Qeios

Constructing a Set of Kronecker-Pauli Matrices

Christian RAKOTONIRINA

Institut Supérieur de Technologie d'Antananarivo, IST-T
and
Laboratoire de la Dynamique de l'Atmosphère, du Climat et des Océans, DyACO,
Université d'Antananarivo,
Madagascar

E-mail address: rakotonirinachristianpierre@gmail.com
https://orcid.org/0000-0003-0123-4813

Abstract. In quantum physics, the choice of basis is crucial for formulation. The generalization of the Pauli matrices via Kronecker product, called Kronecker-Pauli matrices, is typically restricted to for 2^n dimensional systems. This paper explores extending this generalization to N-dimensional systems, where N is a prime integer, in order to construct $N\times N$-Kronecker-Pauli matrices. We begin by examining the specific cases of 3×3 and 5×5 Kronecker-Pauli matrices, with the goal of the purpose constructing a set of $N\times N$-Kronecker-Pauli matrices for any prime integer N.

Keywords:Pauli matrices, Kronecker-Pauli matrices, Swap operator, unitary basis, qutrit, qudit, Weyl operator basis.

1. Introduction

In quantum physics the choice of basis for formulation is important. For higher-level system, it is normal the generalization of the Pauli matrices by tensor or Kronecker product

$$(\sigma_{j_1} \otimes \sigma_{j_2} \otimes \dots \sigma_{j_n})_{j_1,j_2,\dots,j_n=0,1,2,3}$$

where σ_0 is the 2×2-unit matrix and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices.

However the generalization in this sense applies only to 2^n level systems. These matrices are referred to as $2^n \times 2^n$-KPMs (Kronecker-Pauli matrices). The work in [1] extended some of these matrices' properties [2] to three-dimensional systems, leading to the 3×3-KPMs and offering a path to generalizing this for any dimension. In this road to generalization it is demonstrated that the tensor product of two sets of KPMs is a set of KPMs.

The set $\mathcal{K}_3$ of 3×3-KPMs which are not traceless, even up to phases factor, does not form a group which excludes it from being considered as a Pauli group. Neverthless, the set of traceless matrices $\mathcal{K}_3 \times \mathcal{K}_3 \otimes \left\{1, \omega, \omega^2\right\} = \tau\mathcal{K}_3 \otimes \left\{1, \omega, \omega^2\right\}$, for $\tau \in \mathcal{K}_3$ with $\omega = e^{\frac{2i\pi}{3}}$ and $\omega^2 = e^{\frac{4i\pi}{3}}$ forms a group. This group corresponds to the Weyl-Heisenberg group for the three-dimensional case, according to its definition in [3].

https://doi.org/10.32388/F5DGRH.2

The objective of this paper is to demonstrate that the method for constructing a set of 5×5-KPMs described in [1] can be extended to produce a set of $N\times N$-KPMs for any prime integer N. In other words, we aim to define a set of N^2 matrices that satisfy the following properties.

Definition 1. *We define the set of $N \times N$-KPMs as a set of $N \times N$-matrices, denoted as $\mathcal{K_N} = (\Sigma_i)_{0\leq i\leq N^2-1}$ that satisfy the following conditions:*

i) $S_{N\otimes N} = \frac{1}{N}\sum_{i=0}^{N^2-1}\Sigma_i\otimes\Sigma_i$ *(the swap operator relation);*
ii) $\Sigma_i^\dagger = \Sigma_i$, *for any* $i \in \{0,\ 1,\ \ldots,\ N^2-1\}$ *(hermiticity);*
iii) $\Sigma_i^2 = I_N$, *for any* $i \in \{0,\ 1,\ \ldots,\ N^2-1\}$ *(square root of unit);*
iv) $Tr\left(\Sigma_j^\dagger\Sigma_k\right) = N\delta_{jk}$ *for any* $j,\ k \in \{0,\ 1,\ \ldots,\ N^2-1\}$ *(orthogonality).*

In this definition, an analogous of the relationship i) of the swap operator or tensor commutation matrix with the KPMs is satisfied by the generalized Gell-Mann matrices and the unit matrix [4].
The KPMs are hermitians and according to ii) and iii), they are unitaries. Like the set of the generalized Gell-Mann matrices and the identity, the $N \times N$-KPMs are generators of the unitary group $U(N)$.
It is straightforward to show that for $\Sigma \in \mathcal{K_N}$,

i) the basis $\Sigma\mathcal{K_N} = (\Sigma\Sigma_i)_{0\leq i\leq N^2-1}$ is an unitary basis, containing the identity matrix $\mathbb{I}$, and all elements, except the identity, are traceless
ii) the elements of $\Sigma\mathcal{K_N}$ are mutually orthogonals.

Thus, the elements of $\Sigma\mathcal{K_N}$ satisfy the general properties required to be a matrix basis which is used for the Bloch vector decomposition of qudits [5].
As unitary matrices, the KPMs could serve as quantum gates in 1-qudit quantum circuit. For instance, three gates are defined as elementary gates [6, 7] for 1-qutrit quantum circuit:

$$X^{(01)} = \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\ X^{(02)} = \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix},\ X^{(12)} = \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}.$$

These are among the 3×3-KPMs.

Given these considerations, it is clear that $N \times N$-KPMs merit further study. The paper is organized as the following. In the second section we will give the definition of what we call an inverse-symmetric matrices and some of their properties, which we need for constructing a set of KPMs. In the third section a study of the 3×3-KPMs is given in comparing them with the matrices of the Weyl operator basis. The fourth section is for the way of constructing a set of KPMs, in starting at first the 5-dimensional case.

2. Inverse-symmetric Matrices

To construct Kronecker-Pauli matrices, we first introduce the concept of inverse-symmetric matrices and some of their properties [1].

Definition 2. *Let us call inverse-symmetric an invertible complex matrix $A = \left(A_j^i\right)$ such that*

$$A_i^j = \frac{1}{A_j^i} \quad \textit{if}\ A_j^i \neq 0$$

If a permutation matrix is symmetric, then it is inverse-symmetric.

Proposition 1. *The Kronecker product of two inverse-symmetric matrices is itself inverse-symmetric.*

Proposition 2. *For any $n \times n$ inverse-symmetric matrix A, with only n non zero elements, $A^2 = I_n$.*

3. 3×3-Kronecker-Pauli matrices

3.1. **Weyl Operator Basis.** In this subsection, we present what is Weyl operator basis (See for example, [8–10]) in order to show its relationship with the 3×3-KPMs in the case of 3-dimension, the qutrit case.

Definition 3. *The following d^2 operators*

$$U_{nm} = \sum_{k=0}^{d-1} e^{\frac{2i\pi}{d}kn} |\, k\rangle \langle (k+m) mod\, d| \ , \quad n, m = 1, , 2, \dots, d-1$$

are called Weyl operators.

For the case of 3-dimension the matrices of the Weyl operators are the following

$$U_{00} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, U_{01} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, U_{02} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$U_{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, U_{11} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{pmatrix}, U_{12} = \begin{pmatrix} 0 & 0 & 1 \\ \omega & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix},$$
$$U_{20} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, U_{21} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega^2 \\ \omega & 0 & 0 \end{pmatrix}, U_{22} = \begin{pmatrix} 0 & 0 & 1 \\ \omega^2 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}.$$

3.2. 3×3**-KPMs and the Weyl Operator Basis.** In this subsection we compare the 3×3-Kronecker-Pauli matrices, formed by the cubic roots of unit that are inverse-symmetric matrices

$$\tau_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & \omega^2 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega^2 \\ 0 & \omega & 0 \end{pmatrix},$$
$$\tau_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \tau_5 = \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix}, \quad \tau_6 = \begin{pmatrix} 0 & 0 & \omega^2 \\ 0 & 1 & 0 \\ \omega & 0 & 0 \end{pmatrix},$$
$$\tau_7 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau_8 = \begin{pmatrix} 0 & \omega & 0 \\ \omega^2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau_9 = \begin{pmatrix} 0 & \omega^2 & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

with the matrices of the Weyl operator basis.

For τ_k, $k = 1, 2, \dots, 9$, we can check that the set $\tau_k \mathcal{K}_3$ contains the unit matrix and is equal to the set of the matrices in the Weyl operator basis up to the phases $\omega = e^{\frac{2i\pi}{3}}$ and $\omega^2 = e^{\frac{4i\pi}{3}}$. For example ($k = 1$),

$\tau_1\tau_1 = U_{00},\ \tau_1\tau_2 = U_{20},\ \tau_1\tau_3 = U_{10},\ \tau_1\tau_4 = U_{02},\ \tau_1\tau_5 = \omega U_{12},\ \tau_1\tau_6 = \omega U_{22}$

$\tau_1\tau_7 = U_{01},\ \tau_1\tau_8 = \omega^2 U_{11},\ \tau_1\tau_9 = \omega^2 U_{21}$

4. Constructing the $N \times N$-Kronecker Pauli matrices for N prime integer

Proposition 3. *If $\mathcal{K}_{\mathcal{N}}$ and $\mathcal{K}_{\mathcal{M}}$ are sets of KPMs, then $\mathcal{K}_{\mathcal{N}} \otimes \mathcal{K}_{\mathcal{M}}$ is also a set of KPMs.*

According to this proposition, it remains for us to construct the $N \times N$-KPMs, for N prime integer. After that, we will, by Kronecker product, have set of $n \times n$-KPMs for any integer n.

Example 1. *To construct a set of 5×5-KPMs, we begin by decomposing 5×5-ones matrix into the sum of five symmetric permutation matrices, each having only one entry of unit ("1") in the diagonal, as the follows:*

$$\begin{pmatrix} 1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1 \end{pmatrix} = \begin{pmatrix} 1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0 \end{pmatrix} + \begin{pmatrix} 0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0 \end{pmatrix} + \begin{pmatrix} 0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0 \end{pmatrix} + \begin{pmatrix} 0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0 \end{pmatrix} + \begin{pmatrix} 0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1 \end{pmatrix}$$

Then, for each term of this sum by replacing the "ones" by the quintic roots of unit 1, $\eta = e^{\frac{2i\pi}{5}}$, $\eta^2 = e^{\frac{4i\pi}{5}}$, $\eta^3 = e^{\frac{6i\pi}{5}}$, $\eta^4 = e^{\frac{8i\pi}{5}}$, in keeping the only unit in the diagonal and in keeping that the matrices are inverse-symmetrics, we have additional four matrices i.e five matrices with the matrix taken from the sum. Thus, we have twenty five inverse-symmmetric matrices.

$$\chi_1 = \begin{pmatrix} 1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 1&0&0&0&0\\0&0&0&0&\eta\\0&0&0&\eta^2&0\\0&0&\eta^3&0&0\\0&\eta^4&0&0&0 \end{pmatrix}, \quad \chi_3 = \begin{pmatrix} 1&0&0&0&0\\0&0&0&0&\eta^2\\0&0&0&\eta^4&0\\0&0&\eta&0&0\\0&\eta^3&0&0&0 \end{pmatrix},$$

$$\chi_4 = \begin{pmatrix} 1&0&0&0&0\\0&0&0&0&\eta^3\\0&0&0&\eta&0\\0&0&\eta^4&0&0\\0&\eta^2&0&0&0 \end{pmatrix}, \quad \chi_5 = \begin{pmatrix} 1&0&0&0&0\\0&0&0&0&\eta^4\\0&0&0&\eta^3&0\\0&0&\eta^2&0&0\\0&\eta&0&0&0 \end{pmatrix},$$

$$\chi_6 = \begin{pmatrix} 0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0 \end{pmatrix}, \quad \chi_7 = \begin{pmatrix} 0&\eta&0&0&0\\\eta^4&0&0&0&0\\0&0&0&0&\eta^2\\0&0&0&1&0\\0&0&\eta^3&0&0 \end{pmatrix}, \quad \chi_8 = \begin{pmatrix} 0&\eta^2&0&0&0\\\eta^3&0&0&0&0\\0&0&0&0&\eta\\0&0&0&1&0\\0&0&\eta^4&0&0 \end{pmatrix},$$

$$\chi_9 = \begin{pmatrix} 0&\eta^3&0&0&0\\\eta^2&0&0&0&0\\0&0&0&0&\eta^4\\0&0&0&1&0\\0&0&\eta&0&0 \end{pmatrix}, \quad \chi_{10} = \begin{pmatrix} 0&\eta^4&0&0&0\\\eta&0&0&0&0\\0&0&0&0&\eta^3\\0&0&0&1&0\\0&0&\eta^2&0&0 \end{pmatrix},$$

$$\chi_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \chi_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & \eta^2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \eta^3 & 0 & 0 & 0 \\ \eta^4 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \chi_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 0 & 0 & \eta & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \eta^4 & 0 & 0 & 0 \\ \eta^3 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\chi_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 & \eta^3 \\ 0 & 0 & 0 & \eta^4 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \\ \eta^2 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \chi_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 & \eta^4 \\ 0 & 0 & 0 & \eta^3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \eta^2 & 0 & 0 & 0 \\ \eta & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\chi_{16} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \ \chi_{17} = \begin{pmatrix} 0 & 0 & \eta & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \eta^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 0 & 0 & \eta^3 & 0 \end{pmatrix}, \ \chi_{18} = \begin{pmatrix} 0 & 0 & \eta^2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \eta^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & \eta^4 & 0 \end{pmatrix},$$

$$\chi_{19} = \begin{pmatrix} 0 & 0 & \eta^3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \eta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^4 \\ 0 & 0 & 0 & \eta & 0 \end{pmatrix}, \ \chi_{20} = \begin{pmatrix} 0 & 0 & \eta^4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^3 \\ 0 & 0 & 0 & \eta^2 & 0 \end{pmatrix},$$

$$\chi_{21} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \ \chi_{22} = \begin{pmatrix} 0 & 0 & 0 & \eta & 0 \\ 0 & 0 & \eta^2 & 0 & 0 \\ 0 & \eta^3 & 0 & 0 & 0 \\ \eta^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \ \chi_{23} = \begin{pmatrix} 0 & 0 & 0 & \eta^2 & 0 \\ 0 & 0 & \eta & 0 & 0 \\ 0 & \eta^4 & 0 & 0 & 0 \\ \eta^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\chi_{24} = \begin{pmatrix} 0 & 0 & 0 & \eta^3 & 0 \\ 0 & 0 & \eta^4 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \\ \eta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \ \chi_{25} = \begin{pmatrix} 0 & 0 & 0 & \eta^4 & 0 \\ 0 & 0 & \eta^3 & 0 & 0 \\ 0 & \eta^2 & 0 & 0 & 0 \\ \eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have verified, with the help of SCILAB software [1], that the set of these twenty five matrices constitute a set of 5×5-KPMs. But we can have another decomposition of the 5×5-ones matrix as sum of five symmetric permutation matrices with only one unit in the diagonal, namely

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

If we replace the "ones" in these symmetric permutations matrices by the five quintic roots of unit, but in keeping that they are inverse symmetrics, it is obvious that the properties ii) and iii) of hermiticity and of square root of unit are satisfied by the twenty five obtained matrices. It remains for us to study the properties i)

and iv) of the relationship with the swap operator and the orthogonality. But to study it, let us consider the general case for any N dimension, with N a prime integer. Instead of studying the relationship between matrices let us study the relationship between operators whose matrices with respect to the standard basis $(\mid 0\rangle, \mid 1\rangle, \ldots, \mid N^2-1\rangle)$ are the matrices in question.
The following lemma for the swap operator $S_{N\otimes N}$ should be helpful for this study.

Lemma 1.

$$S_{N\otimes N} = \sum_{(i,j)} \mid i\rangle \langle j \mid \otimes \mid j\rangle \langle i \mid$$

In order to make the presentation of the following theorem more shorter we take that the matrices of the operators are their matrices with respect to the standard basis $(\mid 0\rangle, \mid 1\rangle, \ldots, \mid N^2-1\rangle)$.

Theorem 1. *Let P_1, P_2, ..., P_N be $N\times N$ operators whose matrices are symmetric permutation matrices with only one unit in the diagonal.*
$\Sigma_0 = P_1$ and Σ_1, Σ_2, ..., Σ_{N-1} are operators whose matrices are obtained in replacing the "ones" in $\Sigma_0 = P_1$ by the N-th roots of unit in keeping that the they are inverse-symetrics. We do the same to the operators P_2, ..., P_N in order to have the operators
$\Sigma_N = P_2$ and Σ_{N+1}, Σ_{N+2}, ..., Σ_{2N-1}
. . . . .
$\Sigma_{N^2-N} = P_N$ and Σ_{N^2-N+1}, Σ_{N^2-N+2}, ..., Σ_{N^2-1}
whose matrices are inverse-symmetrics.
If

(1) *the sum $P_1 + P_2 + ..., +P_N$ is equal to the operator whose matrices is the $N \times N$ ones matrix and*
(2) *for any $l \in \{0, 1, \ldots, N-1\}$, for any $k, j \in \{lN+1, \ldots, lN+N-1\}$, for any two places in a $N\times N$-matrix, non symmetrics with respect to the diagonal if the elements of Σ_k in these two places are $e^{\frac{2i\pi p_k}{N}}$ and $e^{\frac{2i\pi r_k}{N}}$, the elements of Σ_j in these two places are $e^{\frac{2i\pi p_j}{N}}$ and $e^{\frac{2i\pi r_j}{N}}$,*

$$e^{\frac{2i\pi(r_k+p_k)}{N}} \neq e^{\frac{2i\pi(r_j+p_j)}{N}}$$

then

$$S_{N\otimes N} = \frac{1}{N}\sum_{j=0}^{N^2-1} \Sigma_j \otimes \Sigma_j \quad and \quad Tr\left(\Sigma_j^\dagger \Sigma_k\right) = N\delta_{jk}$$

Proof. Let us take the operator Σ_j, with $j \in \{0, 1, 2, \ldots, N-1\}$. Σ_j can be decomposed as sum of elementary operators, a non zero term in this sum is of the form

$$e^{\frac{2i\pi p}{N}} \mid k\rangle \langle l \mid$$

with $p \in \{0, 1, 2, \ldots, N-1\}$. Thus a non zero term of the sum giving $\Sigma_j \otimes \Sigma_j$ is of the form

$$e^{\frac{2i\pi(p+r)}{N}} \mid k\rangle \langle l \mid \otimes \mid m\rangle \langle n \mid$$

with $r \in \{0, 1, 2, \ldots, N-1\}$. If $k = n$ and $l = m$, then

$$e^{\frac{2i\pi(p+r)}{N}} \mid k\rangle \langle l \mid \otimes \mid m\rangle \langle n \mid = \mid k\rangle \langle l \mid \otimes \mid l\rangle \langle k \mid$$

due to the inverse-symmetries. A non zero term of the sum $\sum_{j=0}^{N-1} \Sigma_j \otimes \Sigma_j$ is

$$\sum_{j=0}^{N-1} e^{\frac{2i\pi(p+r)}{N}} |\, k\rangle \langle l \,| \otimes |\, m\rangle \langle n \,| = N |\, k\rangle \langle l \,| \otimes |\, l\rangle \langle k \,|$$

If $k \neq n$ or $l \neq m$,
for the case where $k = l$, then $p = 0$, because only one unit on the diagonal, and

$$e^{\frac{2i\pi(p+r)}{N}} |\, k\rangle \langle l \,| \otimes |\, m\rangle \langle n \,| = e^{\frac{2i\pi r}{N}} |\, k\rangle \langle l \,| \otimes |\, m\rangle \langle n \,|$$

and in the sum $\sum_{j=0}^{N-1} \Sigma_j \otimes \Sigma_j$, there is the following sum

$$\sum_{r=0}^{N-1} e^{\frac{2i\pi r}{N}} |\, k\rangle \langle l \,| \otimes |\, m\rangle \langle n \,| = 0$$

according to the hypothesis (2) and as the sum of the five quintic roots of unit is equal to zero.
For the case where $m = n$, then $r = 0$

$$e^{\frac{2i\pi(p+r)}{N}} |\, k\rangle \langle l \,| \otimes |\, m\rangle \langle n \,| = e^{\frac{2i\pi p}{N}} |\, k\rangle \langle l \,| \otimes |\, m\rangle \langle n \,|$$

and in the sum $\sum_{j=0}^{N^2-1} \Sigma_j \otimes \Sigma_j$, there is the following sum

$$\sum_{p=0}^{N-1} e^{\frac{2i\pi p}{N}} |\, k\rangle \langle l \,| \otimes |\, m\rangle \langle n \,| = 0$$

for the case where $k \neq l$ and $m \neq n$, a term of the sum giving $\Sigma_j \otimes \Sigma_j$ is of the form

$$e^{\frac{2i\pi(p_j+r_j)}{N}} |\, k\rangle \langle l \,| \otimes |\, m\rangle \langle n \,|$$

then in the sum $\sum_{j=0}^{N-1} \Sigma_j \otimes \Sigma_j$, there is the following sum

$$\sum_{j=0}^{N-1} e^{\frac{2i\pi(p_j+r_j)}{N}} |\, k\rangle \langle l \,| \otimes |\, m\rangle \langle n \,|$$

which equal to the null operator 0, according to the hypothesis (2) of the theorem. Then we can conclude that only the elementary operator non null operators in the decomposition of the sum $\sum_{j=0}^{N^2-1} \Sigma_j \otimes \Sigma_j$ are the elementary operators of the form $N |\, k\rangle \langle l \,| \otimes |\, l\rangle \langle k \,|$. Hence, According to the lemma the first part of the conclusion is demonstrated.
Now, let us move on to the second part. For $j, k \in \{0, 1, 2, \ldots, N-1\}$ with $j \neq k$, for $p \neq r$, $p, r \in \{0, 1, 2, \ldots, N\}$, the operator $\Sigma_j \Sigma_k$ contains two terms $e^{\frac{2i\pi}{N}(p_j+p_k)} |\, p\rangle \langle p \,|$ and $e^{\frac{2i\pi}{N}(r_j+r_k)} |\, r\rangle \langle r \,|$, with $e^{\frac{2i\pi}{N} p_j}$ and $e^{\frac{2i\pi}{N} j}$ are elements of the matrix of Σ_j. Suppose

$$e^{\frac{2i\pi}{N}(p_j+p_k)} = e^{\frac{2i\pi}{N}(r_j+r_k)}$$

Then,

$$e^{\frac{2i\pi}{N}(p_j-r_j)} = e^{\frac{2i\pi}{N}(r_k-p_k)}$$

However, the elements $e^{\frac{2i\pi}{N} p_j}$ and $e^{\frac{2i\pi}{N}(-r_j)}$ of the matrix of Σ_j are respectively in the same places as the elements $e^{\frac{2i\pi}{N}(-p_k)}$ and $e^{\frac{2i\pi}{N} r_k}$ of the matrix of Σ_k. That is in contradiction with the hypothesis (2) of the theorem. Thus, the diagonal of the matrix of $\Sigma_j^{\dagger} \Sigma_k$ is formed by the N-th roots of units. Hence, for $j \neq k$,

$\mathrm{Tr}\left(\Sigma_j^\dagger \Sigma_k\right) = 0.$

For $l_1,\ l_2 \in \{0,\ 1,\ \ldots, N-1\}$, with $l_1 \neq l_2$, for $j \in \{l_1 N + 1,\ \ldots, l_1 N + N - 1\}$, $k \in \{l_2 N + 1,\ \ldots, l_2 N + N - 1\}$ it is obvious that all elements in the diagonal of $\Sigma_j^\dagger \Sigma_k$ are equals to zero. Thus, $\mathrm{Tr}\left(\Sigma_j^\dagger \Sigma_k\right) = 0.$ □

We can remark that the theorem help us how to build a set of $N \times N$-Kronecker-Pauli matrices, for a prime integer N. Let us take as an example the continuation of the construction of 5×5-KPMs above.

Example 2. *Let us take one by one the permutation matrices, terms of the decomposition of the 5×5-ones matrix above. For each term, we add four inverse-symmetric matrices obtained in replacing the five units by the quintic roots of unit, in keeping that they are inverse-symmetrics, but according to the hypothesis (2) of the theorem above.*

$$\Sigma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},\ \Sigma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta & 0 & 0 \\ 0 & \eta^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 0 & 0 & \eta^3 & 0 \end{pmatrix},$$

$$\Sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta^2 & 0 & 0 \\ 0 & \eta^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^4 \\ 0 & 0 & 0 & \eta & 0 \end{pmatrix},\ \Sigma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta^3 & 0 & 0 \\ 0 & \eta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & \eta^4 & 0 \end{pmatrix},\ \Sigma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta^4 & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^3 \\ 0 & 0 & 0 & \eta^2 & 0 \end{pmatrix}$$

. . . . . .

Counter-example 1. $$\Sigma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},\ \Sigma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta & 0 & 0 \\ 0 & \eta^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta^2 \\ 0 & 0 & 0 & \eta^3 & 0 \end{pmatrix},$$

$$\Sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta^2 & 0 & 0 \\ 0 & \eta^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & \eta^4 & 0 \end{pmatrix}, \ldots$$

are forbidden to be elements of a set of 5×5-KPMs, because they does not satisfy the hypothesis (2) of the theorem above, even though they are inverse-symmetrics. Actually, the property iv) of the definition of a set of KPMs is not satisfied because $\mathrm{Tr}\left(\Sigma_2^\dagger \Sigma_3\right) \neq 0.$

5. Conclusion

In conclusion, we have shown that for any given element of the 3×3-Kronecker-Pauli matrices, its products with other elements generate a basis equivalent to the Weyl operator basis, up to phase factors $\omega = e^{\frac{2i\pi}{3}}$ and $\omega^2 = e^{\frac{4i\pi}{3}}$. For any prime integer N, we have demonstrated a method for constructing a set of $N \times N$-Kronecker-Pauli matrices. Our study of the $N = 5$ case indicates that the set of $N \times N$-Kronecker-Pauli matrices is not unique.

References

[1] Rakotonirina C. and Rasamizafy S., 3×3-*Kronecker-Pauli Matrices* , Bull. Math. & Stat.Res, Vol.4.Issue3., 2016.

[2] Rakotonirina C. and Rakotondramavonirina J., *Tensor Commutation Matrices and Some Generalizations of the Pauli Matrices*, Gen. Math. Notes, Vol. 23, No. 1, July 2014, pp.1-8, 2014.

[3] Bengtsson I. Życzkowski K., *Geometry of Quantum States, An Introduction to Quantum Entanglement*, Cambridge University Press, pp 313-315, 2017.

[4] Rakotonirina C., *Expression of a tensor commutation matrix in terms of the generalized Gell-Mann matrices*, Int. J. Math. Math. Sci. 2007, 20672 (2007).

[5] R.A. Bertlmann, P. Krammer, *Bloch vectors for qudits*, Journal of Physics A Mathematical General 41(23), 235303 (2008). https://doi.org/10.1088/1751-8113/41/23/235303. arXiv:0806.1174 [quant-ph].

[6] Yao-Min Di and Hai-Rui Wei, *Elementary gates of ternary quantum logic circuit*, arXiv:1105.5485v3 [quant-ph], 2011.

[7] Ankit Khandelwal and M Girish Chandra, *Quantum Image Representation Methods Using Qutrits* , arXiv:2207.09096v2 [quant-ph], 2022.

[8] H. Narnhofer, J. Phys. A: Math. Gen. 39, 7051 (2006).

[9] B. Baumgartner, B. C. Hiesmayr, and H. Narnhofer, Phys. Rev. A 74, 032327 (2006).

[10] B. Baumgartner, B. C. Hiesmayr, and H. Narnhofer, J. Phys. A: Math. Theor. 40, 7919 (2007).