Open Peer Review on Qeios

A New Family of Solids: The Infinite Kepler-Poinsot Polyhedra

Dirk Huylebrouck[1]

1 KU Leuven

Funding: No specific funding was received for this work.

Potential competing interests: No potential competing interests to declare.

Abstract

The so-called Platonic solids have fascinated mathematicians and artists for over 2000 years. It is astonishing that there are only five cases of regular polyhedra, that is, of polyhedra in which regular polygons form the same spatial angles between them in each vertex. In 1619, Kepler added the small and great stellated dodecahedron to this list, but he allowed intersecting faces. Poinsot did so too, in 1809, and discovered the great dodecahedron and great icosahedron. In 20th century, Coxeter and Petrie added three more regular polyhedra, using infinitely repeating elements, based on the truncated tetrahedron, the cube and the truncated octahedron.

The principle of intersecting faces, typical for the Kepler-Poinsot solids, can be combined with the Coxeter-Petrie generalization to the infinite case. Thus, a new regular polyhedron was discovered, based on the cubohemioctahedron but without its square faces. Placed side by side and on top of each other, identical regular hexagons meet in each vertex, always with the same spatial angle. There are 8 of them in each vertex, and so it is not a compound of twice two polyhedra with 4 hexagons in each vertex. The dual of this {6, 8} polyhedron of infinite Kepler-Poinsot type is indeed a {8, 6} polyhedron of infinite Kepler-Poinsot type, if two overlapping squares are considered as one 8/4 octagonal star.

Kepler-Poinsot solids are difficult to interpret, with their intersecting faces, and this infinite case is even more difficult to grasp. The present paper tries to solve this using open faces so that one can see through the solids.

D. Huylebrouck

KU Leuven, Belgium, dirk.huylebrouck@kuleuven.be

Keywords: Platonic and Archimedean solids, Kepler-Poinsot solids, infinite polyhedra.

Introduction

The five regular Platonic solids are very well-known: the tetrahedron, the octahedron, the cube, the icosahedron, and the dodecahedron. These polyhedra combine regular polygons of the same type (equilateral triangles, squares or pentagons) and in each vertex the spatial angle is identical. The so-called Schläfli symbol for the tetrahedron summarizes that equilateral triangles meet, three at each vertex: {3, 3}. In an octahedron, equilateral triangles are used too, but now four at each vertex {3, 4}. A cube groups squares, three at the time: {4, 3}. Similarly, {3, 5} is the Schläfli symbol for the icosahedron and {5, 3} for the dodecahedron (see figure 1).

In this paper, polyhedra are represented with 'open' faces, which is quite unusual today. Leonardo da Vinci used this representation for Luca Pacioli's 'Divine Proportione' but planar faces are now more common. However, for understanding Kepler-Poinsot solids opening the faces will be helpful. Even in the case of the five regular polyhedra it allows to better see the duals hidden inside each polyhedron. These duals are constructed by connecting the centers of each face. Thus, the tetrahedron is self-dual, the dual of the octahedron is the cube, and vice versa, and the dual of the icosahedron is the dodecahedron and reciprocally. They correspond to a switch of the Schläfli symbols.

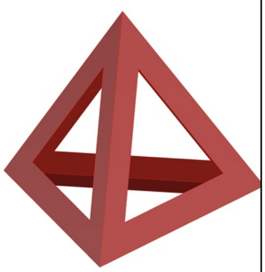
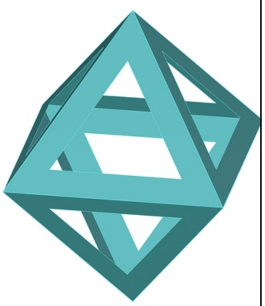
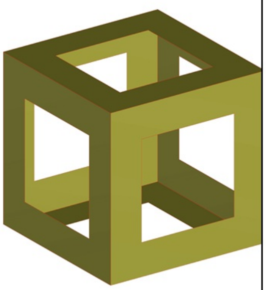
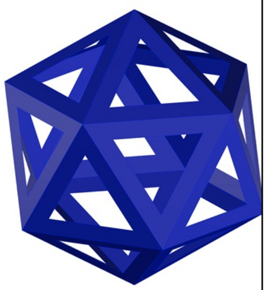
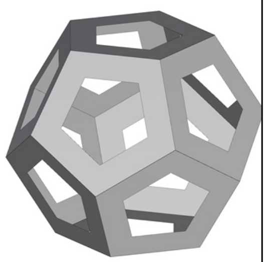
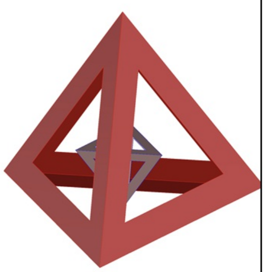
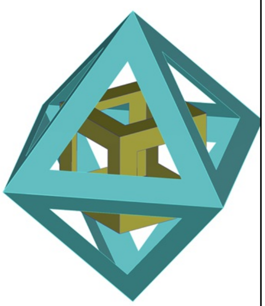
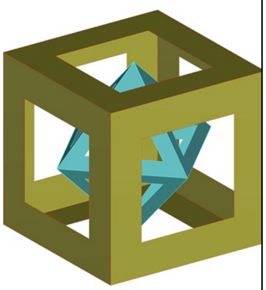
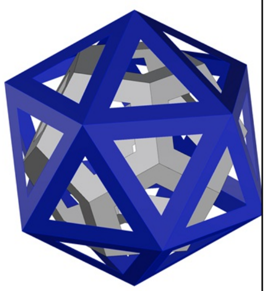
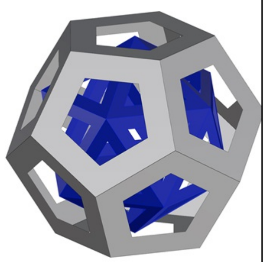
Tetrahedron	Octahedron	Cube	Icosahedron	Dodecahedron
				
{3, 3}	{3, 4}	{4, 3}	{3, 5}	{5, 3}
				

Figure 1. The five Platonic solids and their duals.

Four more regular solids, the so-called Kepler-Poinsot polyhedra, can be obtained when star polygons or intersecting polygons are allowed. Two of them, the small stellated dodecahedron and the great stellated dodecahedron, use pentagonal stars or pentagrams, represented by the symbol 5/2. The other two, the great icosahedron and the great dodecahedron, combine triangles and respectively pentagons around a vertex in the same way as a pentagonal star rotates around its center and thus their number is equally symbolized by 5/2 (see figure 2 and [1]). As the faces of the Kepler-Poinsot polyhedra come close to the center of the polyhedron, the duals are harder to see than in the previous cases. The great stellated dodecahedron and great icosahedron are each other duals but become tiny with respect to the original polyhedron when constructed using the centers of the faces.

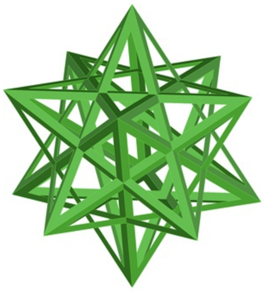
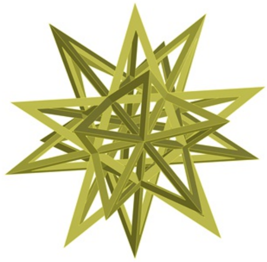
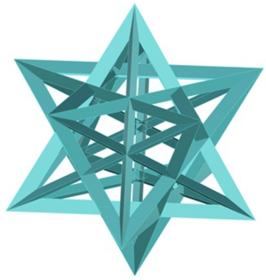
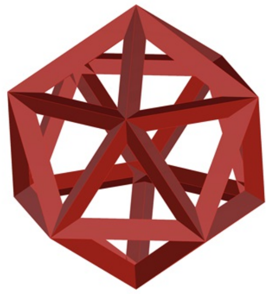
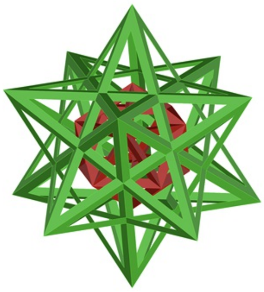
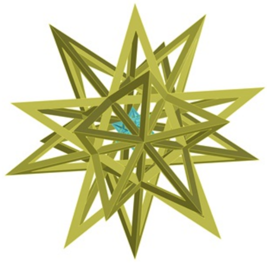
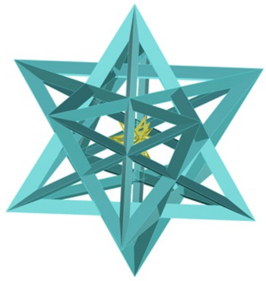
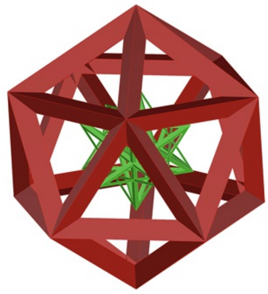
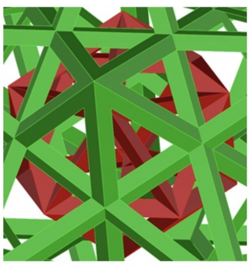
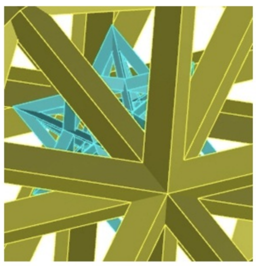
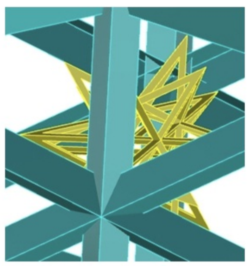
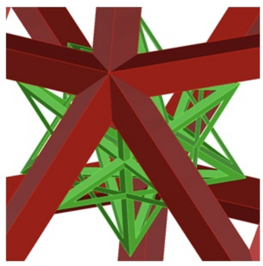
Small stellated dodecahedron	Great stellated dodecahedron	Great icosahedron	Great dodecahedron
			
{5/2, 5}	{5/2, 3}	{3, 5/2}	{5, 5/2}
			
			

Figure 2. The four Kepler-Poinsot solids and their duals with detailed views below.

Coxeter and Petrie extended the concept of a regular polyhedron to infinite polyhedra, where one ‘atom’ is repeated to infinity, and again regular polygons meet in each vertex in identical angles. They obtained three regular configurations, two using hexagons, and one using squares (see figure 3). When the triangles of a truncated tetrahedron are removed, only 4 hexagons remain and arranging them next to each other creates an infinite polyhedron with Schläfli symbol {6, 6}, as six hexagons meet in each vertex. Representing them with open faces allows to see it is a self-dual polyhedron. Removing two opposite faces of a cube can make an arrangement so that six square meet in each vertex. Their centers form a hexagon that will then form open truncated octahedrons. Thus, the {4, 6} has a {6, 4} arrangement as dual and vice-versa.

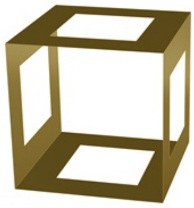
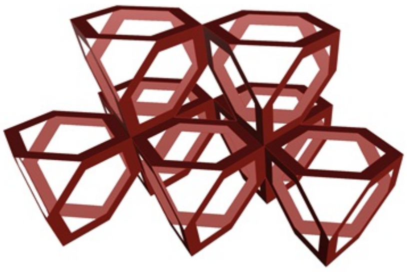
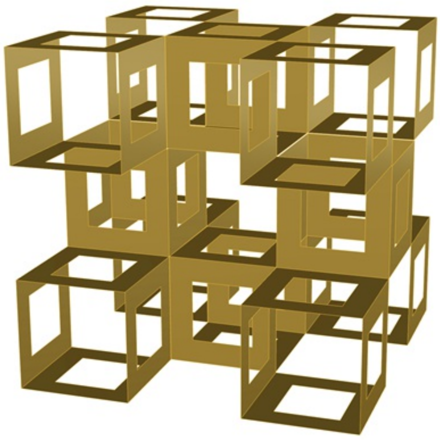
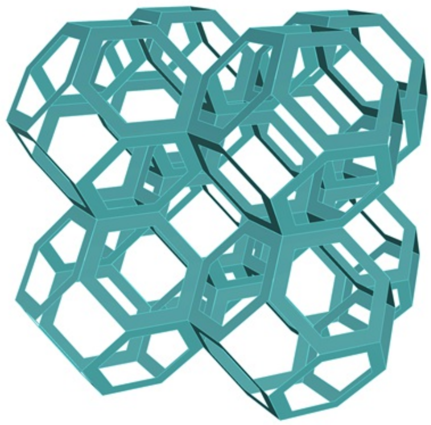
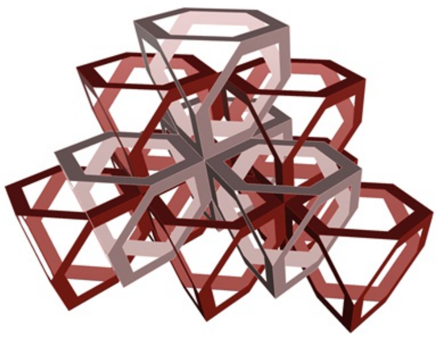
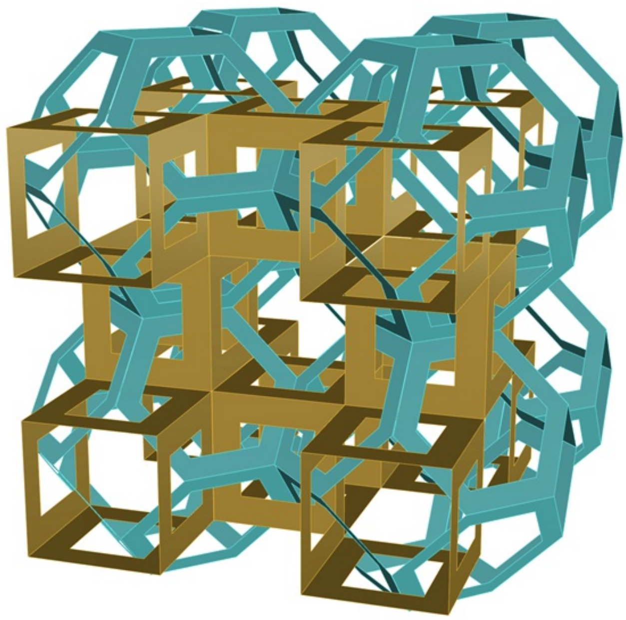
Atom = open truncated tetrahedrons	Atom = open cubes	Atom = open truncated octahedrons
		
		
{6, 6}	{4, 6}	{6, 4}
		

Figure 3. The three Coxeter-Petrie regular infinite polyhedra. (above) with their duals (below).

A New Regular Polyhedron, of Infinite Kepler-Poinsot Type

All previously mentioned polyhedra were regular, in the sense that they only used the same type of regular polygons in the same configuration, using the same spatial angles. Combining two of the above principles, the intersecting faces considered by Kepler and Poinsot and the infinite constructions invented by Coxeter and Petrie, the author could find yet another regular polyhedron (see [2]). However, it is rather difficult to grasp because of this combination and so here it is tried to visualize it better using open faces. Usually, a so-called cubohemioctahedron, having four hexagons with common centers and six squares, is represented in such a way that at first sight one would think there are triangles between the squares. In addition, when making paper models of the cubohemioctahedron, it is easier to use triangles indeed (see figure 4).

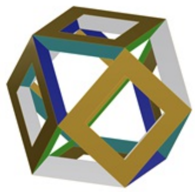
	Cubooctahedron	Cubohemioctahedron	An open cubohemioctahedron
	6 squares, 8 triangles	6 squares, 4 hexagons	4 hexagons
usual representation			
using open faces.			

Figure 4. Representations of the cuboctahedron {4, 3, 4, 3}, the cubohemioctahedron {4, 6, 4, 6} and the open cubohemioctahedron.

If we now place an infinite number of those open cubohemioctahedra next to each other as in Coxeter's cases, we get an infinite polyhedron (see figure 5). It is regular and of Kepler-Poinsot type, because the faces intersect each other. Eight hexagons meet in each vertex, and so its symbol is {6, 8}. It is a new regular polyhedron (see [3], [4], [1], [5]). Asked for a second opinion on the {6, 8}-polyhedron, Branko Grünbaum stated that "it appears to satisfy the conditions imposed explicitly by Coxeter and others", for a shape to be called a polyhedron. "I must say that I never encountered it in the literature," Grünbaum added.

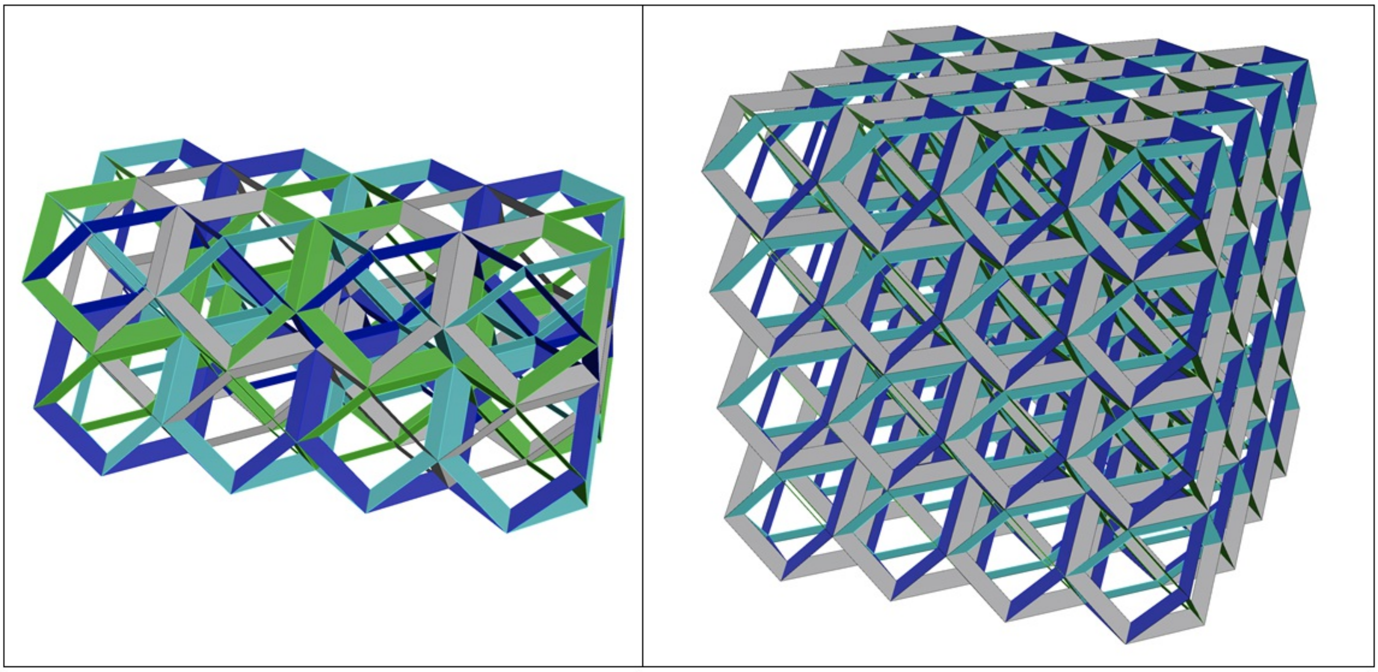


Figure 5. A regular infinite Kepler-Poinsot polyhedron, of which some atoms are shown (left) or many more (right).

The dual of the {6, 8} should be an {8, 6}, that is, a polyhedron in which 6 octagons meet in each vertex. Knowing the dual of the cubohemioctahedron is the hexahemioctacron, with vertices on infinity, one is warned this may be an even more unusual polyhedron. However, if the hexagons are chosen in such a way that they form a {6, 4}, the dual can be formed as for the infinite Coxeter-Petrie polyhedron made by open truncated octahedrons (figure 3 last row and to the right). Putting the four constructions together, each square must be counted twice.

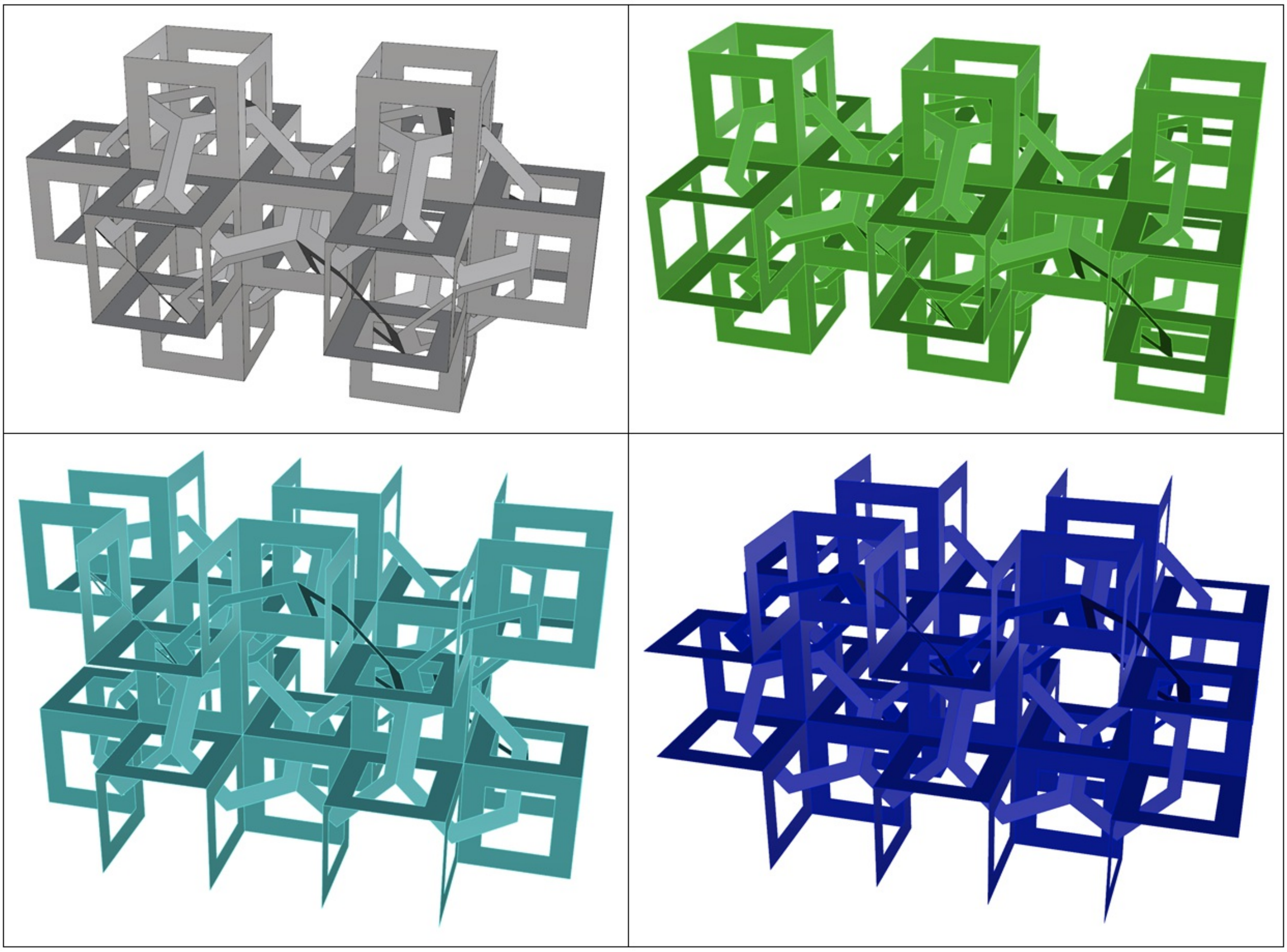


Figure 6. Constructing the dual, ‘color by color’.

Now two overlapping squares can be considered as one ‘octagonal star’. Indeed, the usual octagonal star is represented by 8/3, and often the more unusual star 8/2 is distinguished as well. It consists of two overlapping squares, making an angle of 45° so that the combination indeed has 8 vertices. However, the two squares could also overlap entirely, and when the vertices are counted twice, an ‘octagonal star’ is obtained, which can be called an 8/4.

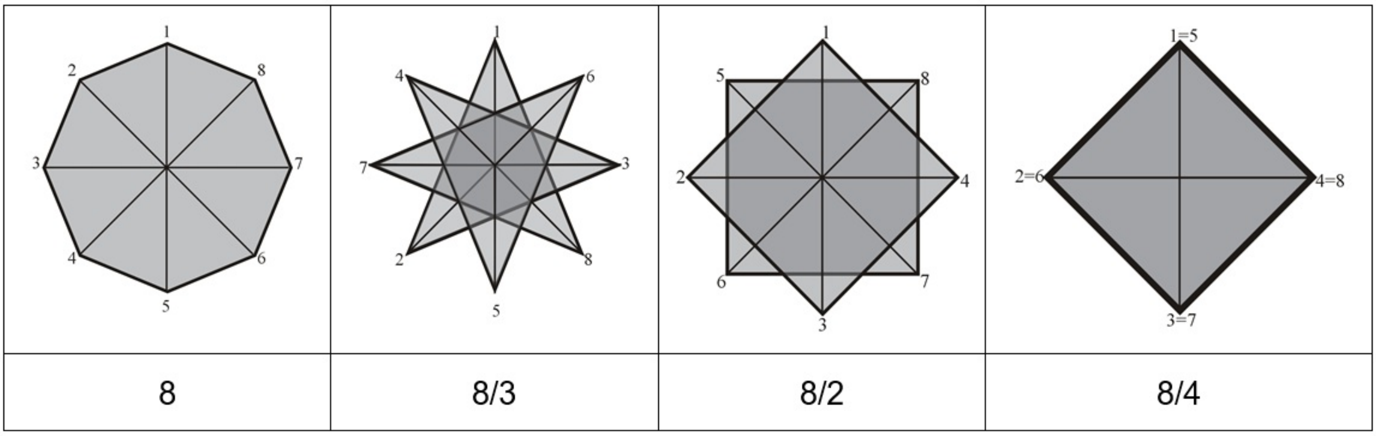


Figure 7. Octagonal stars.

Thus, the dual of the new {6, 8} polyhedron (of infinite Kepler-Poinsot type) is indeed the {8, 6} (of infinite Kepler-Poinsot type).

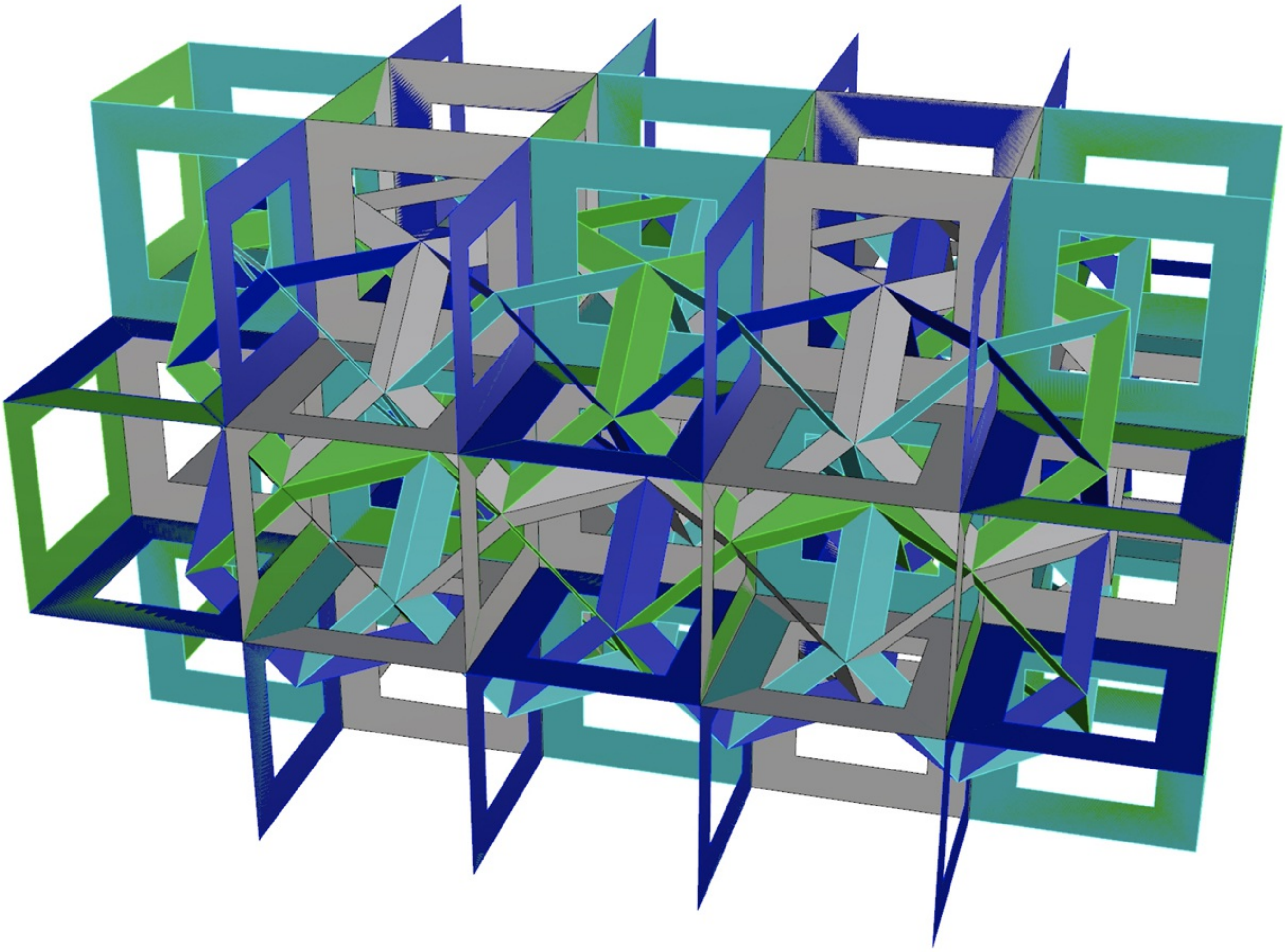


Figure 7. Elements of the {6, 8} polyhedron and its dual, the {8, 6}; the shade on some squares illustrates the overlap.

References

1. [a, b] *A. K. van der Vegt, Order in space, published by VSSD, January 9, 2006.*
2. [^] *D. Huylebrouck, 'A New Regular (Compound) Polyhedron (of Infinite Kepler-Poinsot Type)', The American Mathematical Monthly, Vol. 124, n° 3, 03 2017, pp. 265-268.*
3. [^] *J. H. Conway, H. Burgiel, Ch. Goodman-Strauss, The Symmetries of Things, A K Peters Ltd., 2008.*
4. [^] *H. S. M. Coxeter, Regular Polytopes, 3rd ed., New York: Dover, 1973.*
5. [^] *A. Wachman, M. Burt, M. Kleinman, (1974; 2005), Infinite Polyhedra, Technion Publication Faculty of Architecture and Town Planning, 102 pp.*