sign. The conjugate means one gives the vector components (the space part), $\hat{q} = q_1 i + q_2 j + q_3 k$, the opposite sign:

$$q^* = q_0 - \hat{q} = q_0 - q_1 i - q_2 j - q_3 k \; . \tag{5}$$

It is easy to see that the quantity $q \cdot q^*$ is a scalar number, and all spatial components vanish. From Equations (1) - (5), it can be seen that $q \cdot q^* = q^* \cdot q = \sum_{i=0}^{3} q_i^2$, so the Euclidian norm can be denoted as follows:

$$\|q\| = \sqrt{q^* \cdot q} \; . \tag{6}$$

Consequently the quaternion algebra, $\mathbb{Q}$ is a normed algebra.
The scalar and vector products are operations defined by:

$$\hat{p} \circ \hat{q} = \sum_{i=1}^{3} p_i \, q_i ; \quad \hat{p} \times \hat{q} = \left(p_2 q_3 - p_3 q_2 \right) i + \left(p_3 q_1 - p_1 q_3 \right) j + \left(p_1 q_2 - p_2 q_1 \right) k \tag{7}$$

The vector space $\mathbb{R}^4$ with the multiplication (3) is a noncommutative algebra with unity and it is named quaternion algebra $\mathbb{Q}$. The commutator of two elements, *p* and *q*, is defined by the following:

$$\left[p, q \right] = p \cdot q - q \cdot p = 2 \hat{p} \times \hat{q} \tag{8}$$

and can be looked at as a measure of noncommutativity. Two quaternions commute $\left[p, q \right] = 0$ if, and only if, their vector parts are collinear.
The quaternions typically are represented as the matrices or the exponent functions that have trigonometrical representation: $e^q = e^{q_0} \left(\cos\left|\hat{q}\right| + \hat{q} / \left|\hat{q}\right| \sin\left|\hat{q}\right| \right)$.

Functions of a quaternion variable represent useful physical models, e.g., the electric and magnetic fields described by Maxwell are functions of a quaternion variable [20], section 3.
Let $\Omega \subset \mathbb{R}^3$ be a bounded set. The so-called $\mathbb{Q}$-valued functions may be written as

$$q(x) = q_0(x) 1 + q_1(x) i + q_2(x) j + q_3(x) k, \; x = \left(x_1, x_2, x_3 \right) \in \Omega \, , \tag{9}$$

where the functions $q_0(x), q_l(x),\; l = 1,2,3$ are real-valued.
Similarly, the functions $q(t, x)$, depending on time *t*, may be considered.
We use the Cauchy–Riemann operator *D* in $\mathbb{R}^4$ acting on the quaternion-valued functions

$$D\sigma = \left(-\operatorname{div} \hat{\phi} \right) + \operatorname{grad} \sigma_0 + \operatorname{rot} \hat{\phi}, \quad \sigma = \sigma_0 + \hat{\phi} \, . \tag{10}$$

Properties such as continuity, differentiability, integrability, and so on, have to be possessed by all the components $q_0(t, x), q_l(t, x),\; l = 1,2,3$. In this manner, the Banach, Hilbert, and Sobolev spaces of $\mathbb{Q}$-valued functions can be defined [20], e.g., in the Hilbert space over $\mathbb{Q}$:

$$\mathrm{L}^2(\Omega)=\left\{q:\Omega\to\mathbb{Q}\,\middle|\,\int_\Omega q_0^2\,\mathrm{d}x<\infty,\ \int_\Omega q_l^2\,\mathrm{d}x<\infty,\ l=1,2,3\right\}, \tag{11}$$

and allow introducing the inner product as follows:

$$\langle q_1,q_2\rangle=\int_\Omega q_1\cdot q_2\,\mathrm{d}x,\ \ q_1,q_2\in\mathrm{L}^2(\Omega) \tag{12}$$

Fourier series, Lebesgue measure, Gelfand triples, Laplace transform, and many others on the vector space of $\mathbb{Q}$-valued functions over $\mathbb{Q}$ can be defined in a standard way as in the real and complex cases.

Remark 1. *Hurwitz's theorem says that there are only four normed division algebras:* $\mathbb{R},\mathbb{C},\mathbb{Q}$ *and the octonions algebra. Lagrange's four-square theorem in number theory states that every non-negative integer is the sum of four integer squares. This theorem may have applications in quaternion algebra.*

1.2. The Cauchy Dispacement Field: the Classical Theory of Elasticity and the Properties at the Planck Scale

Cauchy finished the theory of the ideal elastic continuum in 1822 [21], right away Poisson [22] studied the elementary waves. In 1885 Neumann [23] gave the proof of the uniqueness of solutions of some boundary-initial value problems. The rigorous completeness proof was given by Duhem [24]. Cauchy theory is the first real, well posed theory of elasticity using the continuum approach, where the macroscopic phenomena are described in the terms of field variables [25]: the compression $\operatorname{div}\mathbf{u}$, and the twist $\operatorname{rot}\mathbf{u}$. The stress tensor of the ideal elastic continuum is given by

$$\mathbf{T}=\lambda\,\mathrm{tr}(\mathbf{D})\mathbf{I}+2\mu\mathbf{D} \tag{13}$$

where $\mathrm{tr}(\mathbf{D})$ is the trace of the strain tensor, $\mathbf{I}$ is the identity matrix and the two moduli of elasticity, λ and μ, are the material-dependent constants. It was shown by Cauchy and Saint Venant that if the particles composing a regular crystal interact pairwise through central forces, then there is an additional symmetry requiring C44 = C12 that implies the Poisson ratio 0.25 and equal Lamé's coefficients: $\lambda=\mu$ [26]. The identity: $\operatorname{grad}\operatorname{div}\mathbf{u}=\operatorname{div}\operatorname{grad}\mathbf{u}+\operatorname{rot}\operatorname{rot}\mathbf{u}$, implies that the stress tensor becomes:

$$\begin{gathered}\operatorname{div}\mathbf{T}=2\mu\operatorname{grad}\operatorname{div}\mathbf{u}+\mu\operatorname{div}\operatorname{grad}\mathbf{u}=\\ 3\mu\operatorname{grad}\operatorname{div}\mathbf{u}-\mu\operatorname{rot}\operatorname{rot}\mathbf{u}.\end{gathered} \tag{14}$$

Cauchy equation of motion generalizes: (1) the Newton's laws of motion (the conservation of the linear and angular momenta) to an ideal elastic solid, and (2) the concept of stress in terms of the gradients in the displacement field $\mathbf{u}(t,x)\in\mathbb{R}^3$:

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2}=3\mu\operatorname{grad}\operatorname{div}\mathbf{u}-\mu\operatorname{rot}\operatorname{rot}\mathbf{u}+\vec{\mathbf{F}}\in\mathbb{R}^3, \tag{15}$$

where $\vec{\mathbf{F}}$ is the force.

From Equation (15), the vectorial representation of the energy density in the deformation field can be computed [25,27]

$$e=\frac{1}{2}\dot{\mathbf{u}}\circ\dot{\mathbf{u}}+\frac{3}{2}c^2\left(\operatorname{div}\mathbf{u}\right)^2+\frac{1}{2}c^2\operatorname{rot}\mathbf{u}\circ\operatorname{rot}\mathbf{u}\in\mathbb{R}^3, \tag{16}$$

where $\dot{\mathbf{u}}=\partial\mathbf{u}/\partial t$.

In the following, we consider Cauchy continuum with FCC structure. The Young's modulus Y describes tensile elasticity which is axial stiffness of the length of a body to deformation along the axis of the applied tensile force. It is related to Lamé's two moduli of elasticity by

$$Y=\mu(3\lambda+2\mu)/(\lambda+\mu)\overset{\lambda=\mu}{=}2.5\mu. \tag{17}$$

If l_P denotes the dimension of the FCC elementary cell that consists of the four interacting Planck particles showing the mass m_P, the Planck density equals: $\rho_P = 4m_P / l_P^3 = const.$, The computed Planck density, the Young modulus and, the other properties of the Cauchy continuum at the Planck scale are shown in Table 1.

Table 1. The physical constants of the Cauchy continuum (fcc ideal isotropic crystal).

Label Used in This Work	Planck Constants	Symbol for Unit	Value	SI Unit	Reference
Lattice parameter	Planck length	l_P	$1.616229(38) \times 10^{-35}$	m	[32]
Poisson ratio		ν	0.25	-	[26]
Mass of the Planck particle	Planck mass	m_P	$2.176470(51) \times 10^{-8}$	kg	[32]
Duration of the internal process	Planck time	t_P	$5.39116(13) \times 10^{-44}$	s^{-1}	[32]
Transverse wave velocity	Light velocity in vacuum	c	2.99792458×10^8	$m \cdot s^{-1}$	$c = l_P / t_P$ [32]
Planck density		ρ_P	2.062072×10^{97}	$kg \cdot m^{-3}$	[32]
Young modulus, intrinsic energy density		Y	$4.6332447 \times 10^{114}$	$kg \cdot m^{-1} s^{-2}$	$Y = 2.5 \rho_P c^2$

The extreme density allows considering the small deformation limit and the negligible density changes allow assuming, e.g., the constant transverse wave velocity: $c = \sqrt{\mu / \rho_P} = const.$ [25] and Equation (15) becomes :

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{u}}{\partial t^2} = 3 \operatorname{grad} \operatorname{div} \mathbf{u} - \operatorname{rot} \operatorname{rot} \mathbf{u} + \frac{1}{c^2} \vec{\mathbf{F}} \quad \in \mathbb{R}^3. \tag{18}$$

The Cauchy and to the same degree the majority of physical problems cannot be reduced to vectorial models (the vector product does not permit the formulation of algebra with unity, for example, the division operation is not defined). By acting on the equation (18) by rot and div operators we separate the transverse and the longitudinal processes:

$$\begin{matrix} \operatorname{div}\left(\frac{\partial^2 \mathbf{u}}{\partial t^2} = 3c^2 \nabla (\nabla \cdot \mathbf{u}) - c^2 \nabla \times (\nabla \times \mathbf{u}) + \frac{1}{c^2} \vec{\mathbf{F}} \right) \\ \operatorname{rot}\left(\frac{\partial^2 \mathbf{u}}{\partial t^2} = 3c^2 \nabla (\nabla \cdot \mathbf{u}) - c^2 \nabla \times (\nabla \times \mathbf{u}) + \frac{1}{c^2} \vec{\mathbf{F}} \right)_{\nabla \times (\nabla \times \mathbf{u}_\phi) = -\Delta \mathbf{u}_\phi} \end{matrix} \Rightarrow \begin{cases} \frac{\partial^2}{\partial t^2} \operatorname{div} \mathbf{u}_0 = 3c^2 \Delta \operatorname{div} \mathbf{u}_0 + \frac{1}{c^2} \operatorname{div} \vec{\mathbf{F}}, \\ \frac{\partial^2}{\partial t^2} \operatorname{rot} \mathbf{u}_\phi = c^2 \Delta \operatorname{rot} \mathbf{u}_\phi + \frac{1}{c^2} \operatorname{rot} \vec{\mathbf{F}}. \end{cases} \tag{19}$$

The Cauchy equation of motion combined with the Helmholtz decomposition theorem results in four second-order scalar differential equations, “quattro cluster”, and implies the transverse and longitudinal waves in the Cauchy elastic solid. This decomposition does not exist for all vector fields and is not unique.[28].

Remark 2

1. *The mathematical analysis confirms that Cauchy model is well posed, i.e.,* ***,*** *has a solution, the solution is unique and its behavior changes continuously with the initial conditions* [24].
2. *The Hamilton algebra of quaternions and its relation to the four-dimensional space allow combining Cauchy theory with the electrodynamics, gravity and quantum mechanics.*
3. *The Helmholtz decomposition does not exist for all vector fields and is not unique.[29]*

2. The Cauchy Deformation Field in the Quaternion Representation

The Cauchy classical theory of elasticity is an elegant starting point to show the physical reality and the significance and beauty of quaternions. The Hamilton algebra $\mathbb{Q}$ allows recoupling the compression and twist that are separated in (19). Upon denoting $\sigma_0 = \operatorname{div} \mathbf{u}_0 = (\sigma_0, 0, 0, 0)$ and $\hat{\phi} = \operatorname{rot} \mathbf{u}_\phi = (0, \phi_1, \phi_2, \phi_3)$ we get

$$\begin{cases} \dfrac{\partial^2 \sigma_0}{\partial t^2} = 3c^2\, \Delta\sigma_0, \\ \dfrac{\partial^2 \hat{\phi}}{\partial t^2} = \ c^2\, \Delta\hat{\phi}. \end{cases} \quad \Leftrightarrow \quad \left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\sigma - 2c^2\Delta\sigma_0 = 0 \ \in \mathbb{R}^4 \tag{20}$$

The decomposition $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_\phi$ in Equation (18) results in four equations in Equation (19) and implies the existence of the deformation field $\sigma = \sigma_0 + \hat{\phi}$ that represents the twist and compression fields as a superposition of real (scalar compression σ_0) and imaginary (twist vector $\hat{\phi}$) field parts at each point

$$\sigma := \sigma_0 + \hat{\phi} \quad \in \mathbb{Q} \quad \text{and} \quad \sigma^* = \sigma_0 - \hat{\phi} \quad \in \mathbb{Q}. \tag{21}$$

Adding equations in (20) we get the quaternion form of the motion equation

$$\frac{1}{c^2}\frac{\partial^2 \sigma}{\partial t^2} - \Delta\sigma - 2\Delta\sigma_0 = 0, \quad \text{where} \quad \sigma = \sigma_0 + \hat{\phi} \tag{22}$$

Since $\dot{\mathbf{u}} \circ \dot{\mathbf{u}} = \hat{\dot{u}} \circ \hat{\dot{u}} = -\hat{\dot{u}} \cdot \hat{\dot{u}} = \hat{\dot{u}} \cdot \hat{\dot{u}}^*$, where $\hat{u} = \hat{u}_1\, i + \hat{u}_2\, j + \hat{u}_3\, k$ and $\dot{\mathbf{u}} = \left(\dot{u}_1, \dot{u}_2, \dot{u}_3\right)$, the overall energy of the deformation field, the relation (18) takes the form

$$e = \tfrac{1}{2}\hat{\dot{u}} \cdot \hat{\dot{u}}^* + \tfrac{1}{2}c^2\ \sigma \cdot \sigma^* + c^2\sigma_0^2 \ \in \mathbb{R}^4. \tag{23}$$

The above relation is non-symmetric. The kinetic energy has the vectorial form: $\tfrac{1}{2}\hat{\dot{u}} \cdot \hat{\dot{u}}^*$, that can be regarded as $\mathbb{R}^3$ representation, that does not describe the volume changes. Contrary, the deformations have quaternion representation in $\mathbb{R}^4$. We will symmetrize the energy formula (23), where velocity is represented by the imaginary part only and postulate the quaternionic representation of the velocity. The energy density per mass unit of in the quaternionic representation equals

$$e = \tfrac{1}{2}\dot{u} \cdot \dot{u}^* + \tfrac{1}{2}c^2\ \sigma \cdot \sigma^* + c^2\sigma_0^2 \in \mathbb{R}^4, \tag{24}$$

where, e.g., the four-velocity within the particle wave is given by Cauchy-Riemann derivative:

$$\dot{u} = -\frac{\hbar}{m}\mathrm{D}\sigma \quad \text{where} \quad D\sigma = \left(-\operatorname{div}\hat{\phi}\right) + \operatorname{grad}\sigma_0 + \operatorname{rot}\hat{\phi}. \tag{25}$$

In (21) the quaternion potential, i.e., the deformation four-potential, is defined by

$$\begin{array}{ccccc} \sigma & = & \sigma_0 & + & \hat{\phi} \\ \begin{bmatrix} q\text{-potential,} \\ \text{deformation} \end{bmatrix} & = & \begin{bmatrix} \operatorname{div}\mathbf{u}_0 \\ \text{compression} \end{bmatrix} & + & \begin{bmatrix} \operatorname{rot}\mathbf{u}_\phi \\ \text{twist} \end{bmatrix} \end{array} \tag{26}$$

In (24) and (25) we symmetrized the formula (23). The quaternionic velocity represents now the all deformations in $\mathbb{R}^4$. We demonstrate the practical application with example of the particle wave showing the equivalent mass m:

$$\begin{array}{ccccc} \dot{u} & = & \dot{u}_0 & + & \hat{\dot{u}} \\ -\dfrac{\hbar}{m}\mathrm{D}\sigma & = & -\dfrac{\hbar}{m}\operatorname{div}\hat{\phi} & + & \dfrac{\hbar}{m}\left(\operatorname{grad}\sigma_0 + \operatorname{rot}\hat{\phi}\right) \\ \begin{bmatrix} \text{velocity of the} \\ \text{q-potential changes,} \\ \text{deformation velocity} \end{bmatrix} & = & \begin{bmatrix} \text{compression} \\ \text{velocity} \end{bmatrix} & + & \left[\text{twist velocity}\right] \end{array} \tag{27}$$

The overall energy in arbitrary volume Ω follows from Eq. (24):

$$E=\int_\Omega \rho_E(t,x)\,\mathrm{d}x=\int_\Omega \rho_P\left(\tfrac12 \dot u\cdot\dot u^* +\tfrac12 c^2\ \sigma\cdot\sigma^* +c^2\sigma_0^2\right)\mathrm{d}x=mc^2, \qquad (28)$$

where the external potential, e.g., $V(x)$, is not shown.

The energy is conserved, so relation (28) leads to the nonlocal boundary condition for Equation (22) [13].
Remark 3. *The generalized Helmholtz decomposition in* (27) *does not affect 2*[nd] *order equations. It is obligatory in Maxwell electrodynamics and 1*[st] *order equation systems.*

3. Maxwell Equations in the Cauchy Continuum

Upon introducing the potentials definitions: $\varphi:=\sigma_0=\operatorname{div}\mathbf{u}_0$ and $\mathbf{A}:=\hat\phi=\operatorname{rot}\mathbf{u}_\phi$ where they denote the irrotational scalar and solenoidal vector potentials, we get four-potential

$$A=\varphi+\mathbf{A}=\varphi\mathbf{1}+iA_1+jA_2+kA_3\ \in\mathbb{R}^4. \qquad (29)$$

In the system (19), the $\operatorname{rot}\vec{\mathbf{F}}\neq 0$ and $\operatorname{div}\vec{\mathbf{F}}\neq 0$ imply the 4-potential due to the presence and, e.g., the flux of the charged particles:

$$.\ f=f_0+\hat f=f_0\mathbf{1}+i\,f_1+j\,f_2+k\,f_3\ \in\mathbb{R}^4. \qquad (30)$$

The scalar potentials φ, f_0 quantify the compression/expansion, the vectorial potentials $\hat f=i\,f_1+j\,f_2+k\,f_3$ and $\mathbf{A}=iA_1+jA_2+kA_3$, the twists in all three axes. The force $\mathbf{F}=-\mathrm{D}\,f$ in (19) is induced solely by the charged particles and follows from the Cauchy-Riemann derivative of their 4-potential *f*:

$$\begin{array}{ccccc} \mathbf{F} & = & \mathbf{F}_0 & + & \vec{\mathbf{F}} \\ \Downarrow & & \Downarrow & & \Downarrow \\ -\mathrm{D}\,f & = & \operatorname{div}\hat f & + & \left(-\operatorname{grad} f_0-\operatorname{rot}\hat f\right) \end{array} \in\mathbb{R}^4. \qquad (31)$$

The system (19), using definitions $\varphi=\operatorname{div}\mathbf{u}_0$ and $\mathbf{A}=\operatorname{rot}\mathbf{u}_\phi$ and, the relation (31) becomes

$$\begin{cases} \dfrac{1}{c^2}\dfrac{\partial^2\mathbf{A}}{\partial t^2}=\Delta\mathbf{A}+\dfrac{1}{c^2}\operatorname{rot}\operatorname{rot}\hat f, \text{ where } \mathbf{A}=\operatorname{rot}\mathbf{u}_\phi, \\ \dfrac{1}{c^2}\dfrac{\partial^2\varphi}{\partial t^2}=3\Delta\varphi+\dfrac{1}{c^2}\Delta\, f_0 \text{ where } \varphi=\operatorname{div}\mathbf{u}_0, \end{cases} \in\mathbb{R}^4. \qquad (32)$$

By noting that $\varphi=f_0/c^2$ implies $\Delta\varphi=c^{-2}\Delta f_0$, the scalar equation in (32) might be symmetrized:

$$\begin{cases} -\dfrac{1}{c^2}\dfrac{\partial^2\mathbf{A}}{\partial t^2}+\Delta\mathbf{A}=-\dfrac{1}{c^2}\operatorname{rot}\operatorname{rot}\hat f, \\ -\dfrac{1}{c^2}\dfrac{\partial^2\varphi}{\partial t^2}+\Delta\varphi=-\dfrac{3}{c^2}\Delta\, f_0. \end{cases} \qquad (33)$$

The microscopic, vacuum version. As a first step we consider the empty crystal space (no charged particles and the irrotational deformations are negligible) consequently, the systems (32) and (33) reduce to

$$\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}+\operatorname{rot}\operatorname{rot}\mathbf{A}=0\quad\in\mathbb{R}^3 \qquad (34)$$

We introduce definitions:

$$\mathbf{E} := -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \tag{35}$$

$$\mathbf{H} := \operatorname{rot}\mathbf{A}, \tag{36}$$

Upon combining (34) - (36) and by taking the rotation of the definition (35): $\operatorname{rot}\left(c^{-1}\,\partial\mathbf{A}/\partial t = -\mathbf{E}\right)$ the Maxwell system for vacuum follows:

$$\begin{cases} \dfrac{1}{c}\dfrac{\partial \mathbf{E}}{\partial t} - \operatorname{rot}\mathbf{H} = 0, \\ \dfrac{1}{c}\dfrac{\partial \mathbf{H}}{\partial t} + \operatorname{rot}\mathbf{E} = 0. \end{cases} \tag{37}$$

The macroscopic version. The charged particles affect the volume changes, i.e., the particles and their fluxes imply nonnegligible irrotational deformations. By adding equations in system (33) one gets:

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \Delta\right)A = \mu J, \quad \in \mathbb{R}^4, \tag{38}$$

where $\mu J = J_0 + \hat{J} = -3c^{-2}\Delta\, f_0 - c^{-2}\operatorname{rot}\operatorname{rot}\hat{f}$ and $A = \varphi + \mathbf{A}$,

In the condensed representation also: $\Box A^{\alpha} = \mu J^{\alpha}$, where the operator $\Box$ is called D'Lambertian and can be written: $\Box = \partial^{\alpha}\partial_{\alpha} = -\dfrac{\partial^2}{\partial(ct)^2} + \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$.

The time invariant assembly of the charged particles means: $\partial\varphi/\partial t = 0$, and the scalar equation in the system (33) becomes the Gauss law:

$$c^2\Delta\varphi = -3\operatorname{div}\operatorname{grad} f_0. \tag{39}$$

The scalar force $\operatorname{div}\hat{f}$ in the relation (31) results in an expansion/compression process and when compared with the 2nd order scalar equation in system (32), allows relating the vector and scalar potentials

$$\begin{cases} c\dfrac{\partial\varphi}{\partial t} + \operatorname{div}\hat{f} = 0, \\ \hat{f} = c^2\mathbf{A}, \end{cases} \Rightarrow \frac{1}{c}\frac{\partial\varphi}{\partial t} + \operatorname{div}\mathbf{A} = 0. \tag{40}$$

Above relation is the Lorentz gauge that relates the scalar and vector potentials in Maxwell electrodynamics. The notation used in the equations (38)-(40) was introduced by Lanczos who obtained the equivalent relations [30].

4. The bosons, fermions, quarks and their q-potentials

4.1 The Quaternionic Propagators

The coupling of the transverse and the longitudinal waves takes place in the elementary cell of the Cauchy continuum, i.e., at the Planck scale. The quaternionic oscillator controls the acceleration of the all *q*-potential components during the propagation, e.g., in the particle wave in Ω: $\ddot{\sigma}_0, \ddot{\phi}_1, \ddot{\phi}_2, \ddot{\phi}_3$. The function $G_0 \in \mathbb{R}$, will be called the intensity of the oscillator. In the earlier papers, we disregarded that the twists ϕ_1, ϕ_2 and ϕ_3 form the twist vector $\hat{\phi} = \phi_1 i + \phi_2 j + \phi_3 k$ [31] and are controlled by the oscillator G_0. Thus, the relation between the q-potential and its scalar component σ_0 will be corrected and consider the two *q*-potential constituents, σ_0 and $\hat{\phi}$ [31]:

$$\left\langle \frac{\partial^2 \sigma}{\partial t^2} \right\rangle = 2 \left\langle \frac{\partial^2 \sigma_0}{\partial t^2} \right\rangle = 8\pi^2 f_P f \ , \tag{41}$$

and the power of the quaternionic oscillator equals

$$G_0(f) = 8\pi^2 f_P f \ . \tag{42}$$

The particle wave frequency depends on the particle mass, $f = f(m)$, and follows from the $\mathbb{R}^1$ schema, e.g., see Fig. 1 in [15]. The sum of moments of all the Planck masses forming the particle wave in Ω (at the arbitrary time *t* and solely due to the particle wave) equals the momentum of the particle *m* itself. On the other hand, we may estimate the average momentum of the arbitrary single Planck mass m_P in the elementary cell during the whole particle cycle: $T = f^{-1}$. The complete cycle implies that every Planck mass m_P returns to its initial conditions: $\mathbf{u}_P(t) = \mathbf{u}_P(t+T)$ and $\dot{\mathbf{u}}_P(t) = \dot{\mathbf{u}}_P(t+T)$. The overall distance of the Planck mass during the wave cycle *T* equals $2\pi l_P$. Thus, the average momentum of the Planck mass $\overline{p}(m_P)$ during the particle wave cycle equals

$$\overline{p}(m_P) = m_P \frac{2\pi l_P}{T} = 2\pi m_P l_P f . \tag{43}$$

The momentum of the particle wave *m* results from the particle wave propagation velocity, e.g., *c* in the system (50):

$$p(m) = m c \tag{44}$$

The both moments must equal: $p(m) = \overline{p}(m_{P:})$, and the frequency of the particle wave becomes:

$$f = \frac{m c}{2\pi m_p l_p} \times \frac{c}{c} = \frac{m c^2}{2\pi \hbar} \ \text{where} \ \hbar = m_P c l_P \ , \tag{45}$$

where upon using the NIST data [32] for the Planck's natural units m_P, l_P, t_p and the light velocity *c*, the constant $\hbar$ introduced in relation (45) equals the Planck constant [12].

Combining the relations (42), (45) and the definition $f_P = 1/t_P$, the overall power of the quaternionic oscillator when the particle mass is known equals:

$$G_0(m) = 4\pi\, m c^2 / (\hbar t_P) . \tag{46}$$

The oscillator might generate the lower frequencies *f* of the particle wave and, the families of propagators:

$$G_n = \frac{1}{n} G_0(m) = \frac{1}{n} 4\pi\, m c^2 / (\hbar t_P), \ \text{where} \ n = 1, 2, ... \tag{47}$$

where *n* can be interpreted as the measure of the propagator volume, e.g., $l_n = n l_P$.

The quaternionic oscillator $G_0(m)$, controls four propagators:

- the scalar I (spin 0), $G_0(m)\sigma \cdot \sigma^*$,
- the scalar ½ (spin 1/2), $G_0(m)\sigma_0$,

- the vectorial (spin 1/2), $G_0(m)\hat{\phi}$, and
- the quaternion (spin 1/2), $G_0(m)(\sigma_0+\hat{\phi})$.

The above propagators generate the particle wave and simultaneously, the particles produce different force fields that are represented by the Poisson equation:

$$n c^2\Delta(\bullet)+G_0(m)(\bullet)=0 \text{ where } (\bullet) \text{ denotes the quaternion variable} \tag{48}$$

Remark 4. *Substituting* $mc^2=E_0$ *in (45), the Planck–Einstein relation follows:* $E_0=hf$ *, where* $h=2\pi\hbar$ *.*

4.2. Bosons

The family of the scalar 2nd order PDE systems of the spin 0 particles result from Equations (20), (47) and (48) In (49), we show the core set of the three 2nd order PDE and its equivalent, the set of two 2nd order equations: the particle wave and the force field produced by the particle. This schema will be used in the following sections.

$$\begin{cases}\left(\dfrac{\partial^2}{\partial t^2}-c^2\Delta\right)\hat{\phi}=0,\\ \left(\dfrac{\partial^2}{\partial t^2}-3c^2\Delta\right)\sigma_0=0,\\ nc^2\Delta\sigma_0+G_0(m)\sigma\cdot\sigma^*=0,\end{cases} \Leftrightarrow \begin{cases}\left(\dfrac{\partial^2}{\partial t^2}-c^2\Delta\right)\tilde{\sigma}_n+2G_0(m)\sigma\cdot\sigma^*=0,\\ \left((n-1)\dfrac{\partial^2}{\partial t^2}-(n-3)c^2\Delta\right)\sigma_0+2G_0(m)\sigma\cdot\sigma^*=0,\end{cases} \tag{49}$$

where $\tilde{\sigma}_n=n\sigma_0+\hat{\phi}$ and, n denotes integer, $n\neq 0$.
At n = 1, the system (49) results in [14]:

$$\begin{cases}\left(\dfrac{\partial^2}{\partial t^2}-c^2\Delta\right)\hat{\phi}=0,\\ \left(\dfrac{\partial^2}{\partial t^2}-3c^2\Delta\right)\sigma_0=0,\\ c^2\Delta\sigma_0+G_0(m)\sigma\cdot\sigma^*=0,\end{cases} \Leftrightarrow \begin{cases}\left(\dfrac{\partial^2}{\partial t^2}-c^2\Delta\right)\sigma+2G_0(m)\sigma\cdot\sigma^*=0,\\ c^2\Delta\sigma_0+G_0(m)\sigma\cdot\sigma^*=0.\end{cases} \tag{50}$$

Above two systems are identical, five equations and five unknowns: $\sigma_0,\phi_1,\phi_2,\phi_3$ and m. If mass m is unknown it may be treated as the parameter in the Poisson equation above. The equation (50) corresponds to Klein-Gordon equation, i.e., the spin 0 boson particle.

The 2nd order PDE systems on the right hand site of equations (49) and (50) comply with the Cauchy equation of motion, i.e., by adding the Poisson and wave equations, the equation (20) results.

The Poisson equation in (50) describes the irrotational potential σ_0 of the deformation field

$$c^2\Delta\sigma_0=-G_0(m)\sigma\cdot\sigma^*=-4\pi\frac{mc^2}{\hbar t_P}\sigma\cdot\sigma^*, \tag{51}$$

where $\hbar=m_P c l_P$. It can be expressed as a function of the particle mass density: $\rho=m\sigma\cdot\sigma^*/l_P^3$:

$$c^2\Delta\sigma_0=-4\pi\rho\frac{l_P^3}{m_P t_P^2}=-4\pi\rho G\ , \tag{52}$$

using data in Table 1, the gravitational constant equals: $G=l_P^3/(t_P^2 m_P)=6.674082\cdot 10^{-11}\ \left[\mathrm{m^3\cdot kg^{-1}\cdot s^{-2}}\right]$.

The particle mass center, equals its wave energy center. The “space-localized” particle is defined in the sense given by the Bodurov definition [33]:

“*A singularity-free multi-component function* $\sigma = (\sigma_0, \phi_1, \phi_2, \phi_3) \in \mathbb{Q}$ *of the space* $x = (x_1, x_2, x_3)$ *and time t variables will be called space-localized if* $\|\sigma(t,x)\| \to 0$ *sufficiently fast when* $\|x\| \to \infty$, *so that its Hermitean norm*

$$\langle \sigma, \sigma^* \rangle = \int_\Omega \left(\sigma_0^2 + \sum_{l=1}^{3} \phi_l \cdot \phi_l^* \right) \mathrm{d}x = \int_\Omega \sigma \cdot \sigma^* \mathrm{d}x < \infty \tag{53}$$

remains finite for all time.”(50)

4.3. The particles formed by the odd number of quarks

The strong coupling only is considered in the following sections: $n = 1$ in the relation (47).

The vectorial propagator. We begin with vectorial potential, where the term $G_0(m)\hat{\phi}$ fixes the density of the rate of twist change and is called vectorial propagator

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - c^2 \Delta \right) \hat{\phi} = 0, \\ \left(\dfrac{\partial^2}{\partial t^2} - 3c^2 \Delta \right) \sigma_0 = 0, \\ -c^2 \Delta \hat{\phi} + G_0(m) \hat{\phi} = 0, \end{cases} \Leftrightarrow \begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - 3c^2 \Delta \right) \sigma + 2G_0(m) \hat{\phi} = 0, \\ -c^2 \Delta \hat{\phi} + G_0(m) \hat{\phi} = 0. \end{cases} \tag{54}$$

Upon the rearrangement, the particle wave (electron) and the vectorial Poisson equations are evident. The adding equations in system (54) shows that it complies with the Cauchy equation of motion (20):

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - 3c^2 \Delta \right) \sigma + 2G_0(m) \hat{\phi} = 0, \\ -c^2 \Delta \hat{\phi} + G_0(m) \hat{\phi} = 0, \end{cases} \Rightarrow \left(\dfrac{\partial^2}{\partial t^2} - c^2 \Delta \right) \sigma - 2c^2 \Delta \sigma_0 = 0. \tag{55}$$

Note that the wave propagation velocity in system (54) equals the velocity of longitudinal waves in the Cauchy continuum: $c_L = \sqrt{3}c$ [12]. The vectorial Poisson equation in (54) confirms that it’s the 2nd order PDE system for electron.

The quaternionic propagator. In the quaternion propagator, $G_0(m)(\sigma_0 + \hat{\phi})$, the vectorial, $G_0(m)\hat{\phi}$, and scalar, $G_0(m)\sigma_0$, propagators are “merged” and form the strongly coupled system. The rearrangement of the system (56) is shown below and display different forms of the 2nd order PDE systems:

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - c^2 \Delta \right) \hat{\phi} = 0, \\ \left(\dfrac{\partial^2}{\partial t^2} - 3c^2 \Delta \right) \sigma_0 = 0, \\ -c^2 \Delta \hat{\phi} + G_0(m) \hat{\phi} = 0, \\ c^2 \Delta \sigma_0 + G_0(m) \sigma_0 = 0, \end{cases} \Leftrightarrow \begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - c^2 \Delta \right) \hat{\phi} = 0, \\ \left(\dfrac{\partial^2}{\partial t^2} - 3c^2 \Delta \right) \sigma_0 = 0, \\ c^2 \Delta \left(\sigma_0 - \hat{\phi} \right) + G_0(m) \left(\sigma_0 + \hat{\phi} \right) = 0, \end{cases} \Leftrightarrow \begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - 2c^2 \Delta \right) \sigma + G_0(m) \left(\sigma_0 + \hat{\phi} \right) = 0, \\ c^2 \Delta \sigma^* + G_0(m) \sigma = 0. \end{cases} \tag{56}$$

The comparison of the scalar, vectorial and quaternionic propagators shows that the q-propagator offers the strongest coupling, Eq. (56). The quaternionic Poisson equation in (56) reveals that it is the 2nd order PDE system for proton. The sum of equations in (56) shows that system complies with Cauchy equation of motion (20):

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - 2c^2\Delta\right)\sigma + G_0(m)\left(\sigma_0 + \hat{\phi}\right) = 0, \\ c^2\Delta\left(\sigma_0 - \hat{\phi}\right) + G_0(m)\left(\sigma_0 + \hat{\phi}\right) = 0. \end{cases} \Rightarrow \left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\sigma - 2c^2\Delta\sigma_0 = 0. \tag{57}$$

Note that the propagation velocity in system (56) exceeds the transverse wave velocity: $c' = \sqrt{2}\,c$.

The quarks. The comparison of the systems (50), (54) and (56) allows postulating the 2nd order PDE for the quarks from the *up* and *down* groups. Explicitly, the 2nd order system of the *u* quark from the *up* group equals:

$$\begin{cases} \left(\frac{1}{3}\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\sigma + \frac{2}{3}G_0(m)\hat{\phi} = 0, \\ -c^2\frac{2}{3}\Delta\hat{\phi} - \frac{2}{3}G_0(m)\hat{\phi} = 0, \end{cases} \tag{58}$$

and the system of the *d* quark from the *down* group:

$$\begin{cases} \frac{1}{3}\frac{\partial^2\sigma}{\partial t^2} + G_0(m)\sigma_0 - \frac{1}{3}G_0(m)\hat{\phi} = 0, \\ c^2\Delta\left(\sigma_0 + \frac{1}{3}\hat{\phi}\right) - G_0(m)\left(\sigma_0 - \frac{1}{3}\hat{\phi}\right) = 0. \end{cases} \tag{59}$$

The sum of equations in the above quark systems (58) and (59) does not comply with the Cauchy equation of motion (20) and may indicate their short lifetime. The terms $\frac{2}{3}G_0(m)\hat{\phi}$ and $-\frac{1}{3}G_0(m)\hat{\phi}$ in the systems (58) and (59) respectively, are related to the charge, Table 1

Table 1. The basic properties of the quarks in baryons.

Group	Quarks	Charge	Spin
up	*u,c,t*	2/3	1/2
down	*d,s,b*	-1/3	1/2

4.4. The quarks

There are two groups of hadrons: baryons (containing three quarks or three antiquarks); and mesons (containing a quark and an antiquark). In the following we show that systems (54) - (59) comply with the experimental findings shown in Table 1.

Proton is formed be the two up and the single down quarks: $d-u-u$. Thus by computing the sum of two systems (58) and one system (59) we may expect the proton, the system (56):

$$\begin{cases} \frac{1}{3}\frac{\partial^2\sigma}{\partial t^2} + G_0(m)\sigma_0 - \frac{1}{3}G_0(m)\hat{\phi} = 0, \\ c^2\Delta\left(\sigma_0 + \frac{1}{3}\hat{\phi}\right) + G_0(m)\left(\sigma_0 - \frac{1}{3}\hat{\phi}\right) = 0 \end{cases} + 2\times\begin{cases} \left(\frac{1}{3}\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\sigma + \frac{2}{3}G_0(m)\hat{\phi} = 0, \\ -c^2\frac{2}{3}\Delta\hat{\phi} + \frac{2}{3}G_0(m)\hat{\phi} = 0, \end{cases} \tag{60}$$

and the result is in agreement with equations (56) and (57):

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - 2c^2\Delta\right)\sigma + G_0(m)\left(\sigma_0 + \hat{\phi}\right) = 0, \\ c^2\Delta\left(\sigma_0 - \hat{\phi}\right) + G_0(m)\left(\sigma_0 + \hat{\phi}\right) = 0. \end{cases} \tag{61}$$

Neutron is formed by the one up and the two down quarks: $d-d-u$

$$2\times\begin{cases}\dfrac{1}{3}\dfrac{\partial^2\sigma}{\partial t^2}+G_0(m)\sigma_0-\dfrac{1}{3}G_0(m)\hat{\phi}=0,\\ c^2\Delta\left(\sigma_0+\dfrac{1}{3}\hat{\phi}\right)+G_0(m)\left(\sigma_0-\dfrac{1}{3}\hat{\phi}\right)=0,\end{cases}+\begin{cases}\left(\dfrac{1}{3}\dfrac{\partial^2}{\partial t^2}-c^2\Delta\right)\sigma+\dfrac{2}{3}G_0(m)\hat{\phi}=0,\\ -c^2\dfrac{2}{3}\Delta\hat{\phi}+\dfrac{2}{3}G_0(m)\hat{\phi}=0,\end{cases} \tag{62}$$

and the result is in agreement with neutron system (50):

$$\begin{cases}\dfrac{\partial^2\sigma}{\partial t^2}-c^2\Delta\sigma+2G_0(m)\sigma_0=0,\\ c^2\Delta\sigma_0+G_0(m)\sigma_0=0.\end{cases} \tag{63}$$

The systems (54), (61) and (63) represent coupled 2nd order PDE's and show the different coupling strengths. The strongest coupling of the proton is related to its enormously long lifetime, Equation (61).

5. The Quaternion Schrödinger Equation

The vectorial Poisson equation indicates that it's the 2nd order PDE system for electron. We will apply the schema in the system (54) in the integral form of the energy conservation, in Equation (28). We treat the wave as a particle in an arbitrary volume **Ω** [14]. The energy per mass unit, *e*, in the volume occupied by the particle wave defines its overall energy: $E_O=E_p+E_V=\int_\Omega \rho_p e\,\mathrm{d}x$,

$$e=\tfrac{1}{2}\dot{u}\cdot\dot{u}^*+\tfrac{1}{2}c^2\sigma\cdot\sigma^*+c^2\sigma_0^2 \quad \text{where} \quad \sigma^*=\sigma_0-\hat{\phi} \quad , \tag{64}$$

where E_p and E_V denote energies of the particle and of its force field respectively, ρ_P is the Planck mass density.

The 1st step in deriving the Schrödinger equation is the choice of the symmetrization scheme for the particle energy, E_p. Equation (64) can be written in the equivalent form:

$$e=\frac{1}{2}\dot{u}\cdot\dot{u}^*+\frac{3}{2}c^2\sigma\cdot\sigma^*-c^2\hat{\phi}\cdot\hat{\phi}^*, \tag{65}$$

upon comparing with the system (54) we separate the E_p and E_V terms in integral formula

$$E_p+E_V=\rho_P\int_\Omega\left(\frac{1}{2}\dot{u}\cdot\dot{u}^*+\frac{3}{2}c^2\sigma\cdot\sigma^*-c^2\hat{\phi}\cdot\hat{\phi}^*\right)\mathrm{d}x \Leftarrow \begin{cases}E_p=\dfrac{1}{2}\rho_P\int_\Omega\left(\dot{u}\cdot\dot{u}^*+3c^2\sigma\cdot\sigma^*\right)\mathrm{d}x,\\ E_V=\rho_P\int_\Omega\left(-c^2\hat{\phi}\cdot\hat{\phi}^*\right)\mathrm{d}x.\end{cases} \tag{66}$$

The mass of the particle, $m=E_p/c^2$, follows from the particle wave energy in (66)

$$m=\frac{1}{2}\rho_P\int_\Omega\left(3\sigma\cdot\sigma^*+\frac{\dot{u}\cdot\dot{u}^*}{c^2}\right)\mathrm{d}x. \tag{67}$$

The terms $3\sigma\cdot\sigma^*$ and $\dot{u}\cdot\dot{u}^*/c^2$ oscillate and depend on the time and position. The symmetry in (67) allows normalizing the deformation and mass velocity with respect to the overall particle mass:

$$\begin{aligned}&\int_\Omega\frac{3\rho_P}{m}\sigma\cdot\sigma^*\,\mathrm{d}x=\int_\Omega\psi\cdot\psi^*\,\mathrm{d}x=1, \quad \text{where} \quad \psi=\sqrt{\frac{3\rho_P}{m}}\sigma,\\ &\int_\Omega\frac{\rho_P}{mc^2}\dot{u}\cdot\dot{u}^*\,\mathrm{d}x=\int_\Omega\psi\cdot\psi^*\,\mathrm{d}x=1, \quad \text{where} \quad \psi=\sqrt{\frac{\rho_P}{m}}\frac{\dot{u}}{c}.\end{aligned} \tag{68}$$

The quaternionic particle mass density ψ can be called the quaternionic probability because the relation $\int_\Omega\psi\cdot\psi^*\,\mathrm{d}x=1$ in (68) is satisfied. Obviously, terms $\psi=\sqrt{3\rho_P/m}\,\sigma(t,x)$ and $\psi\cdot\psi^*$, vary in time.

We analyze the evolution of the wave as in relations (66) and (67) in the time-invariant potential field, e.g., the particle wave in the field generated by other particles. The overall particle energy is now a sum of the ground and excess energy Q,

$$E=E_p+Q=\int_\Omega\left(\frac{3}{2}\rho_P c^2\sigma\cdot\sigma^*+\frac{1}{2}\rho_P\dot{u}\cdot\dot{u}^*+V(x)\psi\cdot\psi^*\right)\mathrm{d}x \tag{69}$$

We consider the low excess energies, and the impact of Q on the overall particle mass in (67) is marginal. Thus, the relation (69) becomes

$$\begin{aligned}E=E_p+Q&=\int_\Omega\left(\frac{1}{2}mc^2\psi\cdot\psi^*+\frac{1}{2}\rho_P\dot{u}\cdot\dot{u}^*+V(x)\psi\cdot\psi^*\right)\mathrm{d}x\\ &=\frac{1}{2}mc^2+\int_\Omega\left(\frac{1}{2}\rho_P\dot{u}\cdot\dot{u}^*+V(x)\psi\cdot\psi^*\right)\mathrm{d}x.\end{aligned} \tag{70}$$

Both the E_p and m are constant; thus, it is enough to minimize the relation

$$Q = \int_\Omega \left(\frac{1}{2} \rho_P \, \dot{u} \cdot \dot{u}^* + V(x) \psi \cdot \psi^* \right) \mathrm{d}x \tag{71}$$

The above relation contains two unknowns: $\dot{u} = \partial u / \partial t$ and ψ. By relating the local lattice velocity $\dot{u}$ to the force, specifically to the normalized Cauchy–Riemann derivative of the deformation: $l_P D\sigma$, one gets

$$\dot{u} = \frac{p}{m} = -\frac{\hbar}{m} D\sigma \,. \tag{72}$$

By introducing (72) and the normalization (68), the relation (71) becomes the functional

$$Q[\psi] = \int_\Omega \left(\frac{\hbar^2}{6m} (D\psi) \cdot (D\psi)^* + V(x) \, \psi \cdot \psi^* \right) \mathrm{d}x \,. \tag{73}$$

The functional $Q[\psi]$, Eq. (73), was minimized with respect to a quaternion function, such that ψ satisfies the normalization introduced in the relation (68). One may follow the schema used in [14]. In simple terms, we seek a differential equation that has to be satisfied by the ψ function to minimize the energies allowed by (73). Given the functional (73), the conditional extreme is found using the Lagrange coefficients method and the Du Bois Reymond variational lemma [34]. In such a case, ψ satisfies the time-invariant Schrödinger equation satisfied by the particle wave in the ground state of the energy E [14]:

$$-\frac{\hbar^2}{2m} \Delta \psi + V(x) \, \psi = \lambda \psi \,, \tag{74}$$

where a constant factor on the right-hand side can be considered as extra energy of the particle in the presence of the field $V = V(x)$. For $E = \lambda$, Equation (74) is clearly the time-independent Schrödinger equation satisfied by the particle in the ground state of the energy E,

$$-\frac{\hbar^2}{2m} \Delta \psi + V(x) \, \psi = E \psi \,. \tag{75}$$

6. The First-Order PDE in the Cauchy Continuum

The operator quantum mechanics base on the complex number algebra, the matrices, and the matrix algebra. Canonical quantization starts from classical mechanics and assumes that the point particle is described by a "probabilistic wave function". Dirac applied complex combinations of the displacements and velocities in the linear problem of secondary quantization [35] and replaced the second-order Klein–Gordon equation by an array of first-order equations. He recognized the problem of medium for the transmission of waves:

> "*It is necessary to set up an action principle and to get a Hamiltonian formulation of the equations suitable for quantization purposes, and for this the aether velocity is required*" [36].

In this section we follow Dirac comment. We derive the formulae basing on the aether concept. Explicitly, the Cauchy continuum and the quaternionic oscillator $G_\lambda(m)$ for the 1st order PDE and the separated Planck time scale. The 2nd order particle wave equation, e.g., in the electron PDE system (55), contains two parts:

$$\begin{array}{ccccc} \left(\dfrac{\partial^2}{\partial t^2} - c_L^2 \Delta \right) \sigma & + & 2G_0(m) \hat{\phi} & = & 0, \\ \begin{bmatrix} \text{2}^{\text{nd}} \text{ order wave term } \sigma^\mu \sigma_\mu \text{: variable } \sigma \\ \text{and constant wave velocity } c_L = \sqrt{3}\, c \end{bmatrix} & + & \begin{bmatrix} \text{Propagator with oscillator } G_0(m) \\ \text{that runs at two frequencies} \end{bmatrix} & = & 0. \end{array} \tag{76}$$

We will comply with above schema for the 1st order PDE:

$$\begin{array}{ccccc} \left(\dfrac{\partial}{\partial t} - c_L \, D \right) \dfrac{\dot{u}}{c_L} & + & 2G_\lambda(m) \dfrac{\dot{u}}{c_L} & = & 0, \\ \begin{bmatrix} \text{1}^{\text{st}} \text{ order wave term } \sigma_\mu \text{:variable } \dot{u}/c_L \text{,} \\ \text{constant wave velocity } c_L = \sqrt{3}\, c \end{bmatrix} & + & \begin{bmatrix} \text{Propagator with oscillator } G_\lambda(m) \\ \text{that runs at particle wave frequency} \end{bmatrix} & = & 0. \end{array} \tag{77}$$

6.1. The 1st order wave term.

We consider the system (54) and the relation between the wave velocity and the Cauchy–Riemann derivative Equation (72): $\mathrm{D}\sigma = -\dfrac{m}{\hbar} \dot{u}.$ The expression for the overall particle energy, Equation (66), implies:

- the deformation velocity as the alternative variable:

$$\frac{\dot{u}}{c_L} = -\frac{\hbar}{mc_L}\mathrm{D}\sigma, \tag{78}$$

- the longitudinal wave velocity as the wave propagation velocity:

$$c_L = \sqrt{3}\,c. \tag{79}$$

The motionless particle is considered, thus its wave is at a steady state. The 2nd order time derivative of the q-potential in (77) we express as follows:

$$\frac{\partial^2\sigma}{\partial t^2} = \frac{\partial}{\partial t}\cdot\left(\frac{\partial\sigma}{\partial t}\right). \tag{80}$$

The term in the bracket on the right-hand side is the rate of the q-potential changes. We want to express this term by the new variable and separate the time scales. The rate of changes of the deformation potential $\partial\sigma/\partial t$ is due to the wave propagation within the particle space. The propagation process must follow the extremum principle, i.e., it is the brachistochrone problem [37]. The good example of "local principle" approximation is by Derbes [38].

We know that wave path fulfills the extremum principle, i.e., the wave path follows its unique trajectory given by the Cauchy–Riemann derivative $\mathrm{D}\sigma$. The trajectory which has the minimum property globally in the whole volume Ω occupied by the particle must have the same property locally. This path grants the shortest possible travelling time for the waves identified in QQM. Consequently from (78) - (80) we postulate the following:

$$\begin{cases} \dfrac{\partial\sigma}{\partial t} = \dfrac{\partial\mathbf{u}}{\partial t}\left(\dfrac{\partial\sigma}{\partial\mathbf{u}}\right), \\ \dfrac{\partial\mathbf{u}}{\partial t} = c_L \\ \dfrac{\partial\sigma}{\partial\mathbf{u}} = \mathrm{D}\sigma = -\dfrac{m}{\hbar}\dot{u}, \end{cases} \quad\Rightarrow\quad \frac{\partial\sigma}{\partial t} = c_L\mathrm{D}\sigma = \frac{mc_L}{\hbar}\dot{u}.\ \cdot \tag{81}$$

From the relation (78) we get

$$\mathrm{D}\sigma = -\frac{m}{\hbar}\dot{u} \quad\Rightarrow\quad \Delta\sigma = -\mathrm{D}\mathrm{D}\sigma = \frac{m}{\hbar}\mathrm{D}\dot{u}. \tag{82}$$

Combining the relations (81) and (82), we get the 1st order particle wave term consistent with the 2nd order formula (76):

$$\frac{\partial^2\sigma}{\partial t^2} - c_L^2\Delta\sigma \Leftrightarrow \left(\frac{mc_L}{\hbar}\frac{\partial}{\partial t} - \frac{mc_L^2}{\hbar}\mathrm{D}\right)\frac{\dot{u}}{c_L} = \frac{mc_L^2}{\hbar}\left(\frac{\partial}{\partial t} - c_L\mathrm{D}\right)\frac{\dot{u}}{c_L}. \tag{83}$$

Thus, the 1st order particle wave term in (77) equals:

$$\left(\frac{\partial}{\partial t} - c_L D\right)\frac{\dot{u}}{c_L} = 0. \tag{84}$$

6.2. The 1st order quaternionic oscillator.

The power of the 2nd order quaternionic oscillator, $G_0(f) = 8\pi^2 f_P f$, results from two time scales in PK-C. We consider the macro scale only and 1st order PDE equation thus, by eliminating the Planck frequency from the relation (42), results in the power formula of the 1st order quaternionic oscillator when the particle mass is known:

$$G_\lambda(m) = 4\pi f = 2\frac{mc_L^2}{\hbar} = 6\frac{m}{m_P t_P}. \tag{85}$$

By introducing the relations (84) and (85) in the schema (77), the 1st order PDE for electron equals

$$\left(\frac{\partial}{\partial t} - c_L D\right)\frac{\dot{u}}{c_L} - 6\frac{m}{m_P t_P}\hat{\phi} = 0. \tag{86}$$

By substituting (78)

$$\left(\frac{1}{c_L}\frac{\partial}{\partial t} - \mathrm{D}\right)\frac{\hbar}{m}\mathrm{D}\sigma + \frac{6m}{m_P t_P}\hat{\phi} = 0. \tag{87}$$

The relations in (68): $\psi = \sqrt{\rho_P/m}\,\dot{u}/c_L$ and $\psi = \sqrt{3\rho_P/m}\,\sigma$, imply that by multiplying the particle wave equation (86) by $\psi = \sqrt{\rho_P/m}$, it will be expressed as a function of probability

$$\left(\frac{\partial}{\partial t} - c_L D\right)\psi - \frac{6m}{m_P\, t_P}\psi = 0 \tag{88}$$

or

$$\left(\frac{1}{c_L}\frac{\partial}{\partial t} - D\right)\psi - \frac{6m}{m_P l_P}\psi = \left(\partial_\mu - \frac{6m}{m_P l_P}\right)\psi = 0. \tag{89}$$

The comparison of the first-order wave equations in quaternion formulation, Equation (89), with the form in the Dirac algebra formalism:

$$\begin{aligned}
&\text{Dirac:} \quad \left(i\gamma^\mu \partial_\mu - \frac{m}{m_P l_P}\right)\psi(t,x) = 0 \ \text{ where } \ \gamma^\mu\partial_\mu = \frac{1}{c}\frac{\partial}{\partial t} + \alpha_1\frac{\partial}{\partial x} + \alpha_2\frac{\partial}{\partial y} + \alpha_3\frac{\partial}{\partial z},\\
&\text{QQM:} \quad \left(\partial_\mu - \frac{6m}{m_P l_P}\right)\psi(t,x) = 0.
\end{aligned} \tag{90}$$

The Electron Spin. The energy relations (65) are symmetrical and, in the case of the electron:

$$\begin{cases}
E_{\substack{electron\\ particle}} = \dfrac{1}{2}\rho_P \displaystyle\int_\Omega \left(\dot{u}\cdot\dot{u}^* + c_L^2 \sigma\cdot\sigma^*\right)\mathrm{d}x,\\
E_{\substack{electron\\ potential\ field}} = \rho_P \displaystyle\int_\Omega \left(-c^2\hat{\phi}\cdot\hat{\phi}^*\right)\mathrm{d}x.
\end{cases} \tag{91}$$

Particle is stable and its energy must be conserved. Thus, it’s justified to assume that the constrain: $\operatorname{div}\hat{\phi} = 0$ holds for the completed particle cycle. In the static particle we postulate zero dissipation of the twist energy: $\operatorname{div}\hat{\phi} = 0$. It implies the necessity of the spin, $\hat{S}$, the process that will provide the energy conservation:

$$\operatorname{div}\hat{\phi} = \operatorname{div}\left(\hat{\phi}' + \hat{S}\right) = 0\,. \tag{92}$$

The equipartition of energy between the twists, $\left|\hat{\phi}'\right| = \left|\hat{S}\right|$, implies the equipartition of moments: $\left\|\hat{\phi}'\right\| = \left\|\hat{S}\right\|$.

Thus, the overall momentum per mass unit equals: $2\left\|\hat{S}\right\| = \alpha$.

7. Results

In this section we show the spin simulations.

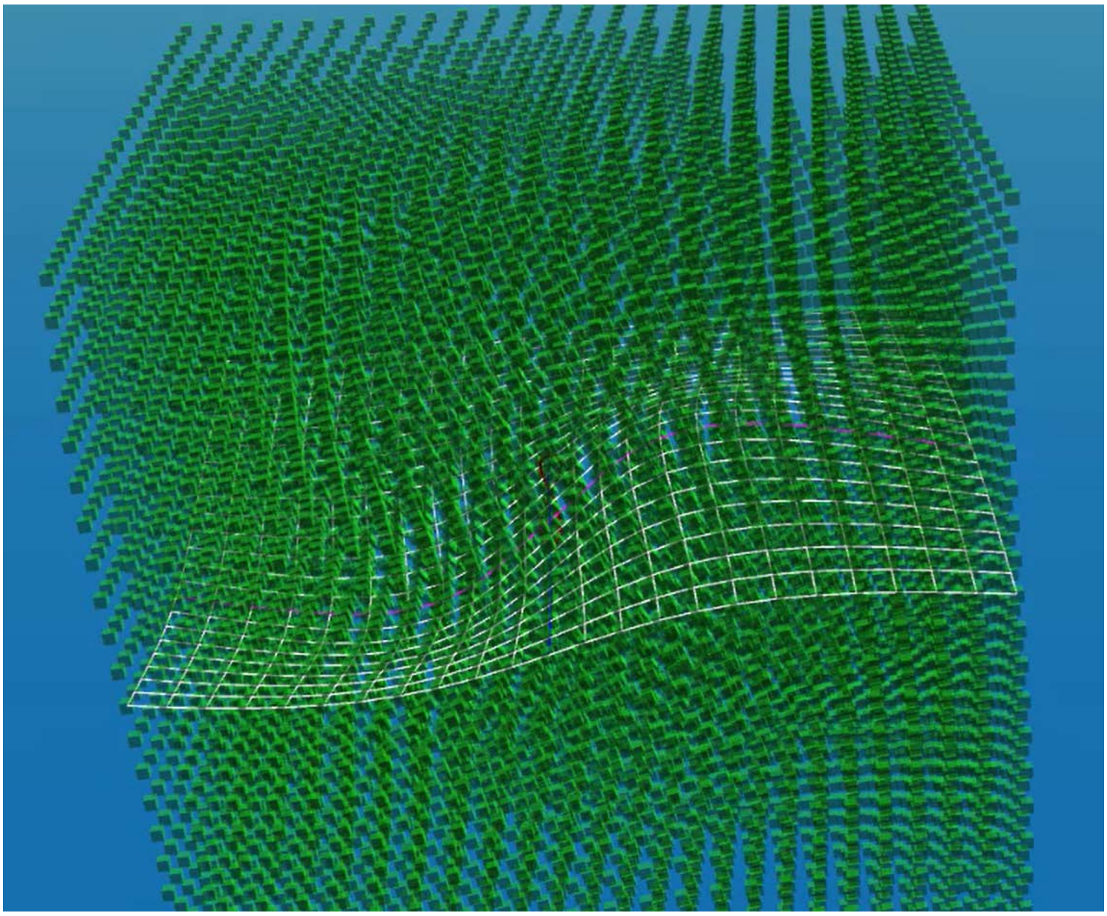

Figure 1: To simplify the visualization of the twist, the planar cut is indicated by the grid lines.

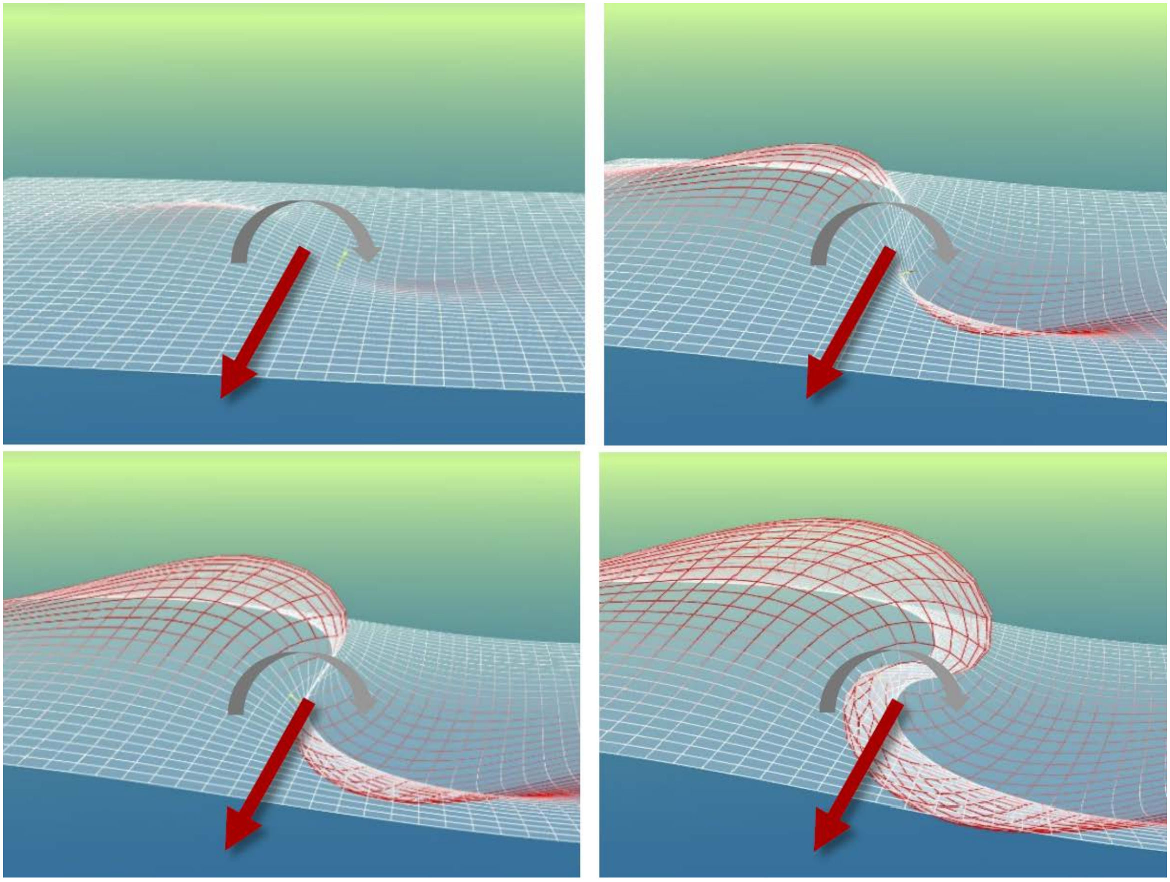

Figure 2. Spin 1/2 has two orthogonal axis of rotation. The first axis (shown in red) can be viewed as "winding of space". The images show a progress, the more and more winding up to T/2.

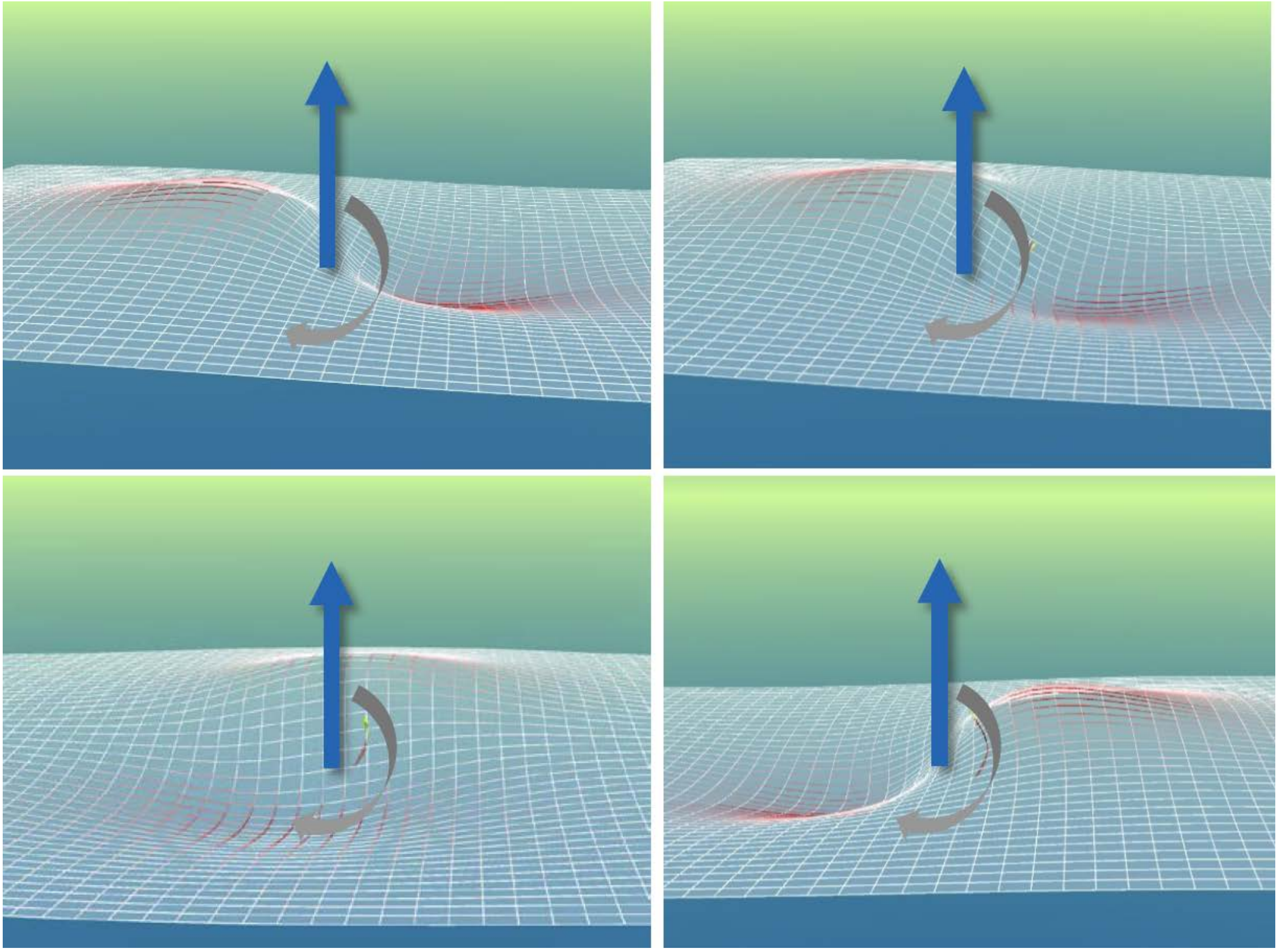

Figure 3: The second axis of rotation (shown in blue) can be viewed as the phase and is a function of time. The images show a spin progress around a central point.

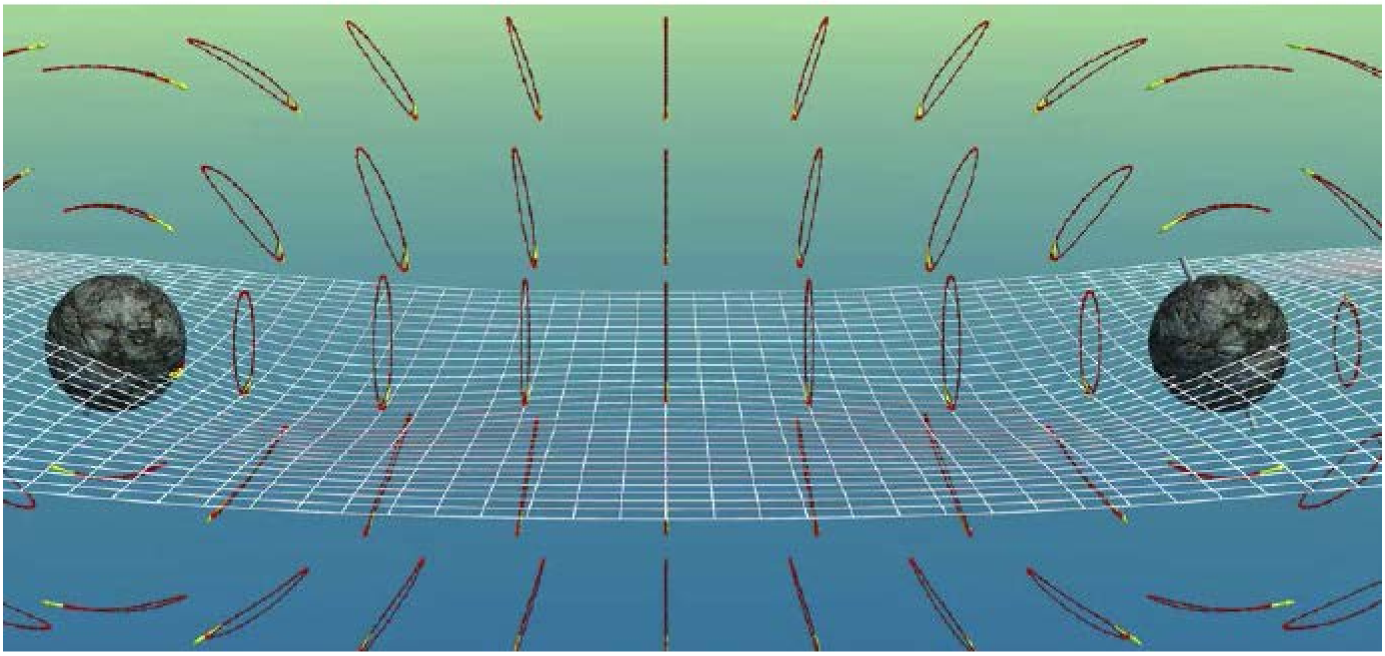

Figure 4. Two spin 1/2 waves are shown next to each other, both pointing in the same direction so that North aligns to South. The red circles indicate the oscillating movement of selected points in the elastic solid. (The spheres are only visual aids to show the center of each spin ½ wave).

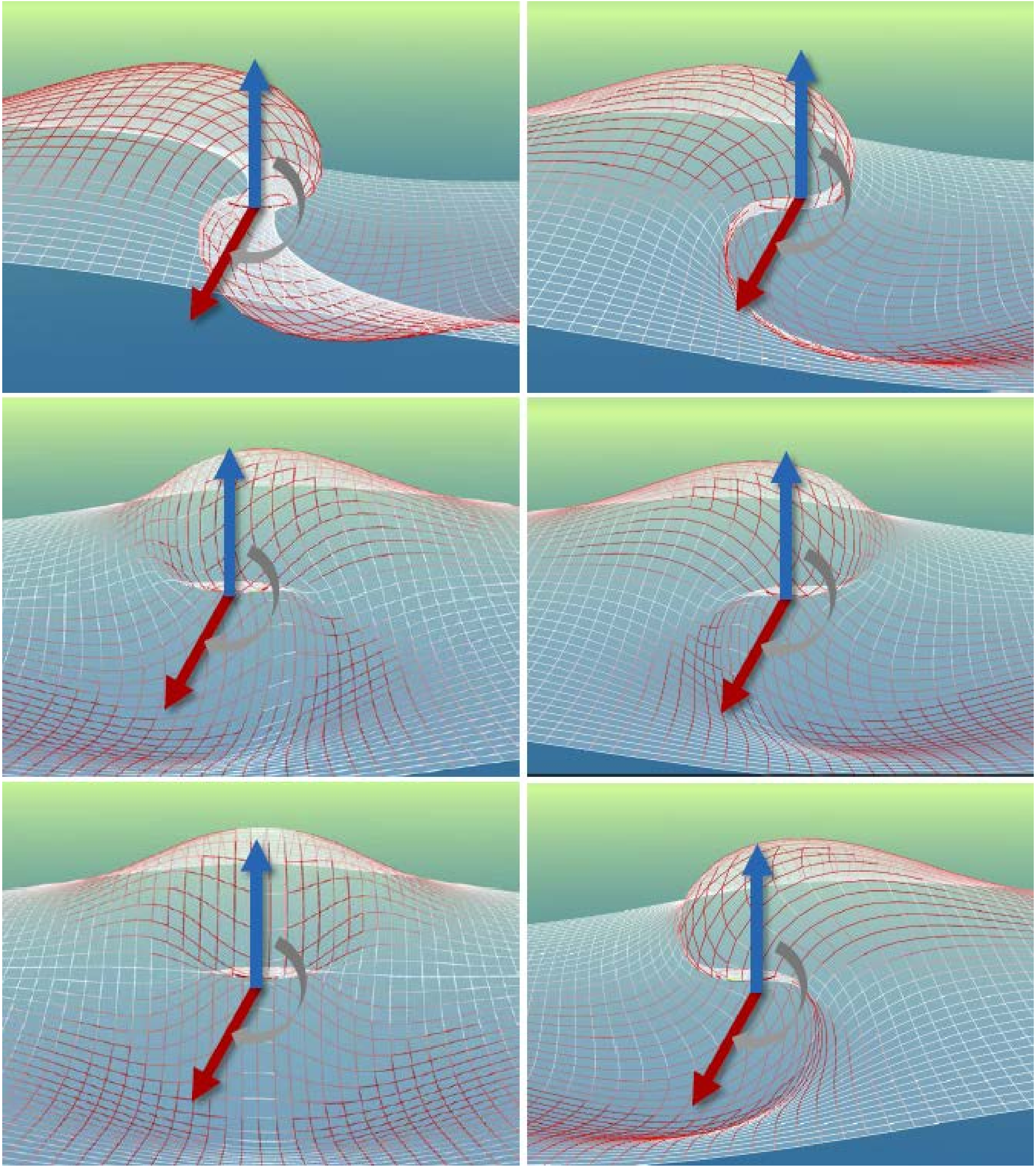

Figure 5: The images show the transverse wave spin ½ at $t \in (0, T/2)$. The wave rotates around a central (blue) axis. Note that the elastic solid stays fully intact. Each point in the elastic solid twists around a central point. The resulting motion results in the well known “Dirac Belt” or “Wine dance” trick, which looks like a rotation, but is in fact only an oscillation of the transverse twists.

7. Conclusions

The presented results are based on the ontological model of the QQM and QFT, on the Cauchy continuum and the Planck unit cell concepts. The major progress is due to the symmetrization of quaternion relations. Explicitly, due to the postulate of the quaternion velocity. It allows considering the momentum of the expanding Cauchy continuum, $\dot{u}_0(t,x)$ and, is the apparent result of the scalar potential of the expansion/compression: $\sigma_0(t,x)$.

The key new results are listed below:

- The vectorial $G_0(m)(\sigma_0+\hat{\phi})$, $G_0(m)\hat{\phi}$ and scalar: $G_0(m)\sigma_0$, $G_0(m)\sigma\cdot\sigma^*$, propagators are postulated and used to generate the 2nd order PDE systems for the proton, electron and neutron.
- The scrupulous assessment of the 2nd order PDE systems allows postulating the two 2nd order PDE systems for the *u* and *d* quarks from the *up* and *down* groups.
- It is shown that both the proton and the neutron obey experimental findings and are formed by three quarks. Namely, the proton and neutron are formed by $d-u-u$ and $d-d-u$ complexes, respectively. All above PDE systems comply with Cauchy equation of motion (20) and can be considered as stable particles.
- The *u* and *d* quarks do not meet the relations of the Cauchy equation of motion. Also experimental efforts to find the individual quarks were without success. Observed were the bound states of the three quarks – the baryons and a quark and an antiquark – the mesons. Wilczek calls it the phenomenological paradox: *"Quarks are Born Free, but Everywhere They are in Chains"* [39]. The inconsistency of the quarks PDE with the Cauchy equation of motion elucidates the observed *Quarks Chains.*

The results indicate the following targets for an immediate future:

- The particles and quarks in the case of higher coupling coefficients: $n>|1|$.
- The ratios between the constants for the different force fields.
- The rigorous derivation of the 1st order PDE basing on the extremum principle.
- The multivalued coordinate transformation to determine the properties of space with curvature and torsion produced by 2nd order PDE systems representing the QFT [40].

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Appendix A

Abbreviations

PDE	partial differential equation
QQM	quaternion quantum mechanics
QFT	Quaternion field theory
$\mathbf{T}$	deformation tensor
λ,μ	Lamé coefficients;
σ'	stress tensors
ρ_E	density of the deformation energy
$\mathbf{u}(u_1,u_2,u_3)$	displacement in $\mathbb{R}^3$
$\sigma(\sigma_0,\phi_1,\phi_2,\phi_3)$	*q-potential* in $\mathbb{R}^4$, the quaternion deformation potential
$\sigma^*\cdot\sigma$	strain energy density

G_0	power of the quaternionic oscillator
$G_0\,\sigma^*\cdot\sigma$	*density of the rate of momentum change, i.e., the quaternionic scalar propagator I*
$G_0\,\sigma$	*density of the rate of momentum change, i.e., the quaternionic scalar propagator II*
$G_0\,\hat{\phi}$	*quaternionic vectorial propagator*
$G_0(m)\left(\sigma\cdot\sigma^*+\hat{\phi}\right)$	*quaternionic q-potential propagator*
$\psi=\sigma\sqrt{\rho_P/m}$	quaternionic particle density, i.e., the particle wave function
$\psi\cdot\psi^*$	probability, i.e., the normalized particle mass density
n	coupling coefficient in the oscillator
l_P	Planck length
$f_P=1/t_P$	Planck frequency, inverse of the Planck time
m_P	Planck mass
$c=l_P/t_P$	transverse wave velocity in elastic continuum
$c_L=\sqrt{3}\,c$	longitudinal wave velocity in elastic continuum
$\rho_P=4m_P/l_P^3$	Planck density, i.e., the mass density of the PK-C
ρ	mass density of the particle $\rho=\rho_E/c^2$, as the equivalent of the energy density ρ_E in the PK-C
$\hbar$	Planck constant in terms of angular frequency
h	Planck constant, $h=2\pi\hbar$
m	equivalent mass of the wave, i.e., mass of the particle
λ	length of the particle wave
f	frequency of the particle wave

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