Qeios

Quaternion Quantum Mechanics: the Baryons, Quarks and their q-potentials

Marek Danielewski[1,*], Lucjan Sapa[2]

1 AGH UST, Mickiewicza 30, 30-059 Kraków, Poland; daniel@agh.edu.pl
2 Faculty of Applied Mathematics, AGH UST, Mickiewicza 30, 30-059 Kraków, Poland; sapa@agh.edu.pl
* Correspondence: daniel@agh.edu.pl

Abstract: The results presented here base on the Planck-Kleinert crystal concept. The rigorous use of the quaternion algebra allows postulating the scalar, vectorial and the quaternion propagators in the ideal elastic continuum. The propagators are used in constructing the proton, electron and the neutron 2nd order partial differential equation systems, PDES's. The results generate the two 2nd order PDES's for the *u* and *d* quarks from the *up* and *down* groups. It was verified that both the proton and the neutron obey experimental findings and are formed by three quarks. The proton and neutron are formed by *d-u-u* and *d-d-u* complexes, respectively. All particle PDES's comply with Cauchy equation of motion and can be considered as stable particles. The u and d quarks do not meet the relations of the Cauchy equation of motion. The inconsistencies of the quarks PDES's with the quaternion forms of the Cauchy equation of motion account for their lifetime and the observed Quarks Chains. That is, explain the Wilczek phenomenological paradox: "Quarks are Born Free, but Everywhere They are in Chains".

Keywords: q-potentials; vectorial potential; proton; quarks chain

1. Introduction

The focus here is the quaternion quantum mechanics, QQM, and the quaternionic field theory, QFT. The quaternion algebra is attributed to many physical systems and laws, sporadically to quantum mechanics. Lanczos dissertation was on a quaternionic field theory of the classical electrodynamics [1,2]. In his derivation of the Dirac's equation [3], there is a doubling in the number of solutions and several concepts that still remain at the front of the fundamental theory. These articles were unnoticed by contemporaries; Lanczos abandoned quaternions and never returned to the quaternionic field theory. Fueter demonstrated that the Cauchy-Riemann type conditions in the quaternion representation are identical in shape to vacuum equations of electrodynamics [4]. Yefremov described the Newtonian mechanics in rotating frame of reference [5] and the motion of non-inertial frames [6]. Adler shows that the Dirac transition amplitudes are quaternion valued [7]. Christianto derived an original wave equation from the correspondence between Dirac equation and Maxwell electromagnetic equations via the biquaternionic representation [8].

The Adler's method of the quaternionizing the quantum mechanics was avoided in the Harari-Shupe's preon model for the composite quarks and leptons [9]. However, the composite fermion states were later identified with the quaternion real components [10]. In spite of the lack of progress in advancing the Harari-Shupe scheme, the substantial progress in the QQM and QFT was made [11,12].

The evolution of the P-KC model and the subsequent development of the QQM are shown [13,14,15]. The QQM presented here is ontological in a sense that it starts with being, that is the Planck-Kleinert ideal regular crystal [14,16]. The basic categories of being and their relations are governed by the quaternion algebra [14]. The stress tensor of Planck-Kleinert crystal is given by

$$\sigma' = \sigma'' / m_P = \left(\lambda_L / m_P \, \mathbf{trD}\right)\mathbf{1} + 2\mu_L / m_P \, \mathbf{D} \quad , \tag{1}$$

where D denotes the deformation tensor (the symmetrical part of the strain tensor) and λ_L and μ_L are the Lamé coefficients of an ideal regular crystal. It was shown by Cauchy and Saint Venant that if the particles composing a regular crystal interact pairwise through central forces, then there is an additional symmetry requiring C44 = C12 that implies the Poisson ratio 0.25 and $\lambda_L = \mu_L$ [17]

https://doi.org/10.32388/E5RTG6

$$\sigma' = \left(\lambda_L / m_P \, \mathbf{trD}\right)\mathbf{1} + 2\lambda_L / m_P \, \mathbf{D} \qquad (2)$$

Using the identity: $\operatorname{grad}\operatorname{div}\mathbf{u} = \operatorname{div}\operatorname{grad}\mathbf{u} + \operatorname{rot}\operatorname{rot}\mathbf{u}$, the stress tensor in the Planck-Kleinert crystal becomes [16]:

$$\operatorname{div}\sigma' = 2\lambda_L / m_P \operatorname{grad}\operatorname{div}\mathbf{u} + \lambda_L / m_P \operatorname{div}\operatorname{grad}\mathbf{u} = 3\lambda_L / m_P \operatorname{grad}\operatorname{div}\mathbf{u} - \lambda_L / m_P \operatorname{rot}\operatorname{rot}\mathbf{u}. \qquad (3)$$

The motivation for writing this paper was to explicate the stress field origin of the QQM and QFT. The Standard Model of elementary particles lack adequate description for the mechanism of quark charges. It is showed here that the quark particle waves do exist and two their PDES, are presented. The further studies in order to verify or refute those propositions are suggested.

1.1. Quaternions

The elements of the quaternion algebra used in the QQM and QFT were already presented in previous papers [13-15]. Only the two definitions are recalled here. In the ideal elastic continuum the quaternion potential, i.e., the deformation four-potential, is defined by

$$\begin{matrix} \sigma & = & \sigma_0 & + & \hat{\phi} \\ \begin{bmatrix} q\text{-potential,} \\ \text{deformation} \end{bmatrix} & = & \begin{bmatrix} \operatorname{div}\mathbf{u}_0 \\ \text{compression} \end{bmatrix} & + & \begin{bmatrix} \operatorname{rot}\mathbf{u}_\phi \\ \text{twist pseudovector} \end{bmatrix}, \end{matrix} \qquad (4)$$

where $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_\phi$ denotes the displacement, $\sigma = \sigma_0 + \hat{\phi} \in \mathbb{H}$ is the q-potential and the constraint $\operatorname{div}\hat{\phi} = 0$ holds.

We use the Cauchy–Riemann operator D in $\mathbb{R}^4$ acting on the quaternion-valued functions

$$D\sigma = \left(-\operatorname{div}\hat{\phi}\right) + \operatorname{grad}\sigma_0 + \operatorname{rot}\hat{\phi}, \quad \sigma = \sigma_0 + \hat{\phi}. \qquad (5)$$

Under the constraint: $\operatorname{div}\hat{\phi} = 0$, D corresponds physically to the nabla operator in $\mathbb{R}^3$:

$$D\sigma = \operatorname{grad}\sigma_0 + \operatorname{rot}\hat{\phi}. \qquad (6)$$

The exponent function has its trigonometrical representation

$$\exp(\sigma) = \left(\cos|\hat{\phi}| + \hat{\phi}/|\hat{\phi}|\sin|\hat{\phi}|\right)\exp(\sigma_0). \qquad (7)$$

1.2. The critical review of the earlier results.

The Cauchy equation of motion and the overall energy density of the deformation field in the quaternion formulation equal [14]

$$\left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\sigma - 2c^2\Delta\sigma_0 = 0, \qquad (8)$$

$$\rho_E = \rho_P\left(\tfrac{1}{2}\hat{u}\cdot\hat{u}^* + \tfrac{1}{2}c^2\,\sigma\cdot\sigma^* + c^2\sigma_0^2\right), \qquad (9)$$

where ρ_E and ρ_P denote the deformation energy and the mass densities in the P-KC respectively, $\hat{u} = u_1 i + u_2 j + u_3 k$ is the mass velocity in the quaternion representation:

$$\hat{u} = -\frac{\hbar}{m}\mathrm{D}\sigma. \qquad (10)$$

The overall energy of the particle wave in arbitrary volume Ω follows from Eq. (9) and is given by integral:

$$\int_\Omega \rho_E(t,x)\,\mathrm{d}x = \int_\Omega \rho_P\left(\frac{1}{2}\hat{u}\cdot\hat{u}^* + \tfrac{1}{2}c^2\,\sigma\cdot\sigma^* + c^2\sigma_0^2\right)\mathrm{d}x = mc^2, \qquad (11)$$

where for the sake of clarity, the external potential, $V(x)$, is not shown [14].

In the previous paper [15] upon substituting $\tilde{\sigma}_0 = \sqrt{3}\sigma_0$, we introduced in (11) the transformed, *q*-potential, $\tilde{\sigma} = \tilde{\sigma}_0 + \hat{\phi} = \sqrt{3}\sigma_0 + \hat{\phi}$, and expressed the particle mass by the symmetrical relation

$$m = \frac{\rho_P}{2c^2}\int_\Omega \left(\hat{u}\cdot\hat{u}^* + c^2\tilde{\sigma}\cdot\tilde{\sigma}^*\right)\mathrm{d}x. \tag{12}$$

The combining (10) and (12) resulted in the energy functional and allowed considering the existence of the stable particle *m* in the potential field $V(x)$. Subsequently, the quaternionic particle density was defined,

$$\psi = \sqrt{\rho_P/_m}\,\tilde{\sigma} \tag{13}$$

and proved that ψ satisfies the time-independent Schrödinger equation [14]

$$-\frac{h^2}{2m}\Delta\psi + V(x)\,\psi = E\psi\,. \tag{14}$$

The quaternionic particle density ψ is also called the quaternionic probability because the relation $\int_\Omega \psi\cdot\psi^*\mathrm{d}x = 1$ holds [15].

Remark. The q-potential definition, $\tilde{\sigma} = \sqrt{3}\sigma_0 + \hat{\phi}$, is incompatible with the derived quaternionic oscillator formula where only integral coupling coefficients *n* are allowed, e.g., $\tilde{\sigma} = (1-n)\sigma_0 + \hat{\phi}$ in [15].

The 2nd order boson PDES's presented in [13,14] base on the postulate of the scalar propagator, $G_0(m)\sigma\cdot\sigma^*$, providing the coupling between the longitudinal and transverse waves. The coupling is evident upon expressing the quaternionic Klein-Gordon system in the equivalent form, e.g.,

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - c^2\Delta\right)\hat{\phi} = 0, \\ \left(\dfrac{\partial^2}{\partial t^2} - 3c^2\Delta\right)\sigma_0 = 0, \\ 2c^2\Delta\sigma_0 + G_0(m)\sigma\cdot\sigma^* = 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - c^2\Delta\right)\sigma + G_0(m)\sigma\cdot\sigma^* = 0, \\ 2c^2\Delta\sigma_0 + G_0(m)\sigma\cdot\sigma^* = 0. \end{cases} \tag{15}$$

Above two systems are identical, five equations and five unknowns: $\sigma_0, \phi_1, \phi_2, \phi_3$ and *m*, see definition (4). If mass *m* is unknown it may be treated as the parameter in the Poisson equation above.

In [15] we further developed the propagator concept and postulated the family of the second-order quaternionic wave equations:

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - c^2\Delta\right)\hat{\phi} = 0, \\ \left(\dfrac{\partial^2}{\partial t^2} - 3c^2\Delta\right)\sigma_0 = 0, \\ 2(1-n)c^2\Delta\sigma_0 + G_0(m)\sigma\cdot\sigma^* = 0 \text{ where } n = 0,2,3,\ldots \end{cases} \tag{16}$$

where *n* is coupling coefficient.

It's evident that at *n* = 0, the coupling (15) for boson particle follows. The propagator term $G_0(m)\sigma\cdot\sigma^*$ in (16) corresponds to the density of the rate of the momentum change, the $G_0(m)$ term is referred to as power of the harmonic oscillator. The coupling coefficient *n* can be elucidated as the radius *R* of quaternionic oscillator in the Cauchy crystal expressed in Planck length: $R = n\,l_P$. In the system (16), the generalized *q*-potential can be introduced

$$\tilde{\sigma}_n = (1-n)\sigma_0 + \hat{\phi} \text{ where } n = 0,2,3,\ldots \tag{17}$$

Upon the $\tilde{\sigma}_n$ substitution into the system (16), the two 2nd order PDE's are evident:

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - c^2\Delta\right)\tilde{\sigma}_n + G_0(m)\sigma\cdot\sigma^* = 0, \\ \left[n\dfrac{\partial^2}{\partial t^2} - (n+2)c^2\Delta\right]\sigma_0 - G_0(m)\sigma\cdot\sigma^* = 0. \end{cases} \tag{18}$$

Remark. The problem of the coupling coefficients $n > |1|$ and resulting different particles is not presented here.
The harmonic oscillator controls the acceleration of *q*-potential in the particle wave. The acceleration of the scalar part σ_0 of the *q*-potential was estimated in [15]:

$$\left\langle \frac{\partial^2\sigma_0}{\partial t^2} \right\rangle = 4\pi^2 f_P f\,. \tag{19}$$

Using the equipartition theorem and common frequency postulate for the all four *q*-potential components: $\sigma_0, \phi_1, \phi_2, \phi_3$, the relation (19) was extended to $\sim^4$:

$$\left\langle \frac{\partial^2\sigma}{\partial t^2} \right\rangle = 4\left\langle \frac{\partial^2\sigma_0}{\partial t^2} \right\rangle = 16\pi^2 f_P f\,. \tag{20}$$

The acceleration of the *q*-potential will be called the power of the quaternionic oscillator in the particle wave:

$$G_0(f) = 16\pi^2 f_P f\,, \tag{21}$$

where f is an unknown particle frequency that may be postulated or computed.
Remark. The power of the oscillator $G_0(f)$, Equation (21), does not take into account both, the constraint $\operatorname{div}\hat{\phi} = 0$ and pseudovector character of twist $\hat{\phi}$.

The 1st order particle wave equation in the quaternion formulation obtained in [15] is consistent with its form in the Dirac algebra formalism. However the 1st order system is generated by the invalidated substitution $n = \sqrt{3}$ in [14]. The 2nd order PDES following the schema (18), where $n = \sqrt{3}$ and $\tilde{\sigma} = \sqrt{3}\sigma_0 + \hat{\phi}$, equals:

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - c^2\Delta\right)\tilde{\sigma} + 2G_0\,\sigma^*\cdot\sigma = 0, \\ \left(\dfrac{\sqrt{3}-1}{3-\sqrt{3}}\dfrac{\partial^2}{\partial t^2} + c^2\Delta\right)\sigma_0 + \dfrac{2}{3-\sqrt{3}}G_0\,\sigma^*\cdot\sigma = 0. \end{cases} \tag{22}$$

The system (22) consists of the two 2nd order scalar PDE's while the vector potential is not present. In the next sections the solutions of the presented problems are given.
Remark. The Equation (22) is mistaken and will be reformulated in the next sections.

2. **The Baryons, Quarks and their q-potentials**

2.1. The Quaternionic Oscillator

The coupling of the transverse and the longitudinal waves takes place in the PKC elementary cell, i.e., at the Planck scale. The quaternionic oscillator controls the acceleration of all the *q*-potential constituents in the particle wave in Ω: $\ddot{\sigma}_0, \ddot{\phi}_1, \ddot{\phi}_2, \ddot{\phi}_3$. The function $G_0 \in$ ° , is called the power of the quaternionic oscillator. In the earlier papers we neglected the facts that twists ϕ_1, ϕ_2 and ϕ_3 form the pseudovector $\hat{\phi} = \phi_1 i + \phi_2 j + \phi_3 k$ [18], and that the constraint $\operatorname{div}\hat{\phi} = 0$ holds. Thus, the relation (19) for the scalar *q*-

potential component σ_0 in $\sim^{1}$ extended for $\sigma_0, \hat{\phi}$ in $\sim^{4}$(20) must be corrected and consider the two independent q-potential constituents, σ_0 and $\hat{\phi}$:

$$\left\langle \frac{\partial^2 \sigma}{\partial t^2} \right\rangle = 2\left\langle \frac{\partial^2 \sigma_0}{\partial t^2} \right\rangle = 8\pi^2 f_P f \tag{23}$$

and the power of the quaternionic oscillator equals

$$G_0(f) = 8\pi^2 f_P f \,. \tag{24}$$

The particle wave frequency depends on the particle mass, $f = f(m)$, and follows from the $\sim^{1}$ schema, see Fig. 1 in [15]. The sum of moments of all the Planck masses forming the particle wave in Ω (at the arbitrary time t and solely due to the particle wave) equals the momentum of the particle m itself. To simplify, we may estimate the average momentum of the arbitrary single Planck mass m_P during the whole particle cycle $T = f^{-1}$. The complete cycle implies that the every Planck mass returns to its initial conditions: $\mathbf{u}_P(t) = \mathbf{u}_P(t+T)$ and $\dot{\mathbf{u}}_P(t) = \dot{\mathbf{u}}_P(t+T)$. The overall distance which the arbitrary mass m_P passes during the wave cycle T equals $2\pi l_P$. The average momentum of the Planck mass $\bar{p}(m_P)$ during the particle wave cycle equals

$$\bar{p}(m_P) = m_P \frac{2\pi l_P}{T} = 2\pi m_P l_P f. \tag{25}$$

The momentum of the particle m results in the same way from the particle wave propagation velocity, e.g., c in the system (15):

$$p(m) = mc \,. \tag{26}$$

The both, the moment (25) and (26) must equal, and the frequency of the particle wave becomes:

$$f = \frac{mc}{2\pi m_P l_P} \times \frac{c}{c} = \frac{mc^2}{2\pi\hbar} \text{ where } \hbar = m_P c l_P. \tag{27}$$

Combining the relations (24), (27) and the definition $f_P = 1/t_P$, the overall power of the quaternionic oscillator when the particle mass is known equals:

$$G_0(m) = 4\pi mc^2 / (\mathrm{h} t_P) \,. \tag{28}$$

By substituting $mc^2 = E_0$ in (27), the Planck–Einstein relation follows: $E_0 = hf$, where $h = 2\pi\hbar$. The family of the scalar 2[nd] order quaternionic wave equations when the corrected propagator is used becomes now:

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\hat{\phi} = 0, \\ \left(\frac{\partial^2}{\partial t^2} - 3c^2\Delta\right)\sigma_0 = 0, \\ nc^2\Delta\sigma_0 + G_0(m)\sigma\cdot\sigma^* = 0, \end{cases} \Leftrightarrow \begin{cases} \left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\sigma_0 + 2G_0(m)\sigma\cdot\sigma^* = 0, \\ \left((n-1)\frac{\partial^2}{\partial t^2} - (n-3)c^2\Delta\right)\sigma_0 + 2G_0(m)\sigma\cdot\sigma^* = 0, \end{cases} \tag{29}$$

where n denotes integer and $n \neq 0$.

It's evident that at $n = 1$, the coupling in (15) for the boson particle follows. The corrected propagator $G_0(m)\sigma\cdot\sigma^*$ results in symmetry of coupling equation in system (29). I t does not have an effect on the scalar 2[nd] order PDES's and gravitational constant in (15)-(18)

.

2.2. The baryon particles formed by the odd number of quarks

The strong coupling only is considered here, e.g., $n=1$ in the system (29). The quaternionic oscillator $G_0(m)$ allows postulating three propagators: the scalar, $G_0(m)\sigma\cdot\sigma^*$, the vectorial, $G_0(m)\hat{\phi}$, and the quaternion, $G_0(m)\left(\sigma\cdot\sigma^*+\hat{\phi}\right)$.

The term $G_0(m)\hat{\phi}$ fixes the density of the rate of twist change and is called vectorial propagator. We postulate the vectorial Poisson equation in system (29): $-c^2\Delta\hat{\phi}+G_0(m)\hat{\phi}=0$. Upon the rearrangement of the new system, the particle wave (electron) and the vectorial Poisson equations are evident:

$$\begin{cases}\left(\dfrac{\partial^2}{\partial t^2}-c^2\Delta\right)\hat{\phi}=0,\\ \left(\dfrac{\partial^2}{\partial t^2}-3c^2\Delta\right)\sigma_0=0,\\ -c^2\Delta\hat{\phi}+G_0(m)\hat{\phi}=0,\end{cases}\Leftrightarrow\begin{cases}\left(\dfrac{\partial^2}{\partial t^2}-3c^2\Delta\right)\sigma+2G_0(m)\hat{\phi}=0,\\ -c^2\Delta\hat{\phi}+G_0(m)\hat{\phi}=0.\end{cases}\tag{30}$$

Note that the wave propagation velocity in system (30) equals the velocity of longitudianl waves in the Couchy continuum: $c_L=\sqrt{3}\,c$ [16]. By adding equations in system (30) it is clear that it complies with the Cauchy equation of motion (8):

$$\begin{cases}\left(\dfrac{\partial^2}{\partial t^2}-3c^2\Delta\right)\sigma+2G_0(m)\hat{\phi}=0,\\ -c^2\Delta\hat{\phi}+G_0(m)\hat{\phi}=0,\end{cases}\Rightarrow\left(\dfrac{\partial^2}{\partial t^2}-c^2\Delta\right)\sigma-2c^2\Delta\sigma_0=0.\tag{31}$$

The above vectorial Poisson equation hints at the Equation (30) as the 2nd order PDES for electron. Note that the wave propagation velocity in electron system in Equation (31) equals the velocity of longitudinal wave in Couchy Equation (15): $c_L\sqrt{3}c$

In the quaternion propagator, $G_0(m)\left(\sigma\cdot\sigma^*+\hat{\phi}\right)$, the vectorial, $G_0(m)\hat{\phi}$, and scalar, $G_0(m)\sigma\cdot\sigma^*$, propagators are "merged" and form the strongly coupled system. The rearrangements of the system (32) is shown below and display different forms of the 2nd order PDES:

$$\begin{cases}\left(\dfrac{\partial^2}{\partial t^2}-c^2\Delta\right)\hat{\phi}=0,\\ \left(\dfrac{\partial^2}{\partial t^2}-3c^2\Delta\right)\sigma_0=0,\\ -c^2\Delta\hat{\phi}+G_0(m)\hat{\phi}=0,\\ c^2\Delta\sigma_0+G_0(m)\sigma\cdot\sigma^*=0,\end{cases}\Leftrightarrow\begin{cases}\left(\dfrac{\partial^2}{\partial t^2}-c^2\Delta\right)\hat{\phi}=0,\\ \left(\dfrac{\partial^2}{\partial t^2}-3c^2\Delta\right)\sigma_0=0,\\ c^2\Delta\left(\sigma_0-\hat{\phi}\right)+G_0(m)\left(\sigma\cdot\sigma^*+\hat{\phi}\right)=0,\end{cases}\Leftrightarrow\begin{cases}\left(\dfrac{\partial^2}{\partial t^2}-2c^2\Delta\right)\sigma+G_0(m)\left(\sigma\cdot\sigma^*+\hat{\phi}\right)=0,\\ c^2\Delta\left(\sigma_0-\hat{\phi}\right)+G_0(m)\left(\sigma\cdot\sigma^*+\hat{\phi}\right)=0.\end{cases}\tag{32}$$

The comparison of the scalar, vectorial and quaternionic propagators shows that the q-propagator offers the strongest coupling, Eq. (32). The quaternionic Poisson equation in (32) reveals that it is the 2nd order PDES for proton. The sum of equations in (32) shows that system complies with Cauchy equation of motion (8):

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - 2c^2\Delta\right)\sigma + G_0(m)\left(\sigma\cdot\sigma^* + \hat{\phi}\right) = 0, \\ c^2\Delta\left(\sigma_0 - \hat{\phi}\right) + G_0(m)\left(\sigma\cdot\sigma^* + \hat{\phi}\right) = 0. \end{cases} \Rightarrow \left(\frac{\partial^2}{\partial t^2} - c^2\Delta\right)\sigma - 2c^2\Delta\sigma_0 = 0. \qquad (33)$$

The scrupulous assessment of systems (15), (30) and (32) allows postulating the 2nd order PDES's for the quarks from the *up* and *down* groups. Explicitly, the 2nd order PDES of the *u* quark from the *up* group equals:

$$\begin{cases} \left(\dfrac{1}{3}\dfrac{\partial^2}{\partial t^2} - c^2\Delta\right)\sigma + \dfrac{2}{3}G_0(m)\hat{\phi} = 0, \\ -c^2\dfrac{2}{3}\Delta\hat{\phi} - \dfrac{2}{3}G_0(m)\hat{\phi} = 0, \end{cases} \qquad (34)$$

and the 2nd order PDES of the *d* quark from the *down* group:

$$\begin{cases} \dfrac{1}{3}\dfrac{\partial^2\sigma}{\partial t^2} + G_0(m)\left(\sigma\cdot\sigma^*\right) - \dfrac{1}{3}G_0(m)\hat{\phi} = 0, \\ c^2\Delta\left(\sigma_0 + \dfrac{1}{3}\hat{\phi}\right) - G_0(m)\left(\sigma\cdot\sigma^* - \dfrac{1}{3}\hat{\phi}\right) = 0. \end{cases} \qquad (35)$$

The sum of equations in the quark systems (34) and (35) does not comply with the Cauchy equation of motion (8) and may indicate their short lifetime.

2.3. The quarks

There are two groups of hadrons: **baryons (containing three quarks or three antiquarks**); and mesons (containing a quark and an antiquark). In the following we show that systems (30) - (35) comply with the experimental findings shown in Table 1.

Table 1. The basic properties of the quarks in baryons.

Group	Quarks	Charge	Spin
up	*u,c,t*	2/3	1/2
down	*d,s,b*	-1/3	1/2

The terms $\frac{2}{3}G_0(m)\hat{\phi}$ and $-\frac{1}{3}G_0(m)\hat{\phi}$ in the systems (34) and (35) respectively, are related to the charge, see Table 1.

Proton is formed be the two up and the single down quarks: $d-u-u$. Thus by computing the sum of two systems (34) and one system (35) we may expect proton, system (32):

$$\begin{cases} \dfrac{1}{3}\dfrac{\partial^2\sigma}{\partial t^2} + G_0(m)\left(\sigma\cdot\sigma^*\right) - \dfrac{1}{3}G_0(m)\hat{\phi} = 0, \\ c^2\Delta\left(\sigma_0 + \dfrac{1}{3}\hat{\phi}\right) + G_0(m)\left(\sigma\cdot\sigma^* - \dfrac{1}{3}\hat{\phi}\right) = 0 \end{cases} + 2\times\begin{cases} \left(\dfrac{1}{3}\dfrac{\partial^2}{\partial t^2} - c^2\Delta\right)\sigma + \dfrac{2}{3}G_0(m)\hat{\phi} = 0, \\ -c^2\dfrac{2}{3}\Delta\hat{\phi} + \dfrac{2}{3}G_0(m)\hat{\phi} = 0, \end{cases} \qquad (36)$$

and the result is in agreement with equation (33):

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t^2} - 2c^2\Delta\right)\sigma + G_0(m)\left(\sigma\cdot\sigma^* + \hat{\phi}\right) = 0, \\ c^2\Delta\left(\sigma_0 - \hat{\phi}\right) + G_0(m)\left(\sigma\cdot\sigma^* + \hat{\phi}\right) = 0. \end{cases} \qquad (37)$$

Neutron is formed by the one up and the two down quarks: $d-d-u$

$$2\times\begin{cases}\dfrac{1}{3}\dfrac{\partial^2\sigma}{\partial t^2}+G_0(m)\left(\sigma\cdot\sigma^*\right)-\dfrac{1}{3}G_0(m)\hat{\phi}=0,\\ c^2\Delta\left(\sigma_0+\dfrac{1}{3}\hat{\phi}\right)+G_0(m)\left(\sigma\cdot\sigma^*-\dfrac{1}{3}\hat{\phi}\right)=0,\end{cases}\quad+\quad\begin{cases}\left(\dfrac{1}{3}\dfrac{\partial^2}{\partial t^2}-c^2\Delta\right)\sigma+\dfrac{2}{3}G_0(m)\hat{\phi}=0,\\ -c^2\dfrac{2}{3}\Delta\hat{\phi}+\dfrac{2}{3}G_0(m)\hat{\phi}=0,\end{cases}\tag{38}$$

and the result is in agreement with neutron system (15):

$$\begin{cases}\dfrac{\partial^2\sigma}{\partial t^2}-c^2\Delta\sigma+2G_0(m)\sigma\cdot\sigma^*=0,\\ c^2\Delta\sigma_0+G_0(m)\sigma\cdot\sigma^*=0.\end{cases}\tag{39}$$

The systems (30), (37) and (39) represent coupled 2nd order PDE's and show the different coupling strengths. The strongest coupling of the proton, Equation (37), is related to its enormously long lifetime.

3. The Quaternion Schrödinger Equation

The vectorial Poisson equation indicates that it's the 2nd order PDES for electron. We will apply this schema in the system (30) in the integral form of the energy conservation. We treat the wave as a particle in an arbitrary volume Ω [14]. The energy per mass unit, *e*, in the volume occupied by the particle wave defines its overall energy: $E_O=E_p+E_V=\int_\Omega \rho_p e\,\mathrm{d}x$,

$$e=\tfrac{1}{2}\hat{u}\cdot\hat{u}^*+\tfrac{1}{2}c^2\sigma\cdot\sigma^*+c^2\sigma_0^2\ \text{ where }\ \sigma^*=\sigma_0-\hat{\phi}\,,\tag{40}$$

where E_P and E_V denote energies of the particle and of its force field respectively, ρ_P is the Planck mass density.

The 1st step in deriving the Schrödinger equation is the choice of the symmetrization scheme for the particle energy, E_P. Equation (40) can be written in the equivalent form:

$$e=\frac{1}{2}\hat{u}\cdot\hat{u}^*+\frac{3}{2}c^2\sigma\cdot\sigma^*-c^2\hat{\phi}\cdot\hat{\phi}^*,\tag{41}$$

upon comparing with the system (30) we separate the E_P and E_V terms in integral formula

$$E_p+E_V=\rho_P\int_\Omega\left(\frac{1}{2}\hat{u}\cdot\hat{u}^*+\frac{3}{2}c^2\sigma\cdot\sigma^*-c^2\hat{\phi}\cdot\hat{\phi}^*\right)\mathrm{d}x\Leftarrow\begin{cases}E_p=\dfrac{1}{2}\rho_P\displaystyle\int_\Omega\left(\hat{u}\cdot\hat{u}^*+3c^2\sigma\cdot\sigma^*\right)\mathrm{d}x,\\ E_V=\rho_P\displaystyle\int_\Omega\left(-c^2\hat{\phi}\cdot\hat{\phi}^*\right)\mathrm{d}x.\end{cases}\tag{42}$$

The mass of the particle, $m=E_p/c^2$, follows from the particle wave energy in (42)

$$m=\frac{1}{2}\rho_P\int_\Omega\left(3\sigma\cdot\sigma^*+\frac{\hat{u}\cdot\hat{u}^*}{c^2}\right)\mathrm{d}x.\tag{43}$$

The terms $3\,\sigma\cdot\sigma^*$ and $\hat{u}\cdot\hat{u}^*/c^2$ oscillate and depend on the time and position. The symmetry in (43) allows normalizing the deformation and mass velocity with respect to the overall particle mass:

$$\begin{aligned}&\int_\Omega\frac{3\rho_P}{m}\sigma\cdot\sigma^*\,\mathrm{d}x=\int_\Omega\psi\cdot\psi^*\,\mathrm{d}x=1,\ \text{ where }\ \psi=\sqrt{\frac{3\rho_P}{m}}\,\sigma,\\ &\int_\Omega\frac{\rho_P}{mc^2}\hat{u}\cdot\hat{u}^*\,\mathrm{d}x=\int_\Omega\psi\cdot\psi^*\,\mathrm{d}x=1,\ \text{ where }\ \psi=\sqrt{\frac{\rho_P}{m}}\,\frac{\hat{u}}{c}.\end{aligned}\tag{44}$$

The quaternionic particle mass density ψ can be called the quaternionic probability because the relation $\int_\Omega\psi\cdot\psi^*\,\mathrm{d}x=1$ in (44) is satisfied. Obviously, terms $\psi=\sqrt{3\rho_P/m}\,\sigma(t,x)$ and $\psi\cdot\psi^*$, vary in time.

We analyze the evolution of the wave as in relations (42) and (43) in the time-invariant potential field, e.g., the particle wave in the field generated by other particles. The overall particle energy is now a sum of the ground and excess energy *Q*,

$$.\,E=E_p+Q=\int_\Omega\left(\frac{3}{2}\rho_P c^2\sigma\cdot\sigma^*+\frac{1}{2}\rho_P\hat{u}\cdot\hat{u}^*+V(x)\psi\cdot\psi^*\right)\mathrm{d}x.\tag{45}$$

We consider the low excess energies, and the impact of Q on the overall particle mass in (43) is marginal. Thus, the relation (45) becomes

$$E = E_p + Q = \int_\Omega \left(\frac{1}{2} m c^2 \psi \cdot \psi^* + \frac{1}{2} \rho_P \hat{\dot{u}} \cdot \hat{\dot{u}}^* + V(x) \psi \cdot \psi^* \right) \mathrm{d}x$$
$$= \frac{1}{2} m c^2 + \int_\Omega \left(\frac{1}{2} \rho_P \hat{\dot{u}} \cdot \hat{\dot{u}}^* + V(x) \psi \cdot \psi^* \right) \mathrm{d}x. \tag{46}$$

Both the E_p and m are constant; thus, it is enough to minimize the relation

$$Q = \int_\Omega \left(\frac{1}{2} \rho_P \, \hat{\dot{u}} \cdot \hat{\dot{u}}^* + V(x) \psi \cdot \psi^* \right) \mathrm{d}x . \tag{47}$$

The above relation contains two unknowns: $\hat{\dot{u}} = \partial \hat{u} / \partial t$ and ψ. By relating the local lattice velocity $\hat{\dot{u}}$ to the force, specifically to the normalized Cauchy–Riemann derivative of the deformation: $l_P D\sigma$, one gets

$$\hat{\dot{u}} = \frac{\hat{p}}{m} = -\frac{\hbar}{m} D\sigma . \tag{48}$$

By introducing (48) and the normalization (44), the relation (47) becomes the functional

$$Q[\psi] = \int_\Omega \left(\frac{\hbar^2}{6m} (D\psi) \cdot (D\psi)^* + V(x) \, \psi \cdot \psi^* \right) \mathrm{d}x . \tag{49}$$

The functional $Q[\psi]$, Eq. (49), was minimized with respect to a quaternion function, such that ψ satisfies the normalization introduced in the relation (44). We follow the schema used in [14]. In simple terms, we seek a differential equation that has to be satisfied by the ψ function to minimize the energies allowed by (49). Given the functional (49) and the constraint $\mathrm{div}\hat{\phi} = 0$, the conditional extreme is found using the Lagrange coefficients method and the Du Bois Reymond variational lemma [19]. In such a case, ψ satisfies the time-invariant Schrödinger equation satisfied by the particle wave in the ground state of the energy E

$$-\frac{\mathrm{h}^2}{2m} \Delta \psi + V(x) \, \psi = \lambda \psi , \tag{50}$$

where a constant factor on the right-hand side can be considered as extra energy of the particle in the presence of the field $V = V(x)$. For $E = \lambda$, Equation (50) is clearly the time-independent Schrödinger equation satisfied by the particle in the ground state of the energy E,

$$-\frac{\mathrm{h}^2}{2m} \Delta \psi + V(x) \, \psi = E\psi \; . \tag{51}$$

It has to be satisfied together with the condition

$$\mathrm{div}\hat{\psi} = 0 \;\text{ where }\; \psi = \psi_0 + \hat{\psi} . \tag{52}$$

Upon using the NIST data [20] for the Planck's natural units m_P, l_P, t_p and the light velocity c, the constant $\hbar$ introduced in relation (48) equals the Planck constant [16].
The particle mass center, equals its wave energy center. The "space-localized" particle is defined in the sense given by the Bodurov definition [21]: "*A singularity-free multi-component function* $\sigma = (\sigma_0, \phi_1, \phi_2, \phi_3) \in \S$ *of the space* $x = (x_1, x_2, x_3)$ *and time t variables will be called space-localized if* $\|\sigma(t,x)\| \to 0$ *sufficiently fast when* $\|x\| \to \infty$, *so that its Hermitean norm*

$$\langle \sigma, \sigma^* \rangle = \int_\Omega \left(\sigma_0^2 + \sum_{l=1}^{3} \phi_l \cdot \phi_l^* \right) \mathrm{d}x = \int_\Omega \sigma \cdot \sigma^* \mathrm{d}x < \infty \tag{53}$$

remains finite for all time."

4. The First-Order PDE in the P-KC

The operator quantum mechanics base on the complex number algebra, the matrices, and the matrix algebra. Canonical quantization starts from classical mechanics and assumes that the point particle is described by a "probabilistic wave function". Dirac applied complex combinations of the displacements and velocities in the linear problem of secondary quantization [22] and replaced the second-order Klein–Gordon equation by an array of first-order equations. He also recognized the problem of medium for the transmission of waves: *"It is necessary to set up an action principle and to get a Hamiltonian formulation of the equations suitable for quantization purposes, and for this the aether velocity is required"* [23].

In the earlier work [15] we have not separated the Planck and the particle time scales in quaternionic oscillator $G_0(m)$, i.e., both the Planck and the particle frequencies were running oscillator. In the following we derive the proper formula of the quaternionic oscillator $G_\lambda(m)$ for the 1st order PDES and the separated time scales. We base on the concept of the medium as a solid "aether" [16] and implement the quaternion algebra [14]. The 2nd order particle wave equations in QQM, e.g., in the system (31), contains two characteristic terms:

$$\begin{array}{ccccc} \left(\dfrac{\partial^2}{\partial t^2}-3c^2\Delta\right)\sigma & + & 2G_0(m)\hat{\phi} & = & 0, \\ \begin{bmatrix} 2^{\text{nd}}\text{ order wave term } \sigma^\mu\sigma_\mu\text{: variable } \sigma \\ \text{and constant wave velocity } c_L=\sqrt{3}\,c \end{bmatrix} & + & \begin{bmatrix} \text{Propagator with oscillator } G_0(m) \\ \text{that runs at two frequencies} \end{bmatrix} & = & 0. \end{array} \quad (54)$$

We will comply with above schema for the 1st order PDES:

$$\begin{array}{ccccc} \left(\dfrac{\partial}{\partial t}-\sqrt{3}c\,D\right)\dfrac{\hat{u}}{c} & + & 2G_\lambda(m)\dfrac{\hat{u}}{c} & = & 0, \\ \begin{bmatrix} 1^{\text{st}}\text{ order wave term } \sigma_\mu\text{: variable } \hat{u}/c, \\ \text{constant wave velocity } c_L=\sqrt{3}\,c \end{bmatrix} & + & \begin{bmatrix} \text{Propagator with oscillator } G_\lambda(m) \\ \text{that runs at particle wave frequency} \end{bmatrix} & = & 0. \end{array} \quad (55)$$

4.1. The 1st order wave term.

We consider the system (30) and the relation between the wave velocity and the Cauchy–Riemann derivative, $\mathrm{D}\sigma=-\frac{m}{h}\dot{\hat{u}}$. The expression for the overall particle energy, Equation (42), implies:

- the displacement velocity as the alternative variable:

$$\frac{\dot{\hat{u}}}{c}=-\frac{h}{mc}\mathrm{D}\sigma, \quad (56)$$

- the longitudinal wave velocity as the wave propagation velocity:

$$c_L=\sqrt{3}\,c. \quad (57)$$

The stable particle is considered, thus its wave is at a quasi-steady state. The 2nd order time derivative of the q-potential in (55) we express as follows:

$$\frac{\partial^2\sigma}{\partial t^2}=\frac{\partial}{\partial t}\cdot\left(\frac{\partial\sigma}{\partial t}\right). \quad (58)$$

The term in the bracket on the right-hand side is the rate of the q-potential changes. We want to express this term by new variable, i.e., separate the time scales. The rate of changes of the deformation potential $\partial\sigma/\partial t$ is due to the wave propagation within the particle space. The propagation process must follow the extremum principle, i.e., it is the brachistochrone problem [24]. The good example of "local principle" approximation is by Derbes [25].

We know that wave path fulfills the extremum principle, i.e., the wave path follows its unique trajectory given by the Cauchy–Riemann derivative $\mathrm{D}\sigma$. The trajectory which has the minimum property globally in the whole volume Ω occupied by the particle must have the same property locally. This path grants the shortest possible travelling time for the waves identified in QQM. Consequently from (57) we postulate the following:

$$\begin{cases} c_L = \sqrt{3}c, \\ \mathrm{D}\sigma = -\dfrac{mc}{\hbar}\dfrac{\hat{u}.}{c}, \end{cases} \Rightarrow \quad \frac{\partial\sigma}{\partial t} = c_L D\sigma = \sqrt{3}\,\frac{mc^2}{\hbar}\frac{\hat{\hat{u}}}{c}\ . \tag{59}$$

From the relation (56) we get

$$\mathrm{D}\sigma = -\frac{m}{\hbar}\hat{\hat{u}} \quad \Rightarrow \quad \Delta\sigma = -DD\sigma = \frac{mc}{\hbar}D\frac{\hat{\hat{u}}}{c}. \tag{60}$$

Combining the relations (59) and (60), we get the 1st order particle wave term consistent with the 2nd order formula (54):

$$\frac{\partial^2\sigma}{\partial t^2} - 3c^2\Delta\sigma \Leftrightarrow \left(\sqrt{3}\,\frac{mc^2}{h}\frac{\partial}{\partial t} - \sqrt{3}\,\frac{mc^2}{h}D\right)\frac{\dot{\hat{u}}}{c} = \sqrt{3}\,\frac{mc^2}{h}\left(\frac{\partial}{\partial t} - cD\right)\frac{\dot{\hat{u}}}{c}. \tag{61}$$

Thus, the 1st order particle wave term in (55) equals:

$$\left(\frac{\partial}{\partial t} - D\right)\frac{\dot{\hat{u}}}{c} = 0. \tag{62}$$

4.2. The 1st order quaternionic oscillator.

The power of the 2nd order quaternionic oscillator, Equation (28), follows from two time scales in PK-C, namely from the relations (24) and (27): $G_0(f) = 8\pi^2 f_P f$ and $f = mc^2/(2\pi\hbar)$. Combining the relations (24), (27) and removing the Planck frequency results in the power formula of the 1st order quaternionic oscillator when the particle mass is known:

$$G_\lambda(m) = 4\pi f = 2\frac{mc^2}{h} = 2\frac{m}{m_P t_P}. \tag{63}$$

By introducing the relations (62) and (63) in the schema (55), the 1st order PDE for electron equals

$$\left(\frac{\partial}{\partial t} - cD\right)\frac{\dot{\hat{u}}}{c} - 2\frac{m}{m_P t_P}\frac{\hat{u}}{c} = 0. \qquad \left(\frac{\partial}{\partial t} - cD\right)\frac{\dot{\hat{u}}}{c} - 2\frac{m}{m_P t_P}\frac{\hat{u}}{c} = 0. \tag{64}$$

Relation (44), $\psi = \sqrt{\rho_P/m}\,\hat{u}/c$, implies that by multiplying the particle wave equation (64) by $\psi = \sqrt{\rho_P/m}$, it will be expressed as a function of probability

$$\left(\frac{\partial}{\partial t} - cD\right)\psi - 2\frac{m}{m_P t_P}\psi = 0 \tag{65}$$

or

$$\left(\frac{1}{c}\frac{\partial}{\partial t} - D\right)\psi - 2\frac{m}{m_P l_P}\psi = \left(\partial_\mu - 2\frac{m}{m_P l_P}\right)\psi = 0. \tag{66}$$

Equation (66) may require the σ_0 time dependence. This dependence results from the continuity equation presented in [15]. The comparison of the first-order wave equations in quaternion formulation, Equation (66), with the form in the Dirac algebra formalism:

$$\begin{aligned} &\text{Dirac:} && \left(i\gamma^\mu\partial_\mu - \frac{m}{m_P l_P}\right)\psi(t,x) = 0 \ \text{ where } \gamma^\mu\partial_\mu = \frac{1}{c}\frac{\partial}{\partial t} + \alpha_1\frac{\partial}{\partial x} + \alpha_2\frac{\partial}{\partial y} + \alpha_3\frac{\partial}{\partial z}, \\ &\text{Quaternion:} && \left(\partial_\mu - \frac{m}{m_P l_P}\right)\psi(t,x) = 0. \end{aligned} \tag{67}$$

5. Conclusions

The new results of the QQM and QFT make firmer the concept of the P-KC. The fine tuning of our model allowed obtaining new results and next targets:

- The symmetrical formula of the scalar force field: $nc^2\Delta\sigma_0 + G_0(m)\sigma\cdot\sigma^* = 0$, is consistent with the scalar coupling between transverse and longitudinal waves in [13] and [14].

- The quaternion, $G_0(m)\left(\sigma\cdot\sigma^{*}+\hat{\phi}\right)$, scalar, $G_0(m)\sigma\cdot\sigma^{*}$, and vectorial, $G_0(m)\hat{\phi}$ propagators are postulated and used to generate the 2nd order partial differential equation systems, PDES's, for the proton, electron and the neutron
- The scrupulous assessment of the 2nd order PDE systems allows postulating the two 2nd order PDES for the *u* and *d* quarks from the *up* and *down* groups.
- It was verified that both the proton and the neutron obey experimental findings and are formed by three quarks. Namely, the proton and neutron are formed by $d-u-u$ and $d-d-u$ complexes, respectively. All above systems comply with Cauchy equation of motion (8) and can be considered as stable particles.
- The *u* and *d* quarks do not meet the relations of the Cauchy equation of motion. Also experimental efforts to find the individual quarks were without success. Observed were the bound states of the three quarks – the baryons and a quark and an antiquark – the mesons. Wilczek calls it the phenomenological paradox: *"Quarks are Born Free, but Everywhere They are in Chains"* [26]. The inconsistency of the quarks PDES with the quaternion forms of the Cauchy equation of motion might account for the observed *Quarks Chains.*
- The gravitational waves propagate at the velocity of the transverse wave in Cauchy continuum, c.
- The electron wave propagate at the velocity of the longitudinal wave in Cauchy continuum, √3c.

The results indicate the following targets for an immediate future:

- The particles and quarks in the case of higher coupling coefficients: $n>|1|$.
- The ratios between the constants for the different force fields.
- The rigorous derivation of the 1st order PDES's basing on the extremum principle.
- The multivalued coordinate transformation to determine the properties of space with curvature and torsion produced by 2nd order PDES's of the QFT [27].

Acknowledgements: The ideas reported here were developed during several discussions with Chantal Roth. Her criticism, corrections of errors and, occasional enthusiastic acceptance of the features of earlier versions were essential in the present QQM formulation. I owe her my profound thanks (MD).

Abbreviations

P-KC	Planck-Kleinert crysta
PDE	partial differential equation
PDES	partial differential equation systems
QQM	quaternion quantum mechanics
QFT	Quaternion field theory
D	deformation tensor
λ_L,μ_L	Lamé coefficients;
σ',σ''	stress tensors
ρ_E	density of the deformation energy
$\mathbf{u}(u_1,u_2,u_3)$	displacement in $\sim^{3}$
$\sigma(\sigma_0,\phi_1,\phi_2,\phi_3)$	*q-potential* in $\sim^{4}$, the quaternion deformation potential
$\sigma^{*}\cdot\sigma$	strain energy density
G_0	power of the quaternionic oscillator

$G_0\,\sigma^*\cdot\sigma$	density of the rate of momentum change, i.e., the quaternionic scalar propagator
$G_0\,\hat{\phi}$	quaternionic vector propagator
$G_0(m)\left(\sigma\cdot\sigma^*+\hat{\phi}\right)$	quaternionic q-potential propagator
$\psi=\sigma\sqrt{\rho_P/m}$	quaternionic particle density, i.e., the particle wave function
$\psi\cdot\psi^*$	probability, i.e., the normalized particle mass density
n	coupling coefficient in the propagator
l_P	Planck length
$f_P=1/t_P$	Planck frequency, inverse of the Planck time
m_P	Planck mass
$c=l_P/t_P$	transverse wave velocity in elastic continuum
$c_L=\sqrt{3}\,c$	longitudinal wave velocity in elastic continuum
$\rho_P=4m_P/l_P^3$	Planck density, i.e., the mass density of the PK-C
ρ	mass density of the particle $\rho=\rho_E/c^2$, as the equivalent of the energy density ρ_E in the PK-C
$\hbar$	Planck constant in terms of angular frequency
h	Planck constant, $h=2\pi\hbar$
m	equivalent mass of the wave, i.e., mass of the particle
λ	length of the particle wave
f	frequency of the particle wave

References

1 Lanczos, C. Die Funktionentheoretischen Beziehungen der Maxwellschen Æthergleichungen Ein Beitrag zur Relativitäts und Elektronentheorie; In C. Lanczos Collected Published Papers with Commentaries; Davis, W.R., Chu, M.T., Dolan, P., McCornell, J.R., Norris, L.K., Ortiz, E., Plemmons, R.J., Ridgeway, D., Scaife, B.K.P., Stewart, W.J., et al. Eds.; *North Carolina State University*: *Raleigh*, CA, USA, **1998** *Volume VI* , pp. A1–A82.

2 Lanczos, C. Electricity as a natural property of Riemanian geometry. *Phys. Rev*. **1932**, *39*, 716–736.

3 Lanczos, C. Die Wellenmechanik als Hamiltonsche Dynamik des Funktionraumes. Eine neue Ableitung der Dirak-schengleich ung (Wave mechanics as Hamiltonian dynamics of function space. A new derivation of Dirac's equation). *Zeits. Phys*. **1933**, *81*, 703–732.

4 Fueter R. *Comm. Math. Helv*., **1934–1935**, *v. B7*, 307–330.

5 Yefremov A. P. *Grav. and Cosmology*, **1996**, v. 2(1), 77–83.

6 Yefremov A. P. Acta Phys. *Hung., Series — Heavy Ion*s, **2000,** *V.11(1–2),* 147–153.

7 Adler S. L. Quaternionic quantum mechanics and Noncommutative dynamics, 1996 arXiv: hep-th/9607008.

8 Christianto V. A new wave quantum relativistic equation from quaternionic representation of Maxwell-Dirac equation as an alternative to Barut-Dirac equation, *Electronic Journal of Theoretical Physics*, **2006**, *v. 3*, no. 12.

9 Harari, H. A schematic model of quarks and leptons, *Physics Letters B*. **86** (1979) 83–86; doi:10.1016/0370-2693(79)90626-9.

10 Adler S. L., Composite leptons and quarks constructed as triply occupied quasiparticles in quaternionic quantum mechanics, *Phys. Let. B* **332** (*1994)* 358-365; arXiv: hep-th/9404134.

11 Horwitz, L.P. and Biedenharn, L.C. , *Ann. Phys*. **157** (*1984*) 432.

12 Adler, S.L., Quaternionic Quantum Mechanics and Quantum Fields (Oxford Univ. Press 1995).

13 Danielewski, M. and Sapa, L. Nonlinear Klein–Gordon equation in Cauchy–Navier elastic solid. *Cherkasy Univ. Bull. Phys. Math. Sci*. **2017**, *1*, 22–29.

14 Danielewski, M. and L. Sapa, Foundations of the Quaternion Quantum Mechanics, *Entropy* **22** (*2020*) 1424; DOI:10.3390/e22121424.

15 Danielewski, M., Sapa, L. and Roth, Ch. Quaternion Quantum Mechanics II: Resolving the Problems of Gravity and Imaginary Numbers, Symmetry **15** (**2023)** 1672; https://doi.org/10.3390/sym15091672; https://www.mdpi.com/journal/symmetry

16 M. Danielewski, The Planck-Kleinert Crystal, Z. Naturforsch. **62a**, 564-568 (2007).

17 M. P. Marder, Condensed Matter Physics (John Wiley & Sons, NY 2000), pp. 287-303.

18 Feynman, R., Polar and axial vectors, Feynman Lectures in Physics, Vol. 1, §52-5; pp.52-56.

19 Zeidler, E. *Nonlinear Functional Analysis and Its Applications II/A: Linear Monotone Operators*; Springer: New York, USA, 1990; p. 18.

20 National Institute of Standards and Technology, Available online: http://physics.nist.gov (accessed on Nov 10th 2018).

21 Bodurov, T. Generalized Ehrenfest Theorem for Nonlinear Schrödinger Equations, *Int. J. Theor. Phys.* 1988, *37*, 1299–1306, doi:10.1023/A:1026632006040

22 Dirac, P.A.M. Is there an aether? *Nature* **1952**, *169*, 702.

23 Snoswell, M. Personal communications, 2022.

24 Reid, C., Hilbert (Springer-Verlag, Berlin 1969) p. 68.

25 Derbes, D., Feynman's derivation of the Schrödinger equation, *Am. J. Phys*, **64** (*1996*) 881-884.

26 Wilczek, F., Nobel Lecture: Asymptotic freedom: From paradox to paradigm, *Rev. of Modern Physics*, **77**, *(2005*) 857-870.

27 H. Kleinert, Gauge Fields in Condensed Matter, Vol. II, Stresses and Defects, (World Scientific, Singapore, 1989) http://www.physik.fu-berlin.de/~kleinert.