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We demonstrate that an evanescent wave spin exists outside a finite quantum well by solving the Dirac equation in a finite cylindrical quantum well. The analytical analysis validates the wavefunction inside an infinite quantum well but recovers a non-zero evanescent wave outside the well. We propose that it is possible to probe or eavesdrop on quantum spin information through the evanescent wave spin without destroying the entire spin state. We argue that a spin-based quantum process or device is deterministic rather than probabilistic.

I. ELECTRON WAVE SPIN

The electron spin has attracted significant attention in recent years due to its potential applications in information sciences and technologies, such as quantum computing [1],[2] and spintronics [3]. If future computers rely on the precise manipulation of the electron spin, it is crucial to develop a deeper understanding of this property beyond a simplistic interpretation as a superluminal spinning particle.

In recent works [4],[5], it has been argued that the electron spin is a wave property that can be fully described by current densities calculated using the Dirac theorem. The authors demonstrate the existence of a stable circulating current density for an electron confined in an infinite quantum well, which exhibits a multi-vortex topology at excited states. The electron wave spin exhibits geometric and topological characteristics when it interacts with an electromagnetic field, leading to various effects, including fractional spin and modified Zeeman splitting.

It is noted that the electron wave spin is investigated in the infinite quantum well, where the wave function outside the well is typically assumed to be zero [6]. Although this model has helped gain much insight into many quantum effects inside the quantum well, the zero wavefunction assumption outside the well is too strong to satisfy the boundary condition for the Dirac wavefunction [7]. In reality, all potentials are finite, and therefore, there exists a non-negligible wavefunction outside the well, even when the eigenenergy of the electron is below the quantum well potential. The wavefunction outside the well shall decay rapidly, thus is evanescent in analogy to the evanescent waves observed in optics, but not vanish even at infinite potentials.

An intriguing question arises as to whether the evanescent electron wave also spins. An intuitive answer to this question suggests that an evanescent wave spin exists due to the continuation of wavefunctions and the prevailing algebraic structure of the Dirac equation. The existence of an evanescent wave spin shall raise intriguing possibilities for probing the spin from outside the well without collapsing the spin inside.

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In this paper, we plan to demonstrate the evanescent wave spin by obtaining analytical expressions for the wavefunctions and current densities in all regions of a Dirac electron in a finite quantum well. Moreover, we will demonstrate that the evanescent wave diminishes at high potentials but remains within the skin depth region. This behavior ensures that the charge and current continuation are maintained at the boundary. Finally, we intend to discuss the implications of these findings on quantum information technologies.

II. DIRAC ELECTRON IN A FINITE QUANTUM WELL

We choose to study a finite cylindrical well due to the inherent circulating behavior of the wave spin. Analytical bi-spinor wavefunctions are obtained for both inside and outside the well.

The Dirac equation in the cylindrical coordinate is

$$\frac{1}{c}\frac{\partial}{\partial t}\psi(\boldsymbol{r},t) = \{-\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} - i\frac{mc^2}{\hbar c}\gamma^0 - i\frac{1}{\hbar c}U(\boldsymbol{r})\}\psi(\boldsymbol{r},t), \quad (1)$$

where (ρ, ϕ, z) represent polar, azimuthal angle and z coordinate, respectively. The operator

$$\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} = \alpha_\rho\frac{\partial}{\partial\rho} + \alpha_\phi\frac{1}{\rho}\frac{\partial}{\partial\phi} + \alpha_z\frac{\partial}{\partial z} \quad (2)$$

contains $\alpha-$matrix in cylindrical coordinate

$$\alpha_\rho = \begin{pmatrix} 0 & 0 & 0 & e^{-i\phi} \\ 0 & 0 & e^{i\phi} & 0 \\ 0 & e^{-i\phi} & 0 & 0 \\ e^{i\phi} & 0 & 0 & 0 \end{pmatrix};$$

$$\alpha_\phi = \begin{pmatrix} 0 & 0 & 0 & -ie^{-i\phi} \\ 0 & 0 & ie^{i\phi} & 0 \\ 0 & -ie^{-i\phi} & 0 & 0 \\ ie^{i\phi} & 0 & 0 & 0 \end{pmatrix};$$

$$\alpha_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad (3)$$

with the following properties

$$\begin{aligned}&\sigma_\rho^2 = \sigma_\phi^2 = \sigma_z^2 = 1,\\ &\sigma_\rho\sigma_\phi = i\alpha_z,\\ &\sigma_\rho\sigma_\phi + \sigma_\phi\sigma_\rho = 0.\end{aligned} \tag{4}$$

We now let the potential

$$U(\boldsymbol{r}) = U(\rho) = \begin{cases} 0, & \rho < R \\ U, & \rho > R \end{cases} \tag{5}$$

represent a finite cylindrical quantum well of potential U and radius R. The wavefunction in the quantum well can be expressed by the separation of variables

$$\psi(\boldsymbol{r}, t) = e^{-i\mathcal{E}t/\hbar} e^{iP_z z/\hbar} \tilde{\psi}(\phi, \rho), \tag{6}$$

where $\mathcal{E}$ is the eigenenergy to be determined by the boundary conditions and the momentum along z-direction is set $P_z = 0$ for our discussion.

Plugging Eq. 6 into Eq. 1 to obtain the equation

$$\{-i\frac{\mathcal{E}}{\hbar c} + \alpha_\rho \frac{\partial}{\partial\rho} + \alpha_\phi \frac{1}{\rho}\frac{\partial}{\partial\phi} + i\frac{mc^2}{\hbar c}\gamma^0 + i\frac{1}{\hbar c}U(\rho)\}\tilde{\psi}(\phi, \rho) = 0. \tag{7}$$

$\tilde{\psi}(\phi, \rho)$ is a four-spinor that can be written as

$$\tilde{\psi}(\phi, \rho) = \begin{pmatrix} \mu_A(\phi, \rho) \\ \mu_B(\phi, \rho) \end{pmatrix}, \tag{8}$$

where $\mu_A(\phi, \rho)$ and $\mu_B(\phi, \rho)$ are the large and small components of the Dirac wavefunctions that follow the equations

$$\begin{aligned}&-i\frac{\mathcal{E} - U(\rho) - mc^2}{\hbar c}\mu_A(\phi, \rho) = \\ &\begin{pmatrix} 0 & e^{-i\phi}\frac{\partial}{\partial\rho} - ie^{-i\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} \\ e^{i\phi}\frac{\partial}{\partial\rho} + ie^{i\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} & 0 \end{pmatrix}\mu_B(\phi, \rho);\\ &-i\frac{\mathcal{E} - U(\rho) + mc^2}{\hbar c}\mu_B(\phi, \rho) = \\ &\begin{pmatrix} 0 & e^{-i\phi}\frac{\partial}{\partial\rho} - ie^{-i\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} \\ e^{i\phi}\frac{\partial}{\partial\rho} + ie^{i\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} & 0 \end{pmatrix}\mu_A(\phi, \rho).\end{aligned} \tag{9}$$

Eqs. 9 are used to solve for the two-spinor wavefunctions $\mu_A(\phi, \rho)$ and $\mu_B(\phi, \rho)$ both inside and outside the well. The four-spinor wavefunction $\tilde{\psi}(\phi, \rho)$ for a spin up electron is then obtained by using Eq. 8

$$\tilde{\psi}(\phi, \rho) = \begin{cases} e^{il\phi}\begin{pmatrix} J_l(\zeta\rho) \\ 0 \\ 0 \\ ie^{i\phi}\frac{\hbar c}{\mathcal{E}+mc^2}\{\frac{1}{2}\zeta\left[J_{l-1}(\zeta\rho) - J_{l+1}(\zeta\rho)\right] - \frac{l}{\rho}J_l(\zeta\rho)\} \end{pmatrix}, & \rho \le R; \\ e^{il\phi}\begin{pmatrix} \kappa K_l(\zeta\rho) \\ 0 \\ 0 \\ ie^{i\phi}\kappa\frac{\hbar c}{\mathcal{E}-U+mc^2}\{\frac{1}{2}\xi\left[-K_{l-1}(\xi\rho) - K_{l+1}(\xi\rho)\right] - \frac{l}{\rho}K_l(\xi\rho)\} \end{pmatrix}, & \rho > R, \end{cases} \tag{10}$$

where J_l and K_l are the Bessel function and modified Bessel functions of order l, respectively. l is the azimuthal quantum number. The constant κ measures the relative magnitude between the wavefunctions inside and outside the quantum well. The wave numbers ζ and ξ are linked to the eigenenergy $\mathcal{E}$ by

$$\begin{aligned}\zeta &= \sqrt{\frac{\mathcal{E}^2 - m^2c^4}{c^2\hbar^2}},\\ \xi &= \sqrt{\frac{m^2c^4 - (\mathcal{E} - U)^2}{\hbar^2c^2}},\end{aligned} \tag{11}$$

where

$$\mathcal{E} - mc^2 < U < mc^2. \tag{12}$$

We now apply the boundary condition of wavefunction continuation at $\rho = R$ to obtain

$$\begin{aligned}&\kappa K_l(\xi R) = J_l(\zeta R)\\ &\kappa\{\frac{1}{2}\xi\left[-K_{l-1}(\xi\rho) - K_{l+1}(\xi\rho)\right] - \frac{l}{\rho}K_l(\xi\rho)\} = \\ &\frac{\mathcal{E} - U + mc^2}{\mathcal{E} + mc^2}\{\frac{1}{2}\zeta\left[J_{l-1}(\zeta\rho) - J_{l+1}(\zeta\rho)\right] - \frac{l}{\rho}J_l(\zeta\rho)\},\end{aligned} \tag{13}$$

from which the eigenenergies $\mathcal{E}_{ln}$ and constant κ can be solved. Here l and n denote the azimuthal and radial quantum numbers, respectively.

Eqs. 10,11,13 are used for numerical calculation and discussion of wave properties in all regions.

III. EVANESCENT WAVE SPIN

To simplify the discussion on the evanescent wave spin, we choose the lowest azimuthal quantum number $l = 0$ to obtain the wavefunction

$$\tilde{\psi}(\phi,\rho) = \begin{cases} \begin{pmatrix} J_0(\zeta\rho) \\ 0 \\ 0 \\ -ie^{i\phi}\frac{\hbar c}{\mathcal{E}+mc^2}\frac{1}{2}\zeta J_1(\zeta\rho) \end{pmatrix}, & \rho \leq R; \\ \begin{pmatrix} \kappa K_0(\zeta\rho) \\ 0 \\ 0 \\ -i\kappa e^{i\phi}\frac{\hbar c}{\mathcal{E}-U+mc^2}\frac{1}{2}\xi K_1(\xi\rho) \end{pmatrix}, & \rho > R. \end{cases} \tag{14}$$

The boundary conditions become

$$\begin{aligned} \kappa K_0(\xi R) &= J_0(\zeta R) \\ \kappa\xi K_1(\xi R) &= \frac{\mathcal{E}-U+mc^2}{\mathcal{E}+mc^2}\zeta J_1(\zeta R), \end{aligned} \tag{15}$$

from which the eigenenergies $\mathcal{E}_{0n}$ and constant κ are solved. We will calculate the lowest state $\mathcal{E}_{01}$ for our discussion.

As an example, we select a quantum well of radius $R = 10$ nm and a series of potentials $U = 0.01, 0.1, 1, 10$ eV. We then calculate the lowest eigenenergy subtracting the rest energy $\mathcal{E}_{01} - mc^2$ and κ by solving Eq. 15. The results are listed in Table I.

Table I shows that at $U = 0.01$ eV, $\mathcal{E}_{01} - mc^2 = 1.53$ (meV) $< U = 10$ (meV), the electron wave tunnels out of the well to be evanescent. Fig. 1 plots the wavefunctions inside and outside the well. Substantial evanescent wave is shown outside the well that satisfies wavefunction continuity at the boundary.

The spin nature of the evanescent wave can be demonstrated by the current density, which is obtained from Eq. 14

$$j_\rho = ec\psi^\dagger\alpha_\rho\psi = 0, everywhere;$$

$$j_\phi = ec\psi^\dagger\alpha_\phi\psi = \begin{cases} \frac{e\hbar c^2}{\mathcal{E}+\mathrm{m}c^2}\zeta J_0(\zeta\rho)J_1(\zeta\rho), & \rho \leq R \\ \kappa^2\frac{e\hbar c^2}{\mathcal{E}-U+\mathrm{m}c^2}\xi K_0(\xi\rho)K_1(\xi\rho), & \rho > R. \end{cases} \tag{16}$$

TABLE I. Eigenenergy $\mathcal{E}_{01}$ and κ_{01}

U (eV)	$\mathcal{E}_{01} - mc^2$(meV)	κ_{01}	$\zeta_{01}(m^{-1})$	$\xi_{01}(m^{-1})$
0.01	1.53	44.1	2.00×10^8	4.71×10^8
0.10	1.95	2.23×10^6	2.26×10^8	1.60×10^9
1.00	2.12	2.32×10^{21}	2.36×10^8	5.12×10^9
10.0	2.18	1.75×10^{69}	2.39×10^8	1.62×10^{10}

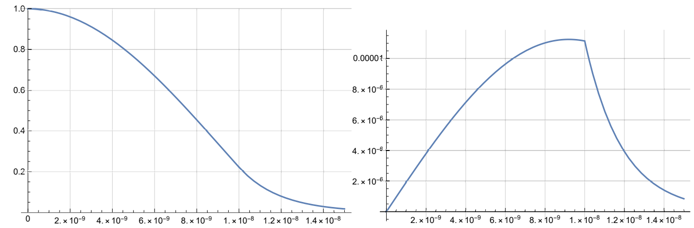

FIG. 1. Plots of large component $\mu_A(\phi,\rho)$ (left) and small component $\mu_B(\phi,\rho)$ (right) wavefunctions (blue) inside a finite cylindrical quantum well of radius $R = 10$ nm and potential $U = 0.01$ eV show that substantial wavefunctions tunnel out of the well but maintain the wavefunction continuity at the boundary.

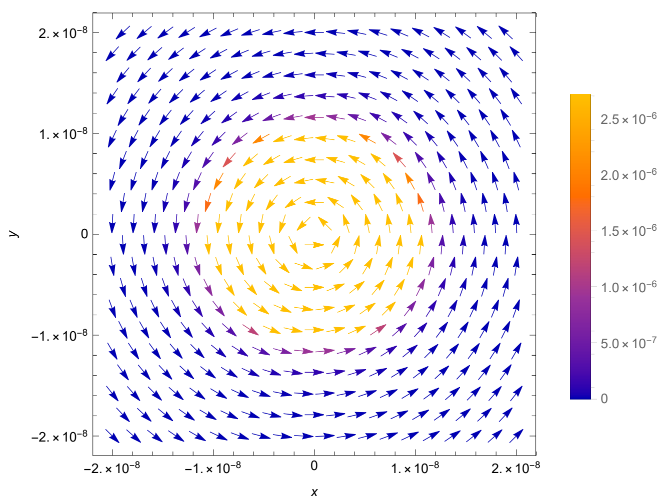

FIG. 2. Vector plot of the current density of a spin-up Dirac electron of eigenenergy $\mathcal{E}_{01}$ in a finite quantum well of $R = 10$ nm and $U = 0.01$ eV. The evanescent wave spins in the same way as the wave spins inside the quantum well.

The result shows that the current density circulates both inside and outside the quantum well, indicated by the sole surviving component j_ϕ in all regions. The evanescent wave spins in the same way as the wave spins inside the quantum well, as illustrated by the vector plot of Fig. 2.

IV. EVANESCENT WAVE SPIN AT INFINITE POTENTIALS

We now analyse evanescent wave spin in high and infinite potentials by examining the wavefunctions that determine the current density.

Table I shows that at $U = 10$ eV, the wave number $\zeta_{01} = 2.39 \times 10^8$ already approaches the wave number for the infinite quantum well, $\zeta_{01}^{\mathrm{inf}} = 2.40 \times 10^8$ from

$$J_0(\zeta_{01}^{\mathrm{inf}}R) = 0, \tag{17}$$

where zero wavefunction is normally assumed outside the well.

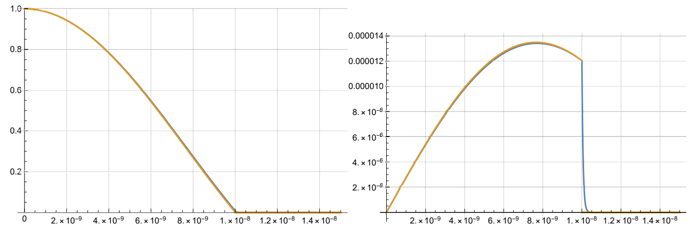

FIG. 3. Plots of large component $\mu_A(\phi,\rho)$ (left) and small component $\mu_B(\phi,\rho)$ (right) wavefunctions (blue) inside a finite cylindrical quantum well of radius $R = 10$ nm and potential $U = 10$ eV show that they are nearly identical to the wavefunctions (orange) inside an infinite quantum well. The wavefunctions outside the quantum well are diminished except for a sharp drop-off of $\mu_B(\phi,\rho)$ to maintain the wavefunction continuity at the boundary.

Therefore, the wavefunctions $\mu_A(\phi,\rho)$ and $\mu_B(\phi,\rho)$ inside the quantum well are nearly the same as the wavefunctions in the infinite quantum well, as illustrated by Fig. 3. However, the wavefunctions outside the quantum well remain non-zero within a narrow region near the boundary known as the skin depth, but decay rapidly. The skin depth becomes narrower as the potential becomes higher, where the evanescent wave resides and spins as expressed by Eq. 16.

The above analysis validates the infinite quantum well model to account for the quantum behavior of the electron inside the well. However, it also reveals that the evanescent wave diminishes but never vanishes to mathematical zero, contrary to the usual assumption. This behavior circumvents the discontinuity problem at the boundary of the infinite quantum well.

V. DISCUSSION ON EVANESCENT WAVE SPIN PROBING

The existence of the evanescent spin wave raises the possibility of coupling wave spin with an electromagnetic field outside the quantum well, analogous to evanescent optical wave sensing [8]. As discussed in our previous paper, fractional spins can be observed through partial wave spin interaction with a confined electromagnetic field. When the field is weak, the wavefunction is perturbed but remains intact according to perturbation theory. Consequently, it is possible to probe or eavesdrop on a spin state inside a quantum well without destroying it through evanescent wave spin coupling with an external field.

This picture conflicts with the particle spin interpretation, where the particle electron carrying the full spin tunnels out of the quantum well with a probability density given by the square of the wavefunction $\psi^\dagger\psi$. If the spin is detected outside the quantum well, the spin information inside is destroyed since the electron that carries the full spin no longer exists within the quantum well.

The conflicting pictures are illustrated in Fig. 4, which comprises two figures for the electron in the same quantum well of $R = 10$ nm and $U = 0.01$ eV.

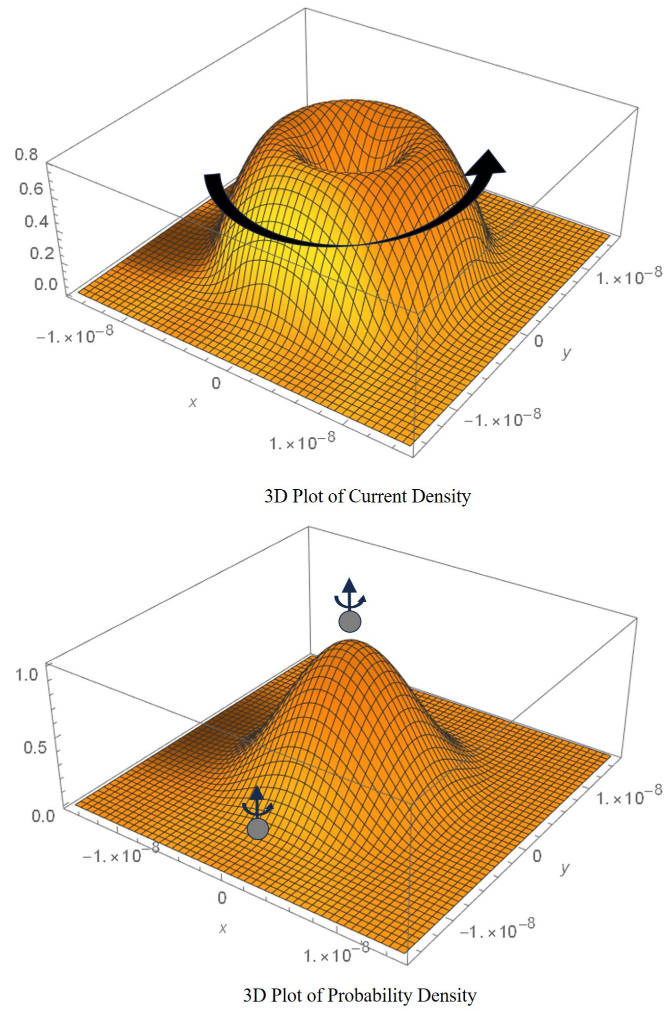

FIG. 4. Upper figure shows the three-dimension distribution of the current density that spins as a whole in all regions for the spin-up electron in a finite quantum well of $R = 10$ nm and $U = 0.01$ eV. The figure illustrates that partial wave spin exists outside the quantum well with full certainty.

Lower figure shows the probability density $\psi^\dagger\psi$ of the particle electron of spin-up (represented by the ball and arrow) in the same quantum well. The figure illustrates that full particle spin tunnels out of the quantum well with partial certainty.

The upper figure illustrates the wave spin picture by showing the spinning current density in all regions. The current density resembles the current density inside an infinite square well discussed previously [4]. Here, partial wave spin exists outside the quantum well but with full certainty.

The lower figure depicts the particle spin picture by showing the probability density and particle electron spin. Here, full particle spin tunnels out of the quantum well but with partial certainty.

Resolving the above conflicting views has significant implications for emerging quantum technologies, such as quantum computing, which is generally regarded as a probabilistic process due to the probability interpretation of the particle electron. However, it is important to note that the wavefunction, which provides the probability distribution $\psi^\dagger\psi$, is itself deterministic as it is a vector in Hilbert space. Concurrently, the charge distribution $e\psi^\dagger\psi$ and current density $ec\psi^\dagger\alpha\psi$ are also deterministic. Consequently, the wave spin is deterministic since it can be fully described by the current density. This suggests that a spin-based quantum process or device is deterministic rather than probabilistic.

VI. CONCLUSIONS

1. We demonstrate that an evanescent wave spin exists outside a finite quantum well by solving the Dirac equation in a finite cylindrical quantum well.

2. The analytical analysis validates the wavefunction inside an infinite quantum well but recovers a non-zero evanescent wave outside the well.

3. We propose that it is possible to probe or eavesdrop on quantum spin information through the evanescent wave spin without destroying the entire spin state.

4. We argue that a spin-based quantum process or device is deterministic rather than probabilistic.

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