Research Article

# σ-Sets and σ-Antisets

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**In this paper we present a brief study of the $\sigma$-set-$\sigma$-antiset duality that occurs in $\sigma$-set theory and we also present the development of the integer space $3^A = \langle 2^A, 2^{A^-} \rangle$ for the cardinals $|A| = 2, 3$ together with its algebraic properties. In this article, we also develop a presentation of some of the properties of fusion of $\sigma$-sets and finally we present the development and definition of a type of equations of one $\sigma$-set variable.**

## 1. $\sigma$-Sets and $\sigma$-Antisets

As we have seen in[1], an $\sigma$-antiset is defined as follows:

**Definition 1.1.** *Let $A$ be a $\sigma$-set, then $B$ is said to be the $\sigma$-antiset of $A$ if and only if $A \oplus B = \emptyset$, where $\oplus$ is the fusion of $\sigma$-sets.*

We must observe that given the definition of the fusion operator $\oplus$ in[1] it is clear that it is commutative and therefore if $B$ is an $\sigma$-antiset of $A$, then it will be necessary that $A$ is also the $\sigma$-antiset of $B$. On the other hand, following Blizard notation,[2] p. 347, we will denote $B$ the $\sigma$-antiset of $A$ as $B = A^-$, in this way we will have that $A = (A^-)^-$.

Continuing with the development of the $\sigma$-sets we have constructed three primary $\sigma$-sets, which are:

| | |
|---|---|
| **Natural Numbers** | $\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$ |
| $0$-**Natural Numbers** | $\mathbb{N}^0 = \{1_0, 2_0, 3_0, 4_0, 5_0, 6_0, 7_0, 8_0, 9_0, 10_0, \ldots\}$ |
| **Antinatural Numbers** | $\mathbb{N}^- = \{1^*, 2^*, 3^*, 4^*, 5^*, 6^*, 7^*, 8^*, 9^*, 10^*, \ldots\}$ |

where $1 = \{\alpha\}$, $1_0 = \{\emptyset\}$ and $1^* = \{\omega\}$, we must clarify that we have changed the letter $\beta$ for the letter $\omega$ for symmetry reasons, we must also remember that:

$$\ldots \in \alpha_{-2} \in \alpha_{-1} \in \alpha \in \alpha_1 \in \alpha_2 \ldots$$

and

$$\ldots \in \omega_{-2} \in \omega_{-1} \in \omega \in \omega_1 \in \omega_2 \in \ldots$$

where both chains have the linear $\in$-root property and are totally different, i.e. they do not have a link-intersection.

On the other hand, we must remember the definition of the space generated by two $\sigma$-sets $A$ and $B$ which is:

**Definition 1.2.** *Let $A$ and $B$ be $\sigma$-sets. The* ***Generated space*** *by $A$ and $B$ is given by*

$$\langle 2^A, 2^B \rangle = \{x \oplus y : x \in 2^A \wedge y \in 2^B\},$$

*where $\oplus$ is the fusion operator.*

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Let us recall a few things about the fusion operator $\oplus$. To begin this brief analysis, we must observe that given $x, y$ two $\sigma$-sets, if $\{x\} \cup \{y\} = \emptyset$ then it will be said that $y$ is the antielement of $x$ and $x$ the antielement of $y$, where the union of pairs $\cup$ axiomatized within the theory of $\sigma$-sets is used, in particular in the completion axioms A and B, which we will call annihilation axioms from now on.

**Notation 1.3.** *Let $x$ be an element of some $\sigma$-set, then we will denote as $x^*$ the antielement of $x$, in the case that it exists.*

Now we move on to define the new operations with $\sigma$-sets which will help us define the fusion of $\sigma$-sets.

**Definition 1.4.** *Let $A$ and $B$ be $\sigma$-sets, then we define the $*$-intersection of $A$ with $B$ by*

$$A \hat{\cap} B = \{x \in A : x^* \in B\}.$$

**Example 1.5.** *Let $A = \{1, 2, 3^*, 4\}$ and $B = \{2, 3, 4^*\}$, then we have that:*

$$A \hat{\cap} B = \{3^*, 4\}$$

*and*

$$B \hat{\cap} A = \{3, 4^*\},$$

*it is clear that the $*$-intersection operator is not commutative.*

**Theorem 1.6.** *Let $A$ be $\sigma$-set, then $A \hat{\cap} A = \emptyset$.*

*Proof.* Let $A$ be a $\sigma$-set, by definition we will have that

$$A \hat{\cap} A = \{x \in A : x^* \in A\}.$$

Suppose now that $A \hat{\cap} A \neq \emptyset$, then there exists an $x \in A$ such that $x^* \in A$, therefore we will have that $x, x^* \in A$, which is a contradiction with Theorem 3.39 (Exclusion of inverses) from [1], so if $A$ is a $\sigma$-set then

$$A \hat{\cap} A = \emptyset.$$

*Regarding Theorem 1.6, we can observe that given a $\sigma$-set $A$ the $\sigma$-set theory does not allow the coexistence of a $\sigma$-element $x$ and its $\sigma$-antielement in the same $\sigma$-set $A$, and this is because $A$ is a $\sigma$-set. However, since $\sigma$-set theory is a $\sigma$-class theory, one can find the $\sigma$-elements together with the $\sigma$-antielements*


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*coexisting without problems in what we call the proper $\sigma$-class, in this way one will have that $\{x, x^*\}$ is a proper $\sigma$-class and not a $\sigma$-set.*

**Theorem 1.7.** *Let $A$ be $\sigma$-set, then $A\hat{\cap}\emptyset = \emptyset$ and $\emptyset\hat{\cap}A = \emptyset$.*

*Proof.* Let $A$ be a $\sigma$-set, by definition we will have that

$$A\hat{\cap}\emptyset = \{x \in A : x^* \in \emptyset\}.$$

Now suppose that $A\hat{\cap}\emptyset \neq \emptyset$, then there exists an $x \in A$ such that $x^* \in \emptyset$, which is a contradiction, hence $A\hat{\cap}\emptyset = \emptyset$. On the other hand, $\emptyset\hat{\cap}A \subseteq \emptyset$ thus we will have that $\emptyset\hat{\cap}A = \emptyset$. □

On the other hand, we will define the $*$-difference between $\sigma$-sets, a fundamental operation to be able to define the fusion between $\sigma$-sets.

**Definition 1.8.** *Let $A$ and $B$ be $\sigma$-sets, then we define the $*$-difference between $A$ y $B$ by*

$$A \circledast B = A - (A\hat{\cap}B),$$

*where $A - B = \{x \in A : x \notin B\}$.*

**Example 1.9.** *Let $A = \{1, 2, 3^*, 4\}$ and $B = \{2, 3, 4^*\}$, thenwe have that:*

$$A\hat{\cap}B = \{3^*, 4\},$$

*therefore*

$$A \circledast B = A - (A\hat{\cap}B) = \{1, 2, 3^*, 4\} - \{3^*, 4\} = \{1, 2\}$$

$$A \circledast B = \{1, 2\}.$$

*We also have to*

$$B\hat{\cap}A = \{3, 4^*\}$$

*therefore*

$$B \circledast A = B - (B\hat{\cap}A) = \{2, 3, 4^*\} - \{3, 4^*\} = \{2\}$$

$$B \circledast A = \{2\}.$$

**Corollary 1.10.** *Let $A$ be $\sigma$-set. Then $A \circledast A = A$.*

*Proof.* Let $A$ be a $\sigma$-set, then by Theorem 1.6 we will have that $A\hat{\cap}A = \emptyset$ therefore


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$$A \divideontimes A = A - (A \hat{\cap} A) = A - \emptyset = A.$$

□

**Corollary 1.11.** *Let $A$ be $\sigma$-set. Then $A \divideontimes \emptyset = A$ and $\emptyset \divideontimes A = \emptyset$.*

*Proof.* Let $A$ be a $\sigma$-set, then by Theorem 1.7 we will have that $A \hat{\cap} \emptyset = \emptyset \hat{\cap} A = \emptyset$ therefore

$$A \divideontimes \emptyset = A - (A \hat{\cap} \emptyset) = A - \emptyset = A$$

*and*

$$\emptyset \divideontimes A = \emptyset - (\emptyset \hat{\cap} A) = \emptyset - \emptyset = \emptyset.$$

□

Now after defining the $*$-intersection and the $*$-difference we can define the fusion of $\sigma$-sets as follows:

**Definition 1.12.** *Let $A$ and $B$ be $\sigma$-sets, then we define the fusion of $A$ and $B$ by*

$$A \oplus B = \{x : x \in A \divideontimes B \vee x \in B \divideontimes A\}.$$

It is clear that the fusion of $\sigma$-sets is commutative by definition. Now let us show an example

**Example 1.13.** *Let $A = \{1, 2, 3^*, 4\}$ and $B = \{2, 3, 4^*\}$, then we have that:*

$$A \oplus B = \{x : x \in A \divideontimes B \vee x \in B \divideontimes A\},$$

$$A \oplus B = \{x : x \in \{1, 2\} \vee x \in \{2\}\},$$

$$A \oplus B = \{1, 2\},$$

therefore we have that

$$\{1, 2, 3^*, 4\} \oplus \{2, 3, 4^*\} = \{2, 3, 4^*\} \oplus \{1, 2, 3^*, 4\} = \{1, 2\}.$$

**Corollary 1.14.** *Let $A$ be a $\sigma$-set, then $A \oplus A = A$.*

*Proof.* Let $A$ be a $\sigma$-set, by definition we have that,

$$A \oplus A = \{x : x \in A \divideontimes A \vee x \in A \divideontimes A\}.$$

Now by corollary 1.10, we have that

$$A \oplus A = \{x : x \in A \vee x \in A\},$$


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$$A \oplus A = \{x : x \in A\},$$

therefore it is clear that $A \subset A \oplus A$ and that $A \oplus A \subset A$, therefore $A \oplus A = A$. □

**Corollary 1.15.** *Let $A$ be $\sigma$-set, then $A \oplus \emptyset = \emptyset \oplus A = A$.*

*Proof.* First we will show that $A \oplus \emptyset = A$. By definition we will have that,

$$A \oplus \emptyset = \{x : x \in A \divideontimes \emptyset \vee x \in \emptyset \divideontimes A\}.$$

Now by the corollary 1.11, we will have that

$$A \oplus \emptyset = \{x : x \in A \vee x \in \emptyset\},$$

$$A \oplus \emptyset = \{x : x \in A\},$$

from this it is clear that $A \subset A \oplus \emptyset$ and that $A \oplus \emptyset \subset A$, in this way $A \oplus \emptyset = A$.

Second we will show that $\emptyset \oplus A = A$. By definition we will have that,

$$\emptyset \oplus A = \{x : x \in \emptyset \divideontimes A \vee x \in A \divideontimes \emptyset\}.$$

Now by the corollary 1.11, we will have that

$$\emptyset \oplus A = \{x : x \in \emptyset \vee x \in A\},$$

$$A \oplus \emptyset = \{x : x \in A\},$$

from this it is clear that $A \subset \emptyset \oplus A$ and that $\emptyset \oplus A \subset A$, in this way $\emptyset \oplus A = A$. □

**Theorem 1.16.** *Let $X$ be a $\sigma$-set, then for all $A, B \in 2^X$, we have that:*

$$A \oplus B = A \cup B,$$

*where $A \cup B = \{x : x \in A \vee x \in B\}$.*

*Proof.* Let $X$ be a $\sigma$-set and $A, B \in 2^X$. Then, by theorem 3.39 of [1] we have that

$$A \hat{\cap} B = B \hat{\cap} A = \emptyset,$$

in this way

$$A \divideontimes B = A \wedge B \divideontimes A = B.$$

Finally $A \oplus B = \{x : x \in A \vee x \in B\} = A \cup B$ □

**Corollary 1.17.** *Let $X$ be a $\sigma$-set, then for all $A \in 2^X$, we have that:*

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$$A \oplus X = X.$$

*Proof.* Let $X$ be a $\sigma$-set and $A \in 2^X$. Then by theorem 1.16 we have that

$$A \oplus X = A \cup X.$$

Now as $A \subset X$, then $A \cup X = X$, therefore

$$A \oplus X = X.$$

□

As we said before the fusion of $\sigma$-sets $\oplus$ is commutative by definition but as we demonstrated in [1][3][4] this operation is not associative.

**Example 1.18.** *Let* $A = \{1^*, 2^*\}$*,* $B = \{1, 2\}$ *and* $C = \{1\}$*, then*

$$A \oplus B \oplus C = \emptyset \oplus C = C$$

*and*

$$A \oplus B \oplus C = A \oplus B = \emptyset,$$

*therefore we have that*

$$(A \oplus B) \oplus C \neq A \oplus (B \oplus C).$$

## 2. Generated space

As we have already indicated in the definition 1.2 we will have that the space generated by two $\sigma$-sets $A$ and $B$ is:

$$\langle 2^A, 2^B \rangle = \{x \oplus y : x \in 2^A \wedge y \in 2^B\}.$$

Now taking into account the duality $\sigma$-set, $\sigma$-antiset we could consider the following example.

**Example 2.1.** *We consider the* $\sigma$*-set* $A = \{1, 2, 3\}$ *and its* $\sigma$*-antiset* $A^- = \{1^*, 2^*, 3^*\}$ *then we obtain the integer space* $3^A$ *where,*

$$3^A = \langle 2^A, 2^{A^-} \rangle.$$

*Is important to observe that*


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$$2^A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,2\}, \{2,3\}, A\}$$

*and*

$$2^{A^-} = \{\emptyset^-, \{1^*\}, \{2^*\}, \{3^*\}, \{1^*,2^*\}, \{1^*,2^*\}, \{2^*,3^*\}, A^-\}.$$

*Also is important to observe that $\emptyset = \emptyset^-$, which is very important for the construction of $3^A$.*

Now considering the definition of generated space,

$$3^A = \langle 2^A, 2^{A^-} \rangle = \{X \oplus Y : X \in 2^A \wedge Y \in 2^{A^-}\},$$

*where the operator $\oplus$ is the fusion of $\sigma$-sets, we will obtain the following matrix:*

| $\oplus$ | $\emptyset$ | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{2,3\}$ | $A$ |
|---|---|---|---|---|---|---|---|---|
| $\emptyset^-$ | $\emptyset_0^0$ | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{2,3\}$ | $A$ |
| $\{1^*\}$ | $\{1^*\}$ | $\emptyset_1^1$ | $\{1^*,2\}$ | $\{1^*,3\}$ | $\{2\}$ | $\{3\}$ | $\{1^*,2,3\}$ | $\{2,3\}$ |
| $\{2^*\}$ | $\{2^*\}$ | $\{1,2^*\}$ | $\emptyset_1^2$ | $\{2^*,3\}$ | $\{1\}$ | $\{1,2^*,3\}$ | $\{3\}$ | $\{1,3\}$ |
| $\{3^*\}$ | $\{3^*\}$ | $\{1,3^*\}$ | $\{2,3^*\}$ | $\emptyset_1^3$ | $\{1,2,3^*\}$ | $\{1\}$ | $\{2\}$ | $\{1,2\}$ |
| $\{1^*,2^*\}$ | $\{1^*,2^*\}$ | $\{2^*\}$ | $\{1^*\}$ | $\{1^*,2^*,3\}$ | $\emptyset_2^4$ | $\{2^*,3\}$ | $\{1^*,3\}$ | $\{3\}$ |
| $\{1^*,3^*\}$ | $\{1^*,3^*\}$ | $\{3^*\}$ | $\{1^*,2,3^*\}$ | $\{1^*\}$ | $\{2,3^*\}$ | $\emptyset_2^5$ | $\{1^*,3\}$ | $\{2\}$ |
| $\{2^*,3^*\}$ | $\{2^*,3^*\}$ | $\{1,2^*,3^*\}$ | $\{3^*\}$ | $\{2^*\}$ | $\{1,3^*\}$ | $\{1,2^*\}$ | $\emptyset_2^6$ | $\{1\}$ |
| $A^-$ | $A^-$ | $\{2^*,3^*\}$ | $\{1^*,3^*\}$ | $\{1^*,2^*\}$ | $\{3^*\}$ | $\{2^*\}$ | $\{1^*\}$ | $\emptyset_3^7$ |

**Table 1.** Integer Space.

It is important to note that from the perspective of $\sigma$-sets we have that $\emptyset = \emptyset^- = \emptyset^i$ with $i \in \{0,1,2,3,4,5,6,7\}$ and $j \in \{0,1,2,3\}$, where the difference of the $\sigma$-emptysets $\emptyset_j^i$ is given by annihilation, which come from the equation $A \oplus A^- = \emptyset$.

From the matrix representation of the integer space $3^A$, we can present another representation of the same integer space. This representation of the integer space $3^A$ is a graphical representation which we show in figure 1.


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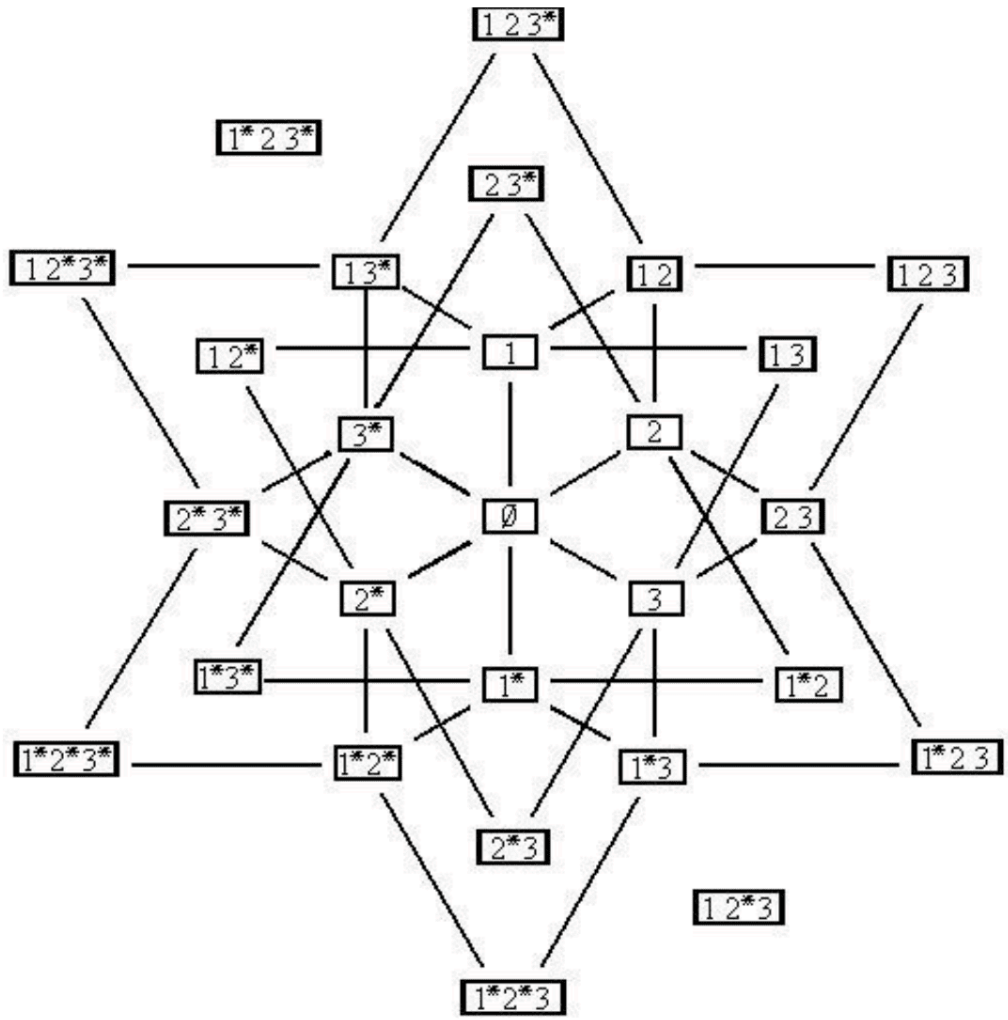


**Figure 1.** Integer Space $3^A$.

Finally as a theoretical result we have a cardinal theorem:

**Theorem 2.2.**

*Let* $A = \{1, 2, 3\}$, *then* $|3^A| = |\langle 2^A, 2^{A^-} \rangle| = 3^3 = 27$.

*Proof.* Let $A = \{1, 2, 3\}$, the proof is the same fusion matrix for this $\sigma$-set. □

We should also note that we have obtained other cardinal results for the integer space $3^A$ with $|A| \in \{0, 1, 2, 3, 4, 5\}$. The cardinal results are the following:


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| $\sigma$-Set | $\sigma$-Antiset | Generated | Cardinal |
|---|---|---|---|
| $A=\emptyset$ | $A^{-}=\emptyset^{-}$ | $\langle 2^{A}, 2^{A^{-}}\rangle$ | $3^{0}=1$ |
| $A=\{1\}$ | $A^{-}=\{1^{*}\}$ | $\langle 2^{A}, 2^{A^{-}}\rangle$ | $3^{1}=3$ |
| $A=\{1,2\}$ | $A^{-}=\{1^{*},2^{*}\}$ | $\langle 2^{A}, 2^{A^{-}}\rangle$ | $3^{2}=9$ |
| $A=\{1,2,3\}$ | $A^{-}=\{1^{*},2^{*},3^{*}\}$ | $\langle 2^{A}, 2^{A^{-}}\rangle$ | $3^{3}=27$ |
| $A=\{1,2,3,4\}$ | $A^{-}=\{1^{*},2^{*},3^{*},4^{*}\}$ | $\langle 2^{A}, 2^{A^{-}}\rangle$ | $3^{4}=81$ |
| $A=\{1,2,3,4,5\}$ | $A^{-}=\{1^{*},2^{*},3^{*},4^{*},5^{*}\}$ | $\langle 2^{A}, 2^{A^{-}}\rangle$ | $3^{5}=243$ |

From these calculations made with the fusion matrix we can obtain the following conjecture.

**Conjecture 2.3.** *Let $A$ be $\sigma$-set such that $|A|=n$, then $|3^{A}|=|\langle 2^{A},2^{A^{-}}\rangle|=3^{n}$.*

On the other hand, as we have already said, we are going to change the notation of $1_{\Theta}$ to $1_{0}$, in this way we will have the set $0$-natural numbers defined as follows:

$1_0=\{\emptyset\}$

$2_0=\{\emptyset,1_0\}$

$3_0=\{\emptyset,1_0,2_0\}$

$4_0=\{\emptyset,1_0,2_0,3_0\}$

and so on, forming the $0$-natural numbers

$$\mathbb{N}^{0}=\{1_0,2_0,3_0,4_0,5_0,6_0,7_0,8_0,9_0,10_0,\ldots\},$$

where one of the important properties of this $\sigma$-set is that it does not annihilate with the natural numbers $\mathbb{N}$ nor with the antinatural numbers $\mathbb{N}^{-}$, in this way we can consider the following example for the generated space.

**Example 2.4.** *We consider the $\sigma$-sets $A=\{1_0,2_0\}$ and $B=\{1,2\}$, therefore the space generated by $A\oplus B$ and $A\oplus B^{-}$ will be:*

$$\langle 2^{A\oplus B},2^{A\oplus B^{-}}\rangle=\{x\oplus y : x\in 2^{A\oplus B}\wedge y\in 2^{A\oplus B^{-}}\}$$

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$$\langle 2^{A\oplus B}, 2^{A\oplus B^-}\rangle = \{\emptyset, \{1_0\}, \{1\}, \{1^*\}, \{2_0\}, \{2\}, \{2^*\}, \{1_0, 2_0\}, \{1_0, 1\}, \{1_0, 1^*\},$$
$$\{1_0, 2\}, \{1_0, 2^*\}, \{2_0, 1\}, \{2_0, 1^*\}, \{2_0, 2\}, \{2_0, 2^*\}, \{1, 2\}, \{1, 2^*\}, \{1^*, 2\}, \{1^*, 2^*\},$$
$$\{1_0, 1, 2\}, \{1_0, 1, 2^*\}, \{1_0, 1^*, 2\}, \{1_0, 1^*, 2^*\}, \{2_0, 1, 2\}, \{2_0, 1, 2^*\}, \{2_0, 1^*, 2\},$$
$$\{2_0, 1^*, 2^*\}, \{1_0, 2_0, 1\}, \{1_0, 2_0, 1^*\}, \{1_0, 2_0, 2\}, \{1_0, 2_0, 2^*\}, \{1_0, 2_0, 1, 2\},$$
$$\{1_0, 2_0, 1, 2^*\}, \{1_0, 2_0, 1^*, 2\}, \{1_0, 2_0, 1^*, 2^*\}\}$$

*In this case the generated space becomes a meta-space generated by $A = \{1_0, 2_0\}$ and $B = \{1, 2\}$ which can be ordered graphically as shown in figure 2.*

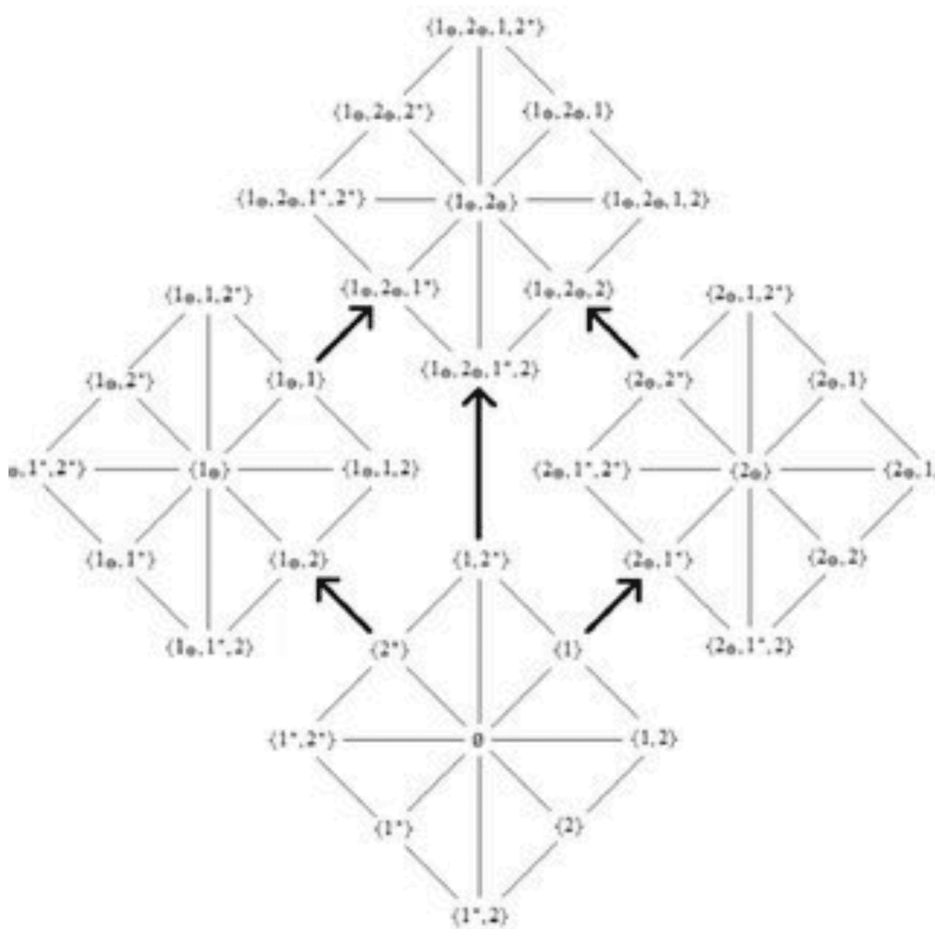

**Figure 2.** Meta-space $\left\langle 2^{A\oplus B}, 2^{A\oplus B^-}\right\rangle$.

Now if we count the number of elements that the meta-space generated by $A = \{1_0, 2_0\}$ and $B = \{1, 2\}$ has, we will have that they are 36, where the prime decomposition of this number is $36 = 2^2 \cdot 3^2$ which is equivalent to the following multiplication of cardinals $36 = 2^{|A|} \cdot 3^{|B|}$, from here we can obtain the following conjecture:

**Conjecture 2.5.** *For all $A \in 2^{\mathbb{N}^0}$ and $B \in 2^{\mathbb{N}}$, then* $\left|\left\langle 2^{A\oplus B}, 2^{A\oplus B^-}\right\rangle\right| = 2^{|A|} \cdot 3^{|B|}$.


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**Example 2.6.** *We consider $A = \{1_0\}$ and $B = \{1, 2\}$, then we obtain that*

$$\left\langle 2^{A\oplus B}, 2^{A\oplus B^-} \right\rangle = \{\emptyset, \{1_0\}, \{1\}, \{1^*\}, \{2\}, \{2^*\},$$
$$\{1_0, 1\}, \{1_0, 2\}, \{1_0, 1^*\}, \{1_0, 2^*\}, \{1, 2\}, \{1, 2^*\}, \{1^*, 2\}, \{1^*, 2^*\}, \{1_0, 1, 2\},$$
$$\{1_0, 1, 2^*\}, \{1_0, 1^*, 2\}, \{1_0, 1^*, 2^*\}\}$$

*Thus, we have that $|A| = 1$ and $|B| = 2$ and $\left|\left\langle 2^{A\oplus B}, 2^{A\oplus B^-} \right\rangle\right| = 2^{|A|} \cdot 3^{|B|} = 2^1 \cdot 3^2 = 18.$*

**Example 2.7.** *We consider $A = \emptyset$ and $B = \{1, 2\}$, then we obtain that*

$$\left\langle 2^{A\oplus B}, 2^{A\oplus B^-} \right\rangle = \{\emptyset, \{1\}, \{1^*\}, \{2\}, \{2^*\}, \{1, 2\}, \{1, 2^*\}, \{1^*, 2\}, \{1^*, 2^*\}\}$$

*Thus, we have that $|A| = 0$ and $|B| = 2$ and $\left|\left\langle 2^{A\oplus B}, 2^{A\oplus B^-} \right\rangle\right| = 2^{|A|} \cdot 3^{|B|} = 2^0 \cdot 3^2 = 9.$*

# 3. Algebraic structure of integer space $3^A$

With respect to the algebraic structure of the Integer Space $3^A$ for all $A \in 2^{\mathbb{N}}$ we think that these structures are related with structures called NAFIL (non-associative finite invertible loops)

**Theorem 3.1.** *Let $A = \{1, 2\}$, then $(3^A, \oplus)$ satisfies the following conditions:*

1. $(\forall X, Y \in 3^A)(X \oplus Y \in 3^A)$,
2. $(\exists! \emptyset \in 3^A)(\forall X \in 3^A)(X \oplus \emptyset = \emptyset \oplus X = X)$,
3. $(\forall X \in 3^A)(\exists! X^- \in 3^A)(X \oplus X^- = X^- \oplus X = \emptyset)$,
4. $(\forall X, Y \in 3^A)(X \oplus Y = Y \oplus X))$.

*Proof.* Let $A = \{1, 2\}$, then we quote the fusion matrix represented in table 2 for $3^{\{1,2\}}$.

| $\oplus$ | $\emptyset$ | $\{1\}$ | $\{2\}$ | $\{1, 2\}$ |
|---|---|---|---|---|
| $\emptyset^-$ | $\emptyset_0^0$ | $\{1\}$ | $\{2\}$ | $\{1, 2\}$ |
| $\{1^*\}$ | $\{1^*\}$ | $\emptyset_1^1$ | $\{1^*, 2\}$ | $\{2\}$ |
| $\{2^*\}$ | $\{2^*\}$ | $\{1, 2^*\}$ | $\emptyset_1^2$ | $\{1\}$ |
| $\{1^*, 2^*\}$ | $\{1^*, 2^*\}$ | $\{2^*\}$ | $\{1^*\}$ | $\emptyset_2^3$ |

**Table 2.** Integer Space $3^{\{1,2\}}$.

From here it is clearly seen that the conditions $(1)$, $(2)$ and $(3)$ of theorem 3.1 are satisfied, where the condition $(4)$ is obvious by definition.

We must clarify that since $\sigma$-set $\emptyset = \emptyset^-$, and also $\emptyset = \emptyset_0^0 = \emptyset_1^1 = \emptyset_1^2 = \emptyset_2^3$, from here we have the condition $(2)$ and the difference is in another dimension, the dimension of annihilation. Here we must clarify that the fusion operation $\oplus$ is not associative. Let $X = \{1^*, 2^*\}$, $Y = \{1, 2\}$ and $Z = \{1\}$ then we will have that $(\{1^*, 2^*\} \oplus \{1, 2\}) \oplus \{1\} = \emptyset \oplus \{1\} = \{1\}$

on the other hand

$\{1^*, 2^*\} \oplus (\{1, 2\} \oplus \{1\}) = \{1^*, 2^*\} \oplus \{1, 2\} = \emptyset$

therefore we have that

$(X \oplus Y) \oplus Z \neq X \oplus (Y \oplus Z)$,

which shows that the structure $(3^A, \oplus)$, is non-associative.

□

We now present a new conjecture.

**Conjecture 3.2.** *Let $A \in 2^{\mathbb{N}}$, then $(3^A, \oplus)$ satisfies the following conditions:*

1. $(\forall X, Y \in 3^A)(X \oplus Y \in 3^A)$,
2. $(\exists! \emptyset \in 3^A)(\forall X \in 3^A)(X \oplus \emptyset = \emptyset \oplus X = X)$,
3. $(\forall X \in 3^A)(\exists! X^- \in 3^A)(X \oplus X^- = X^- \oplus X = \emptyset)$,
4. $(\forall X, Y \in 3^A)(X \oplus Y = Y \oplus X))$.

## 4. $\sigma$-Sets Equations

Continuing with the analysis of the $\sigma$-sets, we now have the development of the equations of $\sigma$-sets of a $\sigma$-set variable, equations that play a very important role when solving a $\sigma$-set equation, now let's define and go deeper into the $\sigma$-sets variables.

We must remember that for every $\sigma$-set $A$ and $B$, the fusion of both is defined as:

$$A \oplus B = \{x : x \in A \divideontimes B \vee x \in B \divideontimes A\}$$

**Definition 4.1.** *Let $A$ be a $\sigma$-set, then $A$ is said to be an entire $\sigma$-set if there exists the $\sigma$-antiset $A^-$.*

**Example 4.2.** Let $A = \{1_0, 2_0, 3_0\}$, then this $\sigma$-set is not an integer, since $A^-$ does not exist, on the other hand the $\sigma$-set $A = \{1, 2, 3, 4\}$, is an integer $\sigma$-set since $A^- = \{1^*, 2^*, 3^*, 4^*\}$ exists which is the $\sigma$-antiset of $A$.

It is clear that if a $\sigma$-set $A$ is integer, then by definition there exists the integer space $3^A$. We should also note that if $A$ is an integer $\sigma$-set, then $[A \cup A^-]$ is a proper $\sigma$-class, for example, consider $A = \{1, 2\}$, then $[A \cup A^-] = [1, 2, 1^*, 2^*]$, is a proper $\sigma$-class.

**Definition 4.3.** *Let $A$ be an integer $\sigma$-set such that $|A| = n$, then $X$ is said to be a $\sigma$-set variable of $3^A$, if and only if*

$$X = \{x_1, x_2, x_3, \ldots, x_m\},$$

*where $m \leq n$ and $x_i$ a variable of the proper class $[A \cup A^-]$.*

**Example 4.4.** *Let $A = \{1, 2, 3\}$ be a $\sigma$-set, it is clear that $A$ is an entire $\sigma$-set since there exists $A^- = \{1^*, 2^*, 3^*\}$ and therefore $3^A$, in this way we will have that*

$$X = \emptyset,$$

$$X = \{x\},$$

$$X = \{x_1, x_2\},$$

$$X = \{x_1, x_2, x_3\},$$

are $\sigma$-sets variables of $3^A$, where $\{x, x_1, x_2, x_3\} \in [1, 2, 3, 1^*, 2^*, 3^*]$.

**Lemma 4.5.** *Let $A$ be an integer $\sigma$-set and $X$ a $\sigma$-set variable of $3^A$, then $A \oplus X = A \cup X$, with $A \subset A \cup X$ and $X \subset A \cup X$.*

*Proof.* Let $A$ be an integer $\sigma$-set and $X$ a $\sigma$-set variable of $3^A$, then

$$A \oplus X = \{x : x \in A \divideontimes X \vee x \in X \divideontimes A\}$$

Now we have that

$$A \divideontimes X = A$$

and

$$X \divideontimes A = X$$


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since $X$ is a $\sigma$-set variable, therefore we will have that

$$A \oplus X = \{x : x \in A \vee x \in X\} = A \cup X.$$

We can also observe that $A \cap X = \emptyset$ since $X$ is a $\sigma$-set variable, therefore $A \subset A \cup X$ and $X \subset A \cup X$.

□

**Example 4.6.** *Let $A = \{1, 2, 3\}$, and $X$ be a $\sigma$-set variable of $3^A$, that is,*

$$X = \emptyset$$

$$X = \{x\},$$

$$X = \{x_1, x_2\},$$

$$X = \{x_1, x_2, x_3\},$$

*are $\sigma$-sets variable of $3^A$, where $\{x, x_1, x_2, x_3\} \in [1, 2, 3, 1^*, 2^*, 3^*]$. then*

$$A \oplus X = \{1, 2, 3\},$$

$$A \oplus X = \{1, 2, 3, x\},$$

$$A \oplus X = \{1, 2, 3, x_1, x_2\},$$

$$A \oplus X = \{1, 2, 3, x_1, x_2, x_3\}$$

After the lemma 4.5 we proceed to analyze some equations of a $\sigma$-set variable and their solutions

Let $A$ be an integer $\sigma$-set, $X$ a $\sigma$-set variable and $M$ and $N$ two $\sigma$-sets of the integer space $3^A$, then an equation of a $\sigma$-set variable will be

$$X \oplus M = N.$$

Now if $M = N$, then the equation becomes

$$X \oplus M = M,$$

and by the corollary 1.17 we have that the solutions are all $X \in 2^M$, where we naturally count $X = \emptyset$, hence we have an equation of a $\sigma$-set variable with multiple solutions.

Now consider $M \neq N$, then the $\sigma$-set equation becomes:

$$X \oplus M = N,$$


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We must remember that the structure in general is not associative, therefore we cannot freely use this property, so to find the solution to the equation we must develop a previous theorem. To develop this theorem we will assume that for every integer $\sigma$-set $A$ then the generated space $\langle 2^A, 2^{A^-} \rangle = 3^A$, and also that $3^A$ satisfies the conjecture 3.2.

**Theorem 4.7.** *Let $A$ be an integer $\sigma$-set, $X$ be a $\sigma$-set variable of $3^A$ and $M \in 3^A$. Then*

$$(X \oplus M) \oplus M^- = X$$

*Proof.* Let $A$ be an entire $\sigma$-set, $X$ be a $\sigma$-set variable of $3^A$ and $M \in 3^A$, then by lemma 4.5 we have that $X \oplus M = X \cup M$, with $X \cap M = \emptyset$.

Therefore we have that

$$\begin{aligned} \circledast (X \oplus M) \oplus M^- &= \{a : a \in (X \oplus M) \divideontimes M^- \vee a \in M^- \divideontimes (X \oplus M)\} \\ &= \{a : a \in (X \cup M) \divideontimes M^- \vee a \in M^- \divideontimes (X \cup M)\} \end{aligned}$$

so

$$(X \cup M) \divideontimes M^- = (X \cup M) - (X \cup M) \hat{\cap} M^- = (X \cup M) - M = X,$$

and

$$M^- \divideontimes (X \cup M) = M^- - M^- \hat{\cap} (X \cup M) = M^- - M^- = \emptyset.$$

Now replacing these calculations in $(\circledast)$ we will have that

$$\begin{aligned} (X \oplus M) \oplus M^- &= \{a : a \in X \vee a \in \emptyset\} \\ (X \oplus M) \oplus M^- &= \{a : a \in X\}, \\ (X \oplus M) \oplus M^- &= X. \end{aligned}$$

□

Now that theorem 4.7 has been proved, we can solve some $\sigma$-set equation for the integer $\sigma$-set $A = \{1, 2\}$, since the generated space is effectively equal to $3^A$, that is, $\langle 2^A, 2^{A^-} \rangle = 3^A$, and also $3^A$ is a non-associative abelian loop.

Let $A = \{1, 2\}$ be an integer set and $M, N \in 3^A$, then the equation

$$X \oplus M = N,$$

has the following solution

$$\begin{aligned} X \oplus M &= N \backslash \oplus M^-. \\ (X \oplus M) \oplus M^- &= N \oplus M^-, \end{aligned}$$

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then by theorem 4.7 we will have that

$$X = N \oplus M^{-}.$$

Let us now show a concrete example for $A = \{1, 2\}$.

**Example 4.8.** *Let $A = \{1, 2\}$ be an integer $\sigma$-set, $M = \{1, 2^*\}$ and $N = \{1\}$, then the equation of a $\sigma$-set variable*

$$X \oplus \{1, 2^*\} = \{1\}$$

*has the following solution.*

$$\begin{aligned} X \oplus \{1, 2^*\} &= \{1\} \setminus \oplus \{1^*, 2\}, \\ (X \oplus \{1, 2^*\}) \oplus \{1^*, 2\} &= \{1\} \oplus \{1^*, 2\}, \\ X &= \{2\}. \end{aligned}$$

*Here we can see that the equation has as solution the $\sigma$-set $S_1 = \{2\}$, since*

$$\{2\} \oplus \{1, 2^*\} = \{1\},$$

*but like the equation $X \oplus M = M$, this one does not have a unique solution since the $\sigma$-set $S_2 = \{1, 2\}$, is also a solution for the equation of a $\sigma$-set variable,*

$$\{1, 2\} \oplus \{1, 2^*\} = \{1\}.$$

*In this way we have two solutions for our equation of a $\sigma$-set variable which are:*

$$S = \{S_1, S_2\} = \{\{2\}, \{1, 2\}\}.$$

Before we conclude our study of $\sigma$-sets and $\sigma$-antisets we are going to present a conjecture regarding the possible solutions of a particular type of equations of a $\sigma$-set variable.

**Conjecture 4.9.** *Let $A$ be an integer $\sigma$-set, $X$ a $\sigma$-set variable of $3^A$, and $M, N \in 3^A$, then a solution of the equation*

$$X \oplus M = N,$$

*is $S = N \oplus M^{-}$.*

Let us look into the following case, where $M = \{1^*\}$ and $N = \{1\}$, so for the equation

$$X \oplus \{1^*\} = \{1\},$$


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there is no element $x$ such that $\{x, 1^*\} = \{1\}$. This lead us into a more general case where $N \subseteq M^-$ or $N \supset M^-$, being the conjecture false, which take us into establishing the following condition in order for this type of equation being solvable.

**Definition 4.10.** *A $\sigma$-set equation $X \oplus M = N$ is said to be* **fusionable** *if $M \hat{\cap} N = \emptyset$.*

With this, let us conclude with a bounded theorem to find some solutions of the $\sigma$-set equation.

**Theorem 4.11.** *Let $A$ be an integer $\sigma$-set, $X$ a $\sigma$-set variable of $3^A$, and $M, N \in 3^A$, then two possible solutions $S = \{S_1, S_2\}$ of the fusionable equation*

$$X \oplus M = N,$$

*are $S_1 = N \oplus R^-$ and $S_2 = R^-$, where $R := M \oplus N^-$.*

*Proof.* For the first solution $S_1$ we have that

$$\begin{aligned} S_1 &= N \oplus R^- \\ &= N \oplus (M \oplus N^-)^- \\ &= N \oplus (N \oplus M^-) \\ &= N \oplus M^-, \end{aligned}$$

where $S_2 = (M \oplus N^-)^- = N \oplus M^-$ because of the result iteration seen above. Hence both results are actually a fusion solution for $X \oplus M = N$, where $S_2 = R^-$ is an exact solution and $S_1 = N \oplus R^-$ is an intersected rest solution. Because of $M \hat{\cap} N = \emptyset$ (Definition 4.10) as the equation $X \oplus M = N$ is fusionable, both $S_1 \oplus M$ and $S_2 \oplus M$ will be fusionable into another $\sigma$-set $N$. □

As we looked above, the solution space is reduced such that the solutions are indeed $N \oplus M^-$, being by consequence possible solutions for the fusionable equation $X \oplus M = N$.

**Example 4.12.** *Let $A = \{1, 2, 3, 4, 5, 6\}$ be an integer $\sigma$-set, $M = \{1, 2, 3^*, 4^*, 5, 6^*\}$ and $N = \{1, 2\}$, then the equation of a $\sigma$-set variable*

$$X \oplus \{1, 2, 3^*, 4^*, 5, 6^*\} = \{1, 2\},$$

*which is fusionable because $M \hat{\cap} N = \{1, 2, 3^*, 4^*, 5, 6^*\} \hat{\cap} \{1, 2\} = \emptyset$.*

*Now, by using Theorem 4.11, let us first obtain*

$$\begin{aligned} R &= (M \oplus N^-)^- \\ &= (\{1, 2, 3^*, 4^*, 5, 6^*\} \oplus \{1, 2\}^-)^- \\ &= (\{1, 2, 3^*, 4^*, 5, 6^*\} \oplus \{1^*, 2^*\})^- \\ &= (\{3^*, 4^*, 5, 6^*\})^- \\ &= \{3, 4, 5^*, 6\}, \end{aligned}$$

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*so we get* $S_1 = N \oplus R^- = \{1,2,3,4,5^*,6\}$ *and* $S_2 = R^- = \{3,4,5^*,6\}$*, which can be easily proved that both solutions gives* $S_1 \oplus M = S_2 \oplus M = N$ *as a resulting* $\sigma$*-set. Hence* $S = \{\{1,2,3,4,5^*,6\},\{3,4,5^*,6\}\}$ *is a solution set for the fusionable equation* $X \oplus M = N$*.*

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