Review of: "New adaptative numerical algorithm for solving partial integro-differential equations"

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Potential competing interests: No potential competing interests to declare.

Referee's report on the manuscript Qeios ID: R4546K

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https://doi.org/10.32388/R4546K

New adaptative numerical algorithm for solving partial integro-differential equations

In my opinion, the problem of the paper is interesting in the context of solving partial integro-differential equations**So, in its actual form, I recommend to accept the publication of the manuscript by addressing the following comments to the authors.**

1. The importance of the research on this topic can be addressed in the introuction?
2. The simulated results provided in the article can be shared the code of your research in git-hub share git-hub account to check the accuracy.

Rebiha Zeghdane[1]

Author(s) details

Declarations

Abstract

The paper introduces an acurrate numerical appraoch based on orthonormal Bernoulli polynomials for solving parabolic partial integro- differential equations (PIDEs). This type of equations arises in physics and engineering. Some operational matrix are given for these polynomials and are also used to obtain the numerical solution. By this approach, the problem is transformed into a nonlinear algebraic system. Convergence analysis is given and some experiment tests are studied to examine the good accuracy of the numerical algorithm, the proposed technique is compared with some other well known methods.

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Keywords: Shifted orthonormal Bernoulli polynomials, parabolic integro- differential equation, collocation method, numerical solution.

1. Introduction

Various approximations by orthonormal family of functions have been investigated in physical sciences, engineering, etc. This type of numerical approximations can be also used in optimal control problems and in general to approximate solutions of dynamical systems. Integral equations arise in many physical problems, diffusion problems, concrete problem of physics and mechanics and some others problems of engineering, different applications of potential theory, synthesis problem, mathematical modelling of economics, population, geophysics, antenna, genetics, communication theory, radiation problems, concerning transport of particles, etc [1][2][3][4][5][6][7]. There are various problems such as differential, integral and partial integro-differential equations which uses polynomial series and orthogonal functions to approximate their numerical solutions [8][9][10][11][12][13][14]. Spectral methods are methods based on polynomial approximation, we can see that the convergence of the approximations is exponentiel when the functions to be approximated are analytic that's means that the order of convergence is limited by the choice of the regularity of the exact solution. In many science fields, systems are described by partial differential equations (PDE). In [15] Fakha et al used Legendre collocation technique for solving parabolic PDE. Radial basis functions are also used for the approximate solutions of nonlinear parabolic type Volterra partial integro-differential equations [14]. In [16], Brunner et al studied the numerical solution of parabolic Volterra integro-differential equations on unbounded spatial domains. For more applications, we can see [17][18][19]. A method based on Hermite-Taylor matrix to solve partial integro-differential equations is given in [9]. A matrix method for solving two-dimensional time-dependent diffusion equations is given by Zogheib et al. [20]. In [12], the author presented a new technique by using Bernoulli operational matrix to solve SDE. In this article, we use pseudo-spectral method based on orthonormal Bernoulli polynomials to approximate the following PIDEs

$$u_t(x,t)+\lambda_1 u_{xx}(x,t)=\lambda_2\int_0^t K_1(x,t,s,u(x,s))\,ds+\lambda_3\int_0^T K_2(x,t,s,u(x,s))\,ds+g(x,t), \tag{1}$$

with initial and boundary conditions

$$u(x,0)=h_0(x),\quad u(0,t)=h_1(t),\quad u(b,t)=h_2(t), \tag{2}$$

where $x\in[0,b]$, $t\in[0,T]$, λ_1,λ_2 and λ_3 are constants and the functions $g(x,t)$, $k_1(x,t,s,u)$ and $k_2(x,t,s,u)$ are supposed to be sufficiently smooth on $\Gamma:=[0,b]\times[0,T]$ and D where $D:=\{(x,t,s): x\in[0,b], s,t\in[0,T]\}$. The existence and uniqueness of equation (1) are given in [21].

To approximate the integrals appeared in (1), we use the Gauss Legendre quadrature on the interval $[-1,1]$ given by

$$\int_{-1}^{1} f(x)\,dx=\sum_{j=0}^{r}\omega_j f(x_j), \tag{3}$$

where x_j are the roots of $L_{r+1}(x)$ and $\omega_j=\frac{2}{(1-x_j^2)[L'_{r+1}(x_j)]^2}$, $j=0,1,\dots,r$. For Gauss Legendre quadrature on $[a,b]$, we have

$$\int_a^b f(x)\,dx=\frac{b-a}{2}\sum_{j=0}^{r}\omega_j f\left(\frac{b-a}{2}x_j+\frac{b+a}{2}\right). \tag{4}$$

2. Othonormal Bernoulli polynomials and approximation

The Bernoulli polynomials $B_n(x)$, are given in [22] and satisfied the following relation

$$\sum_{k=0}^{n}\binom{n+1}{k}B_k(x)=(n+1)x^n,\quad n=0,1,\dots.$$

The Bernoulli polynomials form a complete basis over the interval $[0,1]$[23]. In this paper, we use the shifted OBPs $P_{i,T}(t)$ and $P_{i,b}(x)$ over $[0,T]$ and $[0,b]$ as follows

$$P_{i,b}(x)=\sqrt{\frac{2i+1}{b}}\sum_{k=0}^{i}(-1)^{i-k}\binom{i}{i-k}\binom{i+k}{k}\left(\frac{x}{b}\right)^k,\quad i=0,1,\dots \tag{5}$$

$$\Phi_{N,b}(.)=(P_{0,b}(.),P_{1,b}(.),\dots,P_{N,b}(.))^T, \tag{6}$$

contains first $N+1$ orthonormal Bernoulli polynomials. Any function $u\in L^2[0,b]$ has a best approximation $\hat{u}\in \mathrm{span}\{\Phi_{N,b}(.)\}$ such that

$$\forall v\in \mathrm{span}\{\Phi_{N,b}(.)\},\quad \|u-\hat{u}\|\le\|u-v\|,$$

and

$$u(x)\simeq\hat{u}=\sum_{i=0}^{N}u_iP_{i,b}(x)=U^T\Phi_{b,N}(x), \tag{7}$$

where $U=(u_0,u_1,u_2,\dots,u_N)^T$, and u_i can be computed by the formula

$$u_i=\int_0^b u(x)P_{i,b}(x)dx,\ i=0,1,\dots,N. \quad (8)$$

Any function $u(x,t)$ defined over $[0,b]\times[0,T]$ can be approximated by shifted OBPs as follows:

$$u_{N,M}(x,t)=\sum_{i=0}^{N}\sum_{j=0}^{M}u_{ij}P_{i,b}(x)P_{j,T}(t)=\Phi_{b,N}(x)^T U\Phi_{T,M}(t), \quad (9)$$

where $U=[u_{ij}]$ is a matrix of order $(N+1)\times(M+1)$ with

$$u_{ij}=\int_0^b\int_0^T u(x)P_{i-1,b}(x)P_{j-1,T}(x)dtdx,\ i=1,2,\dots N+1,\ j=1,2,\dots,M+1. \quad (10)$$

3. Pseudo-spectral method for Solving PIDE

In this section, we describe our numerical technique t to solve PIDEs (1). Let the solution of (1) be approximated by the polynomial $u_{N,M}(x,t)$ such that

$$u_{N,M}(x,t)=\sum_{i=0}^{N}\sum_{j=0}^{M}u_{ij}P_{i,b}(x)P_{j,T}(t)=\Phi_{b,N}(x)^T U\Phi_{T,M}(t). \quad (11)$$

In this work, we take $N=M$. It is easy to approximate the derivatives of approximate solution of (1). In view of (11), one can write

$$\frac{\partial}{\partial t}[u_{N,M}(x,t)]=\sum_{i=0}^{N}\sum_{j=0}^{M}u_{ij}P_{i,b}(x)\frac{\partial}{\partial t}[P_{j,T}(t)]=\Phi_{b,N}(x)^T U\frac{\partial}{\partial t}[\Phi_{T,M}(t)], \quad (12)$$

$$\frac{\partial^2}{\partial x^2}[u_{N,M}(x,t)]=\sum_{i=0}^{N}\sum_{j=0}^{M}u_{ij}\frac{\partial^2}{\partial x^2}[P_{i,b}(x)][P_{j,T}(t)]=\frac{\partial^2}{\partial x^2}[\Phi_{b,N}(x)]^T U[\Phi_{T,M}(t)], \quad (13)$$

Now we give some relations for the derivatives of the shifted OBPs, we use technique used by various researchers for solving different kind of integral equations [15][24][25]. Let given the vector defined in (6), we can write

$$\Phi_{b,N}(x)=T_{b,N}\Psi_N(x), \quad (14)$$

where

$$\Psi_N(x)=[1,x,x^2,\dots,x^{N-1},x^N]^T, \quad (15)$$

with $T_{b,N}$ is a lower triangular square matrix of order $N+1$ with entries

$$[T_{b,N}]_{ij}=\begin{cases}(-1)^{i-j}\sqrt{2i-1}\,\frac{1}{b^{j-1/2}}\binom{i-1}{i-j}\binom{i+j-2}{j-1}, & \text{if } i\geq j,\\ 0, & \text{if } i<j.\end{cases}$$

The matrix $T_{b,N}$ is invertible. From equation (14), we have

∂∂xΦb,N(x)=Tb,N∂∂xΨN(x)=Tb,NANΨN(x)=Tb,NANT−1b,NΦb,N(x),(16)
(16)∂∂�Φ�,�(�)=��,�∂∂�Ψ�(�)=��,���Ψ�(�)=��,�����,�−1Φ�,�(�),

where AN�� is a square matrix of order N+1�+1 given by

AN=⎡⎣⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢⎢0100....0020....0003.....…………………….
00000N−200000000N−100000000N00000000⎤⎦⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥⎥,��={000…
0000100…0000020…0000003…0000……0000……�−2000……0�−100……00�0},

and

∂2∂x2Φb,N(x)=Tb,N∂2∂x2ΨN(x)=Tb,NA1NΨN(x)=Tb,NA1NT−1b,NΦb,N(x),(17)
(17)∂2∂�2Φ�,�(�)=��,�∂2∂�2Ψ�(�)=��,���1Ψ�(�)=��,���1��,�−1Φ�,�(�),

where A1N��1 is a square matrix of order N+1�+1 given by

A1N=⎛⎝⎜⎜⎜⎜⎜⎜⎜⎜⎜⎜⎜⎜⎜00(3−1)(3−2)0....000(4−1)(4−2)....0000.....…………………….000000(N−1)
(N−2)00000000N(N−1)0000000000000000⎞⎠⎟⎟⎟⎟⎟⎟⎟⎟⎟⎟⎟⎟⎟.��1=(000…0000000…0000(3−1)(3−2)00…00000(4−1)
(4−2)0…0000……0000……0000……(�−1)(�−2)000……0�(�−1)00).

Then, we get

∂∂t[uN,M(x,t)]=∑i=0N∑j=0MuijPi,b(x)∂∂t[Pj,T(t)]=Φb,N(x)TU∂∂t[ΦT,M(t)]=Φb,N(x)TUTT,MAMT−1T,MΦT,M(t),(18)
(18)∂∂�[��,�(�,�)]=∑�=0�∑�=0�������,�(�)∂∂�[��,�(�)]=Φ�,�(�)��∂∂�[Φ�,�(�)]=Φ�,�(�)
����,�����,�−1Φ�,�(�),

and

∂2∂x2[uN,M(x,t)]=∑i=0N∑j=0Muij∂2∂x2[Pi,b(x)][Pj,T(t)]=∂2∂x2[Φb,N(x)]TU[ΦT,M(t)]=
[Tb,NA1NT−1b,NΦb,N(x)]TU[ΦT,M(t)].(19)(19)∂2∂�2[��,�(�,�)]=∑�=0�∑�=0�����∂2∂�2[��,�(�)]
[��,�(�)]=∂2∂�2[Φ�,�(�)]��[Φ�,�(�)]=[��,���1��,�−1Φ�,�(�)]��[Φ�,�(�)].

Thus, by substituting (18) and (19) into equation (1) we get

Φb,N(x)TUTT,MAMT−1T,MΦT,M(t)+λ1[[Tb,NA1NT−1b,NΦb,N(x)]TU[ΦT,M(t)]]=+λ2∫t0k1(x,t,s,u(x,s))dsλ2∫T0k2(x,t,s,u(x,s))ds
+g(x,t),
(20)Φ�,�(�)����,�����,�−1Φ�,�(�)+�1[[��,���1��,�−1Φ�,�(�)]��[Φ�,�(�)]]=�2∫0��1(
�,�,�,�(�,�))��(20)+�2∫0��2(�,�,�,�(�,�))��+�(�,�),

with x∈[0,b]�∈[0,�] and t∈[0,T]�∈[0,�], We collocate equation (20) at points tj��, j=0,1,…,r1�=0,1,…,�1 which are the Gauss-Legendre noeuds on the interval [0,T][0,�] defined by (3), then we obtain

Φb,N(x)TUTT,MAMT−1T,MΦT,M(tj)+λ1[[Tb,NA1NT−1b,NΦb,N(x)]TU[ΦT,M(tj)]]=+λ2∫tj0K1(x,tj,s,u(x,s))dsλ3∫T0K2(x,tj,s,u(x,s
))ds+g(x,tj),

(21)Φ�,�(�)����,�����,�−1Φ�,�(��)+�1[[��,���1��,�−1Φ�,�(�)]��[Φ�,�(��)]]=�2∫0���1(�,��,�,�(�,�))��(21)+�3∫0��2(�,��,�,�(�,�))��+�(�,��),

The integrals appeared in equation (21) are approximated by using Gauss-Legendre quadrature as follows

$$\int_0^{t_j} K_1(x,t_j,s,u(x,s))\,ds=\sum_{k=0}^{r_1}\beta_k K_1(x,t_j,l_k,u(x,l_k)),\tag{22}$$

(22)∫0���1(�,��,�,�(�,�))��=∑�=0�1���1(�,��,��,�(�,��)),

where $l_k=\frac{t_j+t_j x_k}{2}$ ��=��+����2 and $\beta_k=\frac{t_j(1-x_k)}{2[L'_{r_1+1}(x_k)]^2}$ ��=��(1−��)2[��1+1′(��)]2, $k=0,1,2,\ldots,r_1$ �=0,1,2,…,�1. are Gauss Legendre nodes and weight on $[0,t_j]$[0,��].

$$\int_0^{T} K_2(x,t_j,s,u(x,s))\,ds=\sum_{k=0}^{r_2}\omega_k K_2(x,t_j,s_k,u(x,s_k)),\tag{23}$$

(23)∫0��2(�,��,�,�(�,�))��=∑�=0�2���2(�,��,��,�(�,��)),

where $s_k=\frac{T+Tx_k}{2}$ ��=�+���2 and $\omega_k=\frac{T(1-x_k)}{2[L'_{r_2+1}(x_k)]^2}$ ��=�(1−��)2[��2+1′(��)]2, $k=0,1,2,\ldots,r_2$ �=0,1,2,…,�2 are nodes and weight of Gauss Legendre quadrature on $[0,T]$[0,�]. Substituting equations (22) and (23) in equation (21) we get

$$\Phi_{b,N}(x)^T U T_{T,M} A M_{T}^{-1}\Phi_{T,M}(t_j)+\lambda_1[[T_{b,N} A_1 N_T^{-1}\Phi_{b,N}(x)]^T U[\Phi_{T,M}(t_j)]]=\sum_{k=0}^{r_1}\beta_k K_1(x,t_j,l_k,u(x,l_k))$$

Φ�,�(�)����,�����,�−1Φ�,�(��)+�1[[��,���1��,�−1Φ�,�(�)]��[Φ�,�(��)]]=∑�=0�1���1(�,��,��,�(�,��))

$$+\sum_{k=0}^{r_2}\omega_k K_2(x,t_j,s_k,u(x,s_k))+g(x,t_j).$$

+∑�=0�2���2(�,��,��,�(�,��))+�(�,��).

Let $RES(x,t_j)$ ���(�,��) the residual function given by

$$RES(x,t_j)=\Phi_{b,N}(x)^T U T_{T,M} A M_{T}^{-1}\Phi_{T,M}(t_j)+\lambda_1[[T_{b,N} A_1 N_T^{-1}\Phi_{b,N}(x)]^T U[\Phi_{T,M}(t_j)]]\tag{24}$$

(24)���(�,��)=Φ�,�(�)����,�����,�−1Φ�,�(��)+�1[[��,���1��,�−1Φ�,�(�)]��[Φ�,�(��)]]

$$-\lambda_2\sum_{k=0}^{r_1}\beta_k K_1(x,t_j,l_k,u(x,l_k))-\lambda_3\sum_{k=0}^{r_2}\omega_k K_2(x,t_j,s_k,u(x,s_k))-g(x,t_j)=0.$$

−�2∑�=0�1���1(�,��,��,�(�,��))−�3∑�=0�2���2(�,��,��,�(�,��))−�(�,��)=0.

Using relation (9), initial and boundary conditions given in (6), we get

$$\begin{cases}\Phi_{b,N}(x)^T U\Phi_{T,M}(0)-h_0(x)=0,\\ \Phi_{b,N}(0)^T U\Phi_{T,M}(t)-h_1(t)=0,\\ \Phi_{b,N}(b)^T U\Phi_{T,M}(t)-h_2(t)=0.\end{cases}$$

{Φ�,�(�)���Φ�,�(0)−ℎ0(�)=0,Φ�,�(0)���Φ�,�(�)−ℎ1(�)=0,Φ�,�(�)���Φ�,�(�)−ℎ2(�)=0.

Now, we extract the below $(N+1)\times(M+1)$(�+1)×(�+1) algebraic system

$$\begin{cases}RES(s_i,t_j)=0, & 2\le i\le N,\ 2\le j\le M+1,\\ \Phi_{b,N}(x)^T U\Phi_{T,M}(0)-h_0(s_i)=0, & 1\le i\le N+1,\\ \Phi_{b,N}(0)^T U\Phi_{T,M}(t_j)-h_1(t_j)=0, & 2\le j\le M+1\\ \Phi_{b,N}(b)^T U\Phi_{T,M}(t_j)-h_2(t_j)=0, & 2\le j\le M+1,\end{cases}$$

{���(��,��)=0,2≤�≤�,2≤�≤�+1,Φ�,�(�)���Φ�,�(0)−ℎ0(��)=0,1≤�≤�+1,Φ�,�(0)���Φ�,�(��)−ℎ1(��)=0,2≤�≤�+1Φ�,�(�)���Φ�,�(��)−ℎ2(��)=0,2≤�≤�+1, where $s_i=\frac{b+bx_i}{2}$ and $t_j=\frac{T+Tx_j}{2}$ are respectively the nodes of Gaus-Legendre quadrature on $[0,b]$ and $[0,T]$, with x_i are the corresponding nodes on $[-1,1]$.

4. Error bound of the present method

In [26], the error estimates for some orthogonal systems are given in the norms of the Sobolev spaces $H^\mu(\Omega)$, with $\Omega=I^d\subset R^d$ and I is a bounded open interval of R. In this section, we consider orthogonal approximations in mutiple dimensions. Let $k=(k_1,k_2;\dots,k_d)$, $|k|=\sum_{i=1}^{d}|k_i|$, k_i being any non-negative integers, and $\partial_x^k\Phi=\frac{\partial^{|k|}u}{\partial^{k_1}x_1\dots\partial^{k_d}x_d}\Phi$. For $\mu\ge 0$, we define the Sobolev space

$$H^\mu(\Omega)=\{\Phi,\partial_x^k\Phi\in L^2(\Omega),0\le|k|\le\mu\},\tag{25}$$

with the norm

$$\|\Phi\|_\mu^2=\sum_{k\in N^d,k_1+k_2+\dots k_d<\mu}\int_\Omega\left|\left(\prod_{j=1}^{d}D_j^{k_j}\right)\Phi\right|^2dx,\tag{26}$$

where $D_j=\frac{\partial}{\partial x^{(j)}}$. If $\{\Phi_k\}_{k=0}^{\infty}$ is the system of orthonormal in $L^2(I)$ with $\deg\Phi_k=k$ then the system $\{\Phi_k\}_{k\in N^d}$, where $\Phi_k(x)=\prod_{j=1}^{d}\Phi_{k_j}(x^{(j)})$ is complete and orthonormal in $L^2(\Omega)$ and any $u\in L^2(\Omega)$ is as follows

$$u=\sum_{k\in N^d}u_k\Phi_k,\quad u_k=\langle u,\Phi_k\rangle,\tag{27}$$

with $\|u\|_0^2=\sum_{k\in N^d}|u_k|^2$. Setting,

$$S_N=S_N(\Omega)=\{\text{span}\}\{\Phi_k:k\in N^d,\max(k)<N\},\tag{28}$$

S_N is the set of all polynomials of degree at most N in each variable $x^{(j)},j=1,\dots,d$. Let $P_N:L^2(\Omega)\longrightarrow S_N$ the orthogonal projection on S_N in $L^2(\Omega)$.

Theorem 1. [26] *For any real* $\mu>0$*, there exists a constant* C *such that*

$$\|u-P_Nu\|_0\le CN^{-\mu}\|u\|_\mu,\quad\forall u\in H^\mu(\Omega).\tag{29}$$

For the error estimation of $u-P_N$ in Sobolev space, we need the following lemmas.

Lemma 1. [26] *For any real* μ *and* r *such that* $0<r<\mu$*, there exists a constant* C *such that*

$$\|u\|_\mu\le CN^{2(r-\mu)}\|u\|_r,\quad\forall u\in S_N.\tag{30}$$

Lemma 2. *For the two real* r *and* μ *with* $0\le r\le\mu-1$ *there exists a constant* C *so that for* $j=1,\dots,d$

$$\|(P_ND_j-D_jP_N)u\|_r\le CN^{(2r-\mu+3/2)}\|u\|_\mu,\quad\forall u\in H^\mu(\Omega).\tag{31}$$

(31)‖(����−����)�‖�≤��(2�−�+3/2)‖�‖�,∀�∈��(Ω).

The following theorems gives the error estimation of the approximation of u�

Theorem 2. *For the two real* μ� *and* r� *with* 0≤r≤μ0≤�≤�*, we can get a constant* C� *so that*

‖u−PNu‖r≤CNe(r,μ)‖u‖μ, ∀u∈Hμ(Ω),(32)(32)‖�−���‖�≤���(�,�)‖�‖�, ∀�∈��(Ω),

where

e(r,μ)={2r−μ−1/2,3r/2−μ,r≥10≤r≤1�(�,�)={2�−�−1/2,�≥13�/2−�,0≤�≤1

Proof 1. *For* r=0�=0*, 32 reduces to 29. Now suppose that 32 holds for any integer* r≤m−1�≤�−1 *by inductive hypothesis. Then*

‖u−PNu‖m≤∑j=1d‖Dju−DjPNu‖m−1‖�−���‖�≤∑�=1�‖���−�����‖�−1

≤∑j=1d‖Dju−PNDju‖m−1+∑j=1d‖PNDju−DjPNu‖m−1,≤∑�=1�‖���−�����‖�−1+∑�=1�‖�����−�����‖�−1,

by taking the inductive hypothesis for Dj∈Hμ−1��∈��−1 *and by using 31, we get*

‖u−PNu‖m≤C′Ne(m−1,μ−1)∑j=1d‖Dju‖μ−1+c″Ne(m,μ)‖u‖μ,‖�−���‖�≤�′��(�−1,�−1)∑�=1�‖���‖�−1+�″��(�,�)‖�‖�,

we have e(m−1,μ−1)<e(m,μ)�(�−1,�−1)<�(�,�)*, then we get the results 32.*

Theorem 3. *Suppose that* u(x,t)∈Hμ(Ω)�(�,�)∈��(Ω) *and* u¯(x,t)∈Hμ(Ω)�¯(�,�)∈��(Ω) *with* μ≥0�≥0 *be the exact and the numerical solution of equation (1), respectively. Also, suppose* K1�1 *and* K2�2 *satisfy the following uniform Lipschitz conditions*

|K1(x,t,s,u1)−K1(x,t,s,u2)|≤l1|u1−u2|,|K2(x,t,s,u1)−K2(x,t,s,u2)|≤l2|u1−u2|.(33)(33)|�1(�,�,�,�1)−�1(�,�,�,�2)|≤�1|�1−�2|,|�2(�,�,�,�1)−�2(�,�,�,�2)|≤�2|�1−�2|.

Then, for any real μ� *and* r� *such that* 0≤r<μ0≤�<� *the error bound* EN�� *of the present method is given by*

‖EN‖Hμ(Ω)≤[(C+λ1C)N(2r−μ+3/2)+(l1λ2+l2λ3)cNe(r,μ)‖‖‖u‖Hμ(Ω),(34)(34)‖��‖��(Ω)≤[(�+�1�)�(2�−�+3/2)+(�1�2+�2�3)���(�,�)‖‖�‖��(Ω),

where

e(r,μ)={2r−μ−1/2,3r/2−μ,r≥10≤r≤1.�(�,�)={2�−�−1/2,�≥13�/2−�,0≤�≤1.

Proof 2. *Using equation 1, we get*

||EN||Hr(Ω)=−≤+≤+||ut(x,t)+λ1uxx(x,t)−λ2∫t0K1(x,t,s,u(x,s))ds−λ3∫T0K2(x,t,s,u(x,s))ds(u¯t(x,t)+λ1u¯xx(x,t)−λ2∫t0K1(x,t,s,u¯(x,s))ds−λ3∫T0K2(x,t,s,u¯(x,s))ds)||Hr(Ω)||ut(x,t)−u¯t(x,t)||Hr(Ω)+λ1||uxx(x,t)−u¯xx||Hr(Ω)λ2||

∫t0(K1(x,t,s,u(x,s))−K1(x,t,s,u¯(x,s)))ds||Hr(Ω)+λ3||

∫T0(K2(x,t,s,u(x,s))−K2(x,t,s,u¯(x,s)))ds||Hr(Ω)||ut(x,t)−u¯t(x,t)||Hr(Ω)+λ1||uxx(x,t)−u¯xx||Hr(Ω)λ2∫t0||K1(x,t,s,u(x,s))−K1(x,t,s,u¯(x,s))||Hr(Ω)ds+λ3∫T0||K2(x,t,s,u(x,s))−K2(x,t,s,u¯(x,s))||Hr(Ω)ds,(35)(35)||��||��(Ω)=||

��(�,�)+�1���(�,�)−�2∫0��1(�,�,�,�(�,�))��−�3∫0��2(�,�,�,�(�,�))��−(�¯�(�,�)+�1�¯��(�,�)−�2∫0��1(�,�,�,�¯(�,�))��−�3∫0��2(�,�,�,�¯(�,�))��)||��(Ω)≤||

��(�,�)−�¯�(�,�)||��(Ω)+�1||���(�,�)−�¯��||��(Ω)+�2||

∫0�(�1(�,�,�,�(�,�))−�1(�,�,�,�¯(�,�)))��||��(Ω)+�3||

∫0�(�2(�,�,�,�(�,�))−�2(�,�,�,�¯(�,�)))��||��(Ω)≤||��(�,�)−�¯�(�,�)||��(Ω)+�1||

���(�,�)−�¯��||��(Ω)+�2∫0�||�1(�,�,�,�(�,�))−�1(�,�,�,�¯(�,�))||��(Ω)��+�3∫0�||

�2(�,�,�,�(�,�))−�2(�,�,�,�¯(�,�))||��(Ω)��,

since K1�1 *and* K2�2 *satisfied Lipschitz conditions, then we have*

||EN||Hr(Ω)≤+||ut(x,t)−u¯t(x,t)||Hr(Ω)+λ1||uxx(x,t)−u¯xx||Hr(Ω)l1λ2∫t0||u(x,s))−u¯(x,s))||Hr(Ω)ds+l2λ3∫T0||u(x,s))−u¯(x,s))||Hr(Ω)ds,(36)(36)||��||��(Ω)≤||��(�,�)−�¯�(�,�)||��(Ω)+�1||���(�,�)−�¯��||��(Ω)+�1�2∫0�||

�(�,�))−�¯(�,�))||��(Ω)��+�2�3∫0�||�(�,�))−�¯(�,�))||��(Ω)��,

by using Lemma 31 and theorem 2, we get

||EN||Hr(Ω)≤+≤CN(2r−μ+3/2)||u||Hμ(Ω)+λ1CN(2r−μ+3/2)||u||Hμ(Ω)l1λ2∫t0cNe(r,μ)||u||Hμ(Ω)ds+l2λ3∫T0cNe(r,μ)||u||Hμ(Ω)ds[(C+λ1C)N(2r−μ+3/2)+(l1λ2+l2λ3)cNe(r,μ)]||u||Hμ(Ω),(37)||��||��(Ω)≤��(2�−�+3/2)||�||

��(Ω)+�1��(2�−�+3/2)||�||��(Ω)+�1�2∫0����(�,�)||�||��(Ω)��+�2�3∫0����(�,�)||�||

��(Ω)��(37)≤[(�+�1�)�(2�−�+3/2)+(�1�2+�2�3)���(�,�)]||�||��(Ω),

where

e(r,μ)={2r−μ−1/2,3r/2−μ,r≥10≤r≤1.�(�,�)={2�−�−1/2,�≥13�/2−�,0≤�≤1.

Then if u� *is is infinitely smooth, then* ‖ ‖ EN ‖ H′(Ω)⟶0‖‖��‖�′(Ω)⟶0 *as* N⟶∞�⟶∞.

5. Numerical implementation of the proposed algorithm

In this section, some numerical test equations are considered to shown the accuracy of the presented algorithm, where we have calculated the maximum absolute errors at different time. In these examples, the linear and nonlinear algebraic systems are solved by Newton iterative method and using MATLAB software.

Example 1. *Consider the PIDEs*

ut(x,t)−uxx(x,t)=g(x,t)−∫t0es−tu(x,s)ds,(38)(38)��(�,�)−���(�,�)=�(�,�)−∫0���−��(�,�)��,

with initial and boundary

conditions u(x,0)=x�(�,0)=�*,* u(0,t)=0�(0,�)=0*,* x∈[0,1],u(1,t)=e−t�∈[0,1],�(1,�)=�−�*,* t∈[0,1]�∈[0,1], *and* $g(x,t)=(2t-x^2-t^2x)\exp(-xt)+x(\exp(-t)-\exp(-xt))x^{-1}$. *The analytical solution for this example is* $u(x,t)=xe^{-xt}$. *The numerical experiments are given in table (1).*

t	$N=3$	$N=4$	$N=5$	$N=6$
0.0625	2.4918 E-4	1.6010 E-5	7.4614 E-7	2.5947 E-8
0.1250	3.1382 E-4	1.4463 E-5	3.7456 E-7	6.8578 E-9
0.1875	2.6376 E-4	6.1076 E-6	1.2642 E-7	1.3100 E-8
0.2500	1.5539 E-4	2.3487 E-6	3.5723 E-7	9.0718 E-9
0.3125	6.4499 E-5	7.4193 E-6	2.7735 E-7	2.9712 E-9
0.3750	1.4299 E-5	8.0728 E-6	1.1176 E-7	1.0793E-8
0.4375	2.5590E-4	6.6364 E-5	2.1644 E-7	9.4010E-9
0.500	3.5960 E-4	1.1409 E-5	3.4232 E-7	8.3644 E-9
0.5625	4.6127 E-4	1.8037 E-5	6.0631 E-7	1.6629 E-8
0.6250	6.2660 E-4	2.6913 E-5	1.0077 E-6	3.0685 E-8
0.6875	8.2174 E-4	3.8470 E-5	1.5912 E-6	5.3212E-8
0.7500	1.0486 E-3	5.3175 E-5	2.4089 E-6	8.7621 E-8
0.8125	1.3090 E-3	7.1535 E-5	3.5395E-6	2.1009 E-7
0.8750	1.6050 E-3	9.4094 E-5	5.0403 E-6	3.0975 E-7
0.9375	1.9385 E-3	1.2143 E-4	6.9854 E-6	4.4474 E-7

Table 1. Errors using OBP method for test (1).

Example 2. *Let given the nonlinear equation*

$$u_t(x,t)=u_{xx}(x,t)+g(x,t)-\int_0^1 u^2(x,s)\,ds, \quad (39)$$

with conditions $u(x,0)=x^2$, $u(0,t)=t^2$, $x\in[0,1]$, $u(1,t)=t^2+1$, $t\in[0,1]$. *The function* $g(x,t)$ *is obtained from the analytical solution* $u(x,t)=t^2+x^2$. *The numerical results are presented in figure 1.*

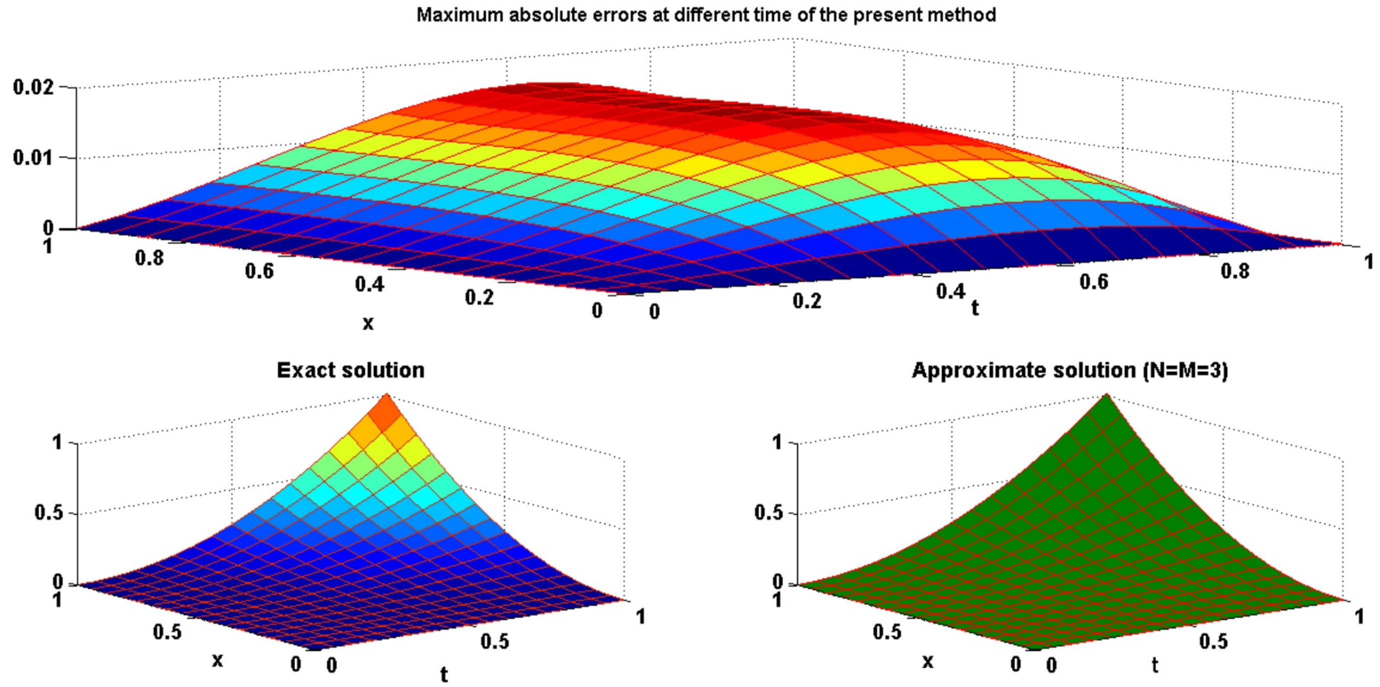

Figure 1. Exact (left) and approximate (right) solutions for example 2 for ($N = M = 3$).

Example 3. [27] *Let consider the following partial integro-differential equation*

$$u_t(x,t)-u_{xx}(x,t)=g(x,t)-\int_0^t e^{x(t-s)}u(x,s)\,ds \quad (40)$$

with conditions

$$u(x,0)=0,\ x\in[0,1],\quad u(0,t)=\sin(t),\ u(1,t)=0,\ t\in[0,1],$$

With $g(x,t)=(1-x^2)\cos(t)+2\sin(t)+(x^2-1)\cos(t)+x\sin(t)-\frac{e^{xt}}{x^2+1}$. *The exact solution is given by* $u(x,t)=(1-x^2)\sin(t)$. *The numerical results of example 3 are summarized in table 2 and figure 2. Table 3 gives a comparison between the proposed method in* [27] *and Cardinal Chebyshev functions* [28]*. Better accuracy than the other methods.*

Example 4. *In this example* [15]*, we take a diffusion problem as*

$$u_t(x,t)=u_{xx}(x,t)+g(x,t)-\int_0^t \frac{t-s+1}{x+1}u(x,s)\,ds,\quad x,t\in[0,1], \quad (41)$$

where $g(x,t)$ *is determined such that the solution is* $u(x,t)=\frac{1-x}{21+t^2}$*. The numerical results for this example are summarized in figure 3. Our numerical tests are better than that given by Legendre multi-wavelets collocation method* [27]*.*

t	$N = 3$	$N = 4$	$N = 5$	$N = 6$
0.0625	2.2589 E-4	4.2169 E-5	1.4692 E-6	1.5646 E-7
0.1250	3.2716 E-4	4.9342 E-5	1.4308 E-6	1.2011 E-7
0.1875	3.3538 E-4	3.9105 E-5	8.5875 E-7	5.3754 E-8
0.2500	2.8047 E-4	2.2820 E-5	3.0315 E-7	1.2997 E-8
0.3125	1.9333 E-4	1.0381 E-5	1.9737 E-7	4.1951 E-9
0.3750	9.6781 E-5	1.1357 E-5	1.4119 E-7	1.5299 E-8
0.4375	1.0161 E-4	8.9784 E-6	1.6239 E-7	1.4767 E-8
0.500	1.2012 E-4	6.8816 E-6	2.3785 E-7	1.0269 E-8
0.5625	1.1537 E-4	6.8278 E-6	2.0673 E-7	1.2452 E-8
0.6250	1.0561 E-4	1.0505 E-6	1.3352 E-7	1.4616 E-8
0.6875	7.0710 E-5	9.1550 E-6	2.0449 E-7	1.2901 E-8
0.7500	9.5995 E-5	6.0418 E-6	2.5585 E-7	1.1463 E-8
0.8125	1.4199 E-4	6.3528 E-6	2.2374 E-7	1.5682 E-8
0.8750	1.1942 E-4	1.2261 E-5	2.2730 E-7	5.8400 E-9
0.9375	7.4179 E-5	8.711 E-6	2.8581 E-7	2.0667 E-8

Table 2. Errors of the present method using OBP method for test (3).

	LMW Collocation Method [27]			Chebyshev CF [28]	Present Method
t	$N = 8$	$N = 16$	$N = 32$	$N = 8$	$N = 8$
0.0625	7.4383 E-5	4.6240 E-6	1.2106 E-5	2.2070 E-8	2.5266 E-10
0.1250					1.2593 E-10
0.1875	7.5155 E- 5	1.2275 E-5	2.4685E-5	1.1514 E- 9	4.6585 E-11
0.2500					3.1922 E-11
0.3125	1.4643 E-4	2.5696 E-5	3.5745 E- 5	4.8570 E-8	2.8864 E-11
0.3750					1.3091 E-11
0.4375	7.5929 E-5	4.2169 E-5	4.5563 E-5	1.4616 E- 9	1.2449 E-11
0.500					1.0695 E-11
0.5625	1.2180 E-4	6.0743 E-5	5.3926 E- 5	1.7855 E-9	1.2261 E-11
0.6250					9.3796 E-12
0.6875	1.0567 E-4	8.1933 E-5	6.0499 E- 5	1.0870 E-7	6.6454 E-12
0.7500					1.3368 E-11
0.8125	4.7215 E- 5	1.0738 E- 4	6.4915 E-5	5.3619 E- 9	1.1561 E-11
0.8750					1.5320 E-11
0.9375	2.1869 E-4	1.3833 E-4	6.6396 E-5	3.8717 E-7	1.2016 E-11

Table 3. Errors of the present method using OBBP method for test (3).

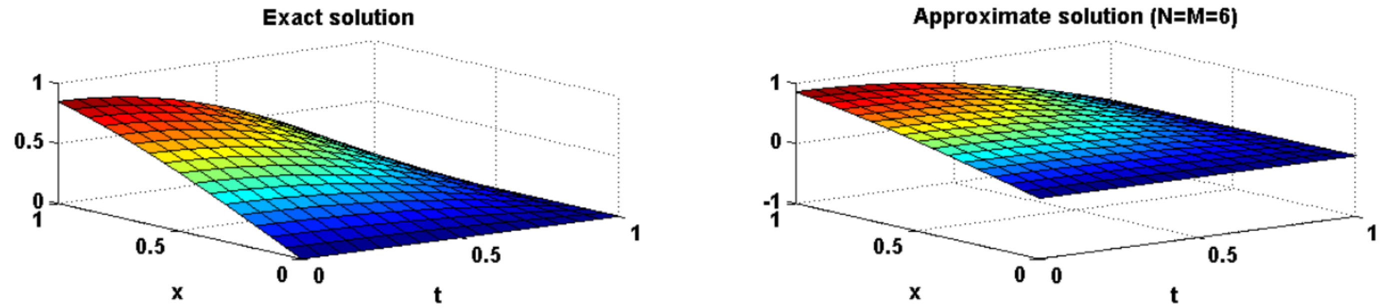

Figure 2. Exact (left) and approximate (right) solutions for example 3 for ($N = M = 6$).

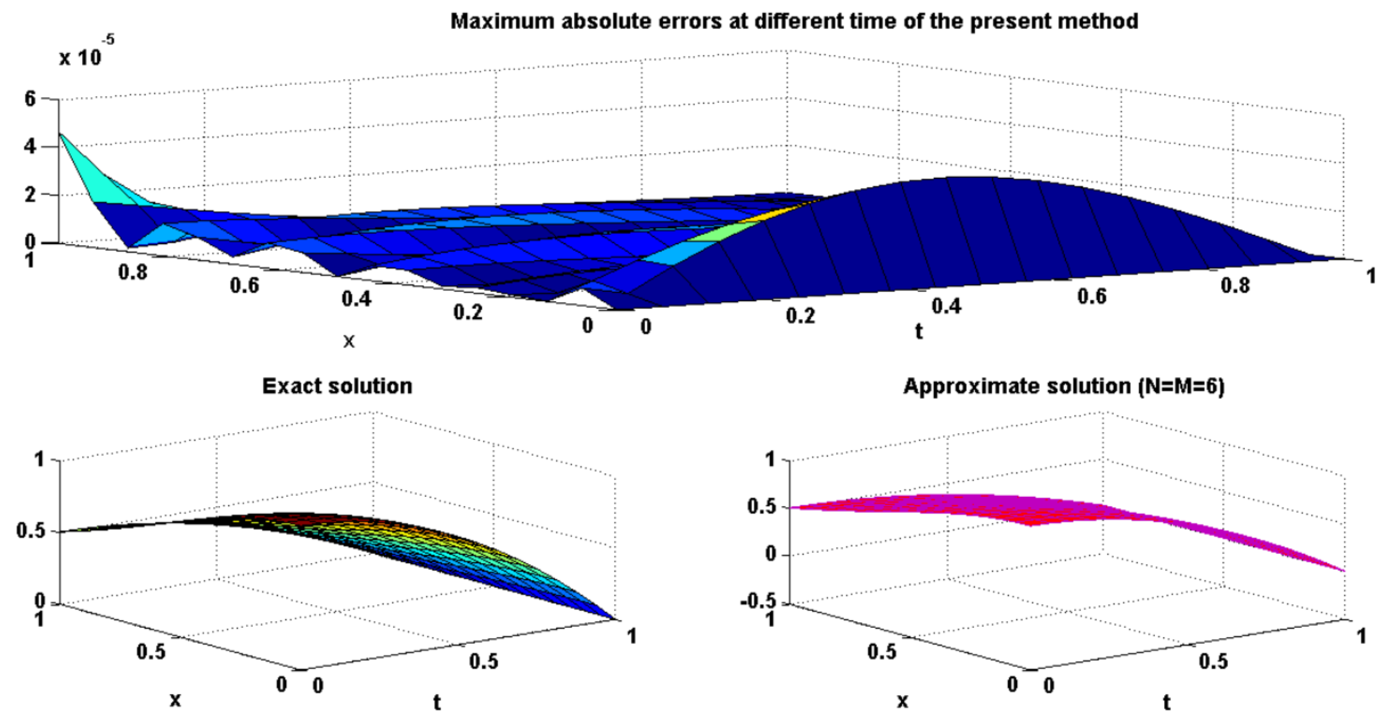

Figure 3. Errors, Exact (left) and approximate (right) solutions for example 4 for ($N = M = 6$).

Example 5. *We are given a linear problem as follows*[27]

$$u_t(x,t)=u_{xx}(x,t)+g(x,t)-\int_0^t u(x,s)\,ds,\quad x,t\in[0,1], \tag{42}$$

subject to the following conditions: $u(x,0)=\frac{1-x^2}{2}$, $u(0,t)=\frac{\cosh(t)}{2+\sinh^2(t)}$ *and* $u(1,t)=0$, *where* $g(x,t)$ *is determined such that the analytical solution is* $u(x,t)=\frac{(1-x^2)\cosh(t)}{2+\sinh^2(t)}$. *The results for this example are given in tables 4-5 and figure 4. The numerical experiments obtained for this example are better than that given by Legendre multi-wavelets collocation method* [27].

t	N=4	N=5	N=8
0.0625	2.3407E-4	5.1581 E-6	3.1127 E-7
0.1250	2.7731 E-4	3.7329 E-6	1.5271 E-7
0.1875	2.2190 E-4	2.1161 E-6	5.2969 E-8
0.2500	1.2981E-4	1.5113 E-6	4.0246 E-8
0.3125	5.9121 E-5	1.3529 E-6	4.0669 E-8
0.3750	6.5468E-5	9.7856 E-7	1.8271 E-8
0.4375	5.3368 E-5	2.8124 E-7	1.6847 E-8
0.500	4.0608 E-5	1.0203 E-6	1.7874 E-8
0.5625	4.0269E-5	1.6929 E-6	1.7609 E-8
0.6250	6.2131 E-5	1.3648 E-6	1.7117 E-8
0.6875	5.4164 E-5	7.2680 E-7	1.0399 E-8
0.7500	3.7198 E-5	1.8706 E-6	2.1713 E-8
0.8125	3.7310 E-5	2.2765 E-6	1.8944 E-8
0.8750	7.1458E-5	1.6466 E-6	2.6507 E-8
0.9375	4.7623 E-5	2.8961 E-6	2.1335 E-8

Table 4. Errors of the present method using OBBP method for test (5).

Example 6. *Here, we take the following PIDE*

$$u_t(x,t)+u_{xx}(x,t)=g(x,t)-\int_0^t e^{s-t}u(x,s)\,ds \qquad (43)$$

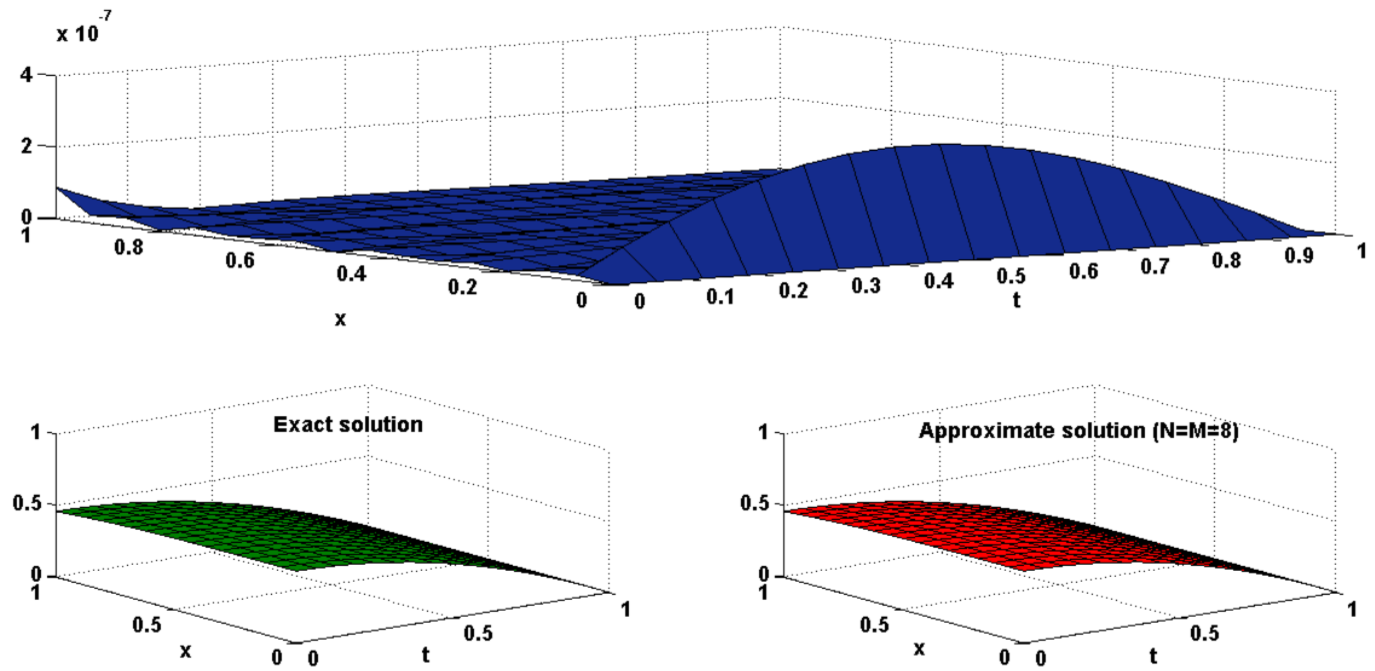

Figure 4. Errors, Exact (left) and approximate (right) solutions for example 5 for ($N = M = 8$).

	Legendre multiwavelets Method [27]			Present Method
t	$N = 8 \times 8$	$N = 16 \times 16$	$N = 64 \times 64$	Present method N=8
0.1	1.8049 E-6	9.2110 E-7	1.1342 E-7	2.2094 E-7
0.2	1.3464 E-5	2.3295 E-6	4.3291 E-8	4.5662 E-8
0.3	7.2956 E-5	1.9806 E-6	1.8598 E-7	4.2211 E-8
0.4	4.4007 E-5	1.7830 E-5	5.6575 E-7	9.7235 E-9
0.5	3.7671 E-4	7.7404 E-5	4.0517 E-6	1.7920 E-8
0.6	4.5192 E-5	9.5238 E-6	1.0850 E-6	1.7605 E-8
0.7	2.3648 E-5	2.0197 E-5	1.1987 E-6	7.2310 E-9
0.8	8.3275 E-5	2.3185 E-5	1.5333 E-6	2.0556 E-8
0.9	1.0790 E-4	1.1975 E-5	7.5777 E-7	1.8646 E-8
1.0	3.4302 E-4	8.6424 E-5	4.5429 E-6	2.5563 E-8

Table 5. Errors using OBBP method for test (5).

with $u(x,0)=x$, $u(0,t)=0$, $x\in[0,1]$, $u(1,t)=e^{-t}$, $t\in[0,1]$, *where* $g(x,t)$ *is determined such that* $u(x,t)=x\exp(-xt)$ *is the analytical solution. We remark that when* N *increases, the error decreases. The errors obtained by our method for* $N=10$ *are presented in 5 and gives a better results than that given by Hermite-Taylor matrix method for* $N=12$ [9] *and radial basis functions* $N=40$ [14].

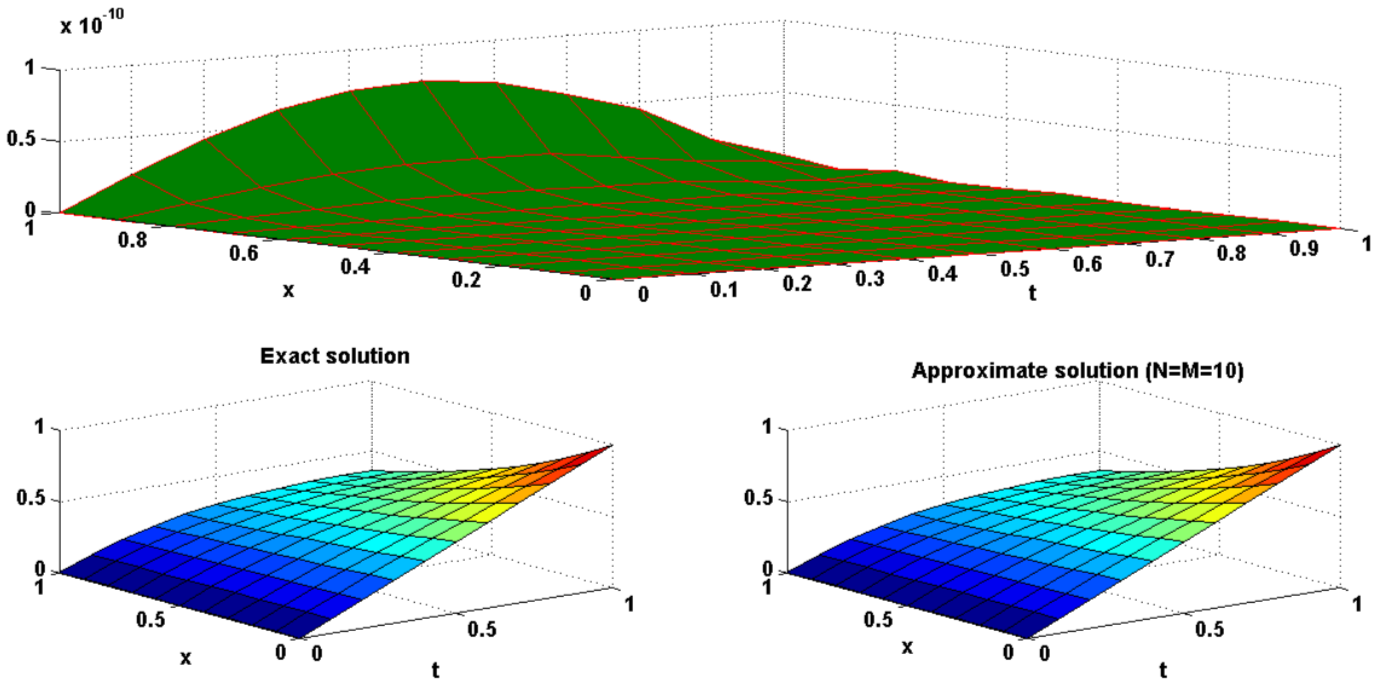

Figure 5. Errors, Exact (left) and approximate (right) solutions for example 6 for ($N = M = 10$).

6. Conclusion

In this article, a new numerical approach was proposed. This approach was utilized to solve partial integro-differential equations with Volterra and Fredholm types. The matrices of orthonormal Bernoulli polynomials were derived and used to obtain the approximate solution of PIDEs. After we take Gauss-Legendre nodes in the intervals [0,b][0,�] and [0,T][0,�] as collocation points. The approach was applied to obtain numerical solutions of some test problems. The numerical results show the high accuracy of the scheduled algorithm. The presented method is easily implementable and simple and can be used for different types of PIDEs and also for differential equations. Many test problems were inserted and compared with other algorithms to appreciate the good efficiency of the proposed methodology. The proposed algorithm can be employed to more dimensions.

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Open Peer Review

Review this Article

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BZ

Bo Zhu posted a **Review**

December 8, 2023

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https://doi.org/10.32388/4SOGB5

This paper consider an acurrate numerical appraoch based on orthonormal Bernoulli polynomials for solving parabolic partial integro-differential equations (PIDEs).
This type of equations arises in physics and engineering.
Some operational matrix are given for these polynomials and are also used to obtain the numerical solution. By this approach, the problem is transformed into a nonlinear algebraic system.
The study of this problem is very meaningful, the method is good. The results improve and extend some relevant results in this area. And also this paper is well organized, contains all the basic concepts that are used, the writing is clear and the overall presentation is satisfactory.
Conclusion Summarizing, the results are interesting in their context. I recommend the paper for publication in Qeios.

See more

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QX

Qiuyan Xu posted a **Review**

December 8, 2023

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https://doi.org/10.32388/6ZF57J

The author proposes a new adaptative numerical algorithm for solving partial integro-differential equations. I recommend the publication of this work after the following commentss are addressed in the revison:

1. Many abbreviations in the text should be given their full names when then first appear.
2. Many symbols should be unified, such as Eq. (6) and (14).
3. Punctuation marks should be added after the formula (15).
4. On page 8, some formula bracket are missed.
5. provide the difference when N is not equal to M.
6. Does the table 3 is completed?

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yT

younes Talaei posted a **Review**

December 7, 2023

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https://doi.org/10.32388/YVS089

This paper addresses a spectral collocation method based on orthonormal Bernoulli polynomials for solving parabolic partial integro-differential equations. Although several numerical examples are provided to demonstrate the accuracy of the method, there is no analysis on the computational complexity, order of convergence and CPU-times.

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MB

Monica Botros posted a **Review**

December 7, 2023

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https://doi.org/10.32388/LXESP1

1. How do the orthonormal Bernoulli polynomials contribute to the accuracy of your numerical algorithm, and how are they utilized in solving PIDEs?
2. The abstract mentions a comparison with other well-known methods. Could you elaborate on the specific methods you compared your approach to and the criteria used for comparison?
3. The introduction mentions various applications of orthonormal functions in solving numerical problems. Could you provide a more detailed explanation of how orthonormal Bernoulli polynomials specifically benefit the solution of parabolic PIDEs?
4. How do you determine the order of the orthonormal Bernoulli polynomials (N+1 and M+1) in the approximation process?
5. In Equation (9), what is the significance of the matrices Φb,N(x), U, and ΦT,M(t), and how are they related to the overall approximation?
6. Could you provide more details on how the pseudo-spectral method is employed to solve PIDEs, and why the choice of N=M is made in your work?
7. In Example 3, a comparison is made with another method. Could you explain the criteria used for comparison and the reasons for the observed better accuracy in your method?

See more

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DV

Dr. Jasvinder singh Virdi posted a **Review**

December 7, 2023

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https://doi.org/10.32388/Z0O97T

Author has tried to calculate theoretically or to investigate the accurate numerical appraoch based on orthonormal

Bernoulli polynomials for solving parabolic partial integro- differential equations (PIDEs).

A a new numerical approach was adopted.

But some modifications do required

1. Last few lines of 2^{nd} paragraph in introduction Author has tried to explain what various methods, that is well known.
2. Equations 1, 10, 18,19,20 etc … needs clear writing with proper format????
3. Further eq 35 things are not clear.

With this modification article may be accepted.

See more

0 0

1 comment

-

DC

Dimplekumar Chalishajar posted a **Review**

December 6, 2023

•••

https://doi.org/10.32388/VGU1UA

- In this article, authors have proposed a new numerical approach to solve partial integro-differential equations with Volterra and Fredholm types. The matrices of orthonormal Bernoulli polynomials were derived and used to obtain the approximate solution of PIDEs. After we take Gauss-Legendre nodes in the intervals [0,b] and [0,T] as collocation points. The numerical results show the high accuracy of the scheduled algorithm. The presented method is easily implementable and simple and can be used for different types of PIDEs and also for differential equations. I have the following concerns:
- Author mentioned that the proposed algorithm can be employed to more dimensions. Please explain in detail.
- Figure 1-4, captions are not made properly. Please relate your captions with the context of the problem.
- Literature survey should be strengthened with the latest developments.
- What are the drawbacks and limitations of the proposed method?

See more

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2 comments

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AK

Alamgir Khan posted a **Review**

December 6, 2023

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https://doi.org/10.32388/8004UJ

Manuscript Title: New adaptative numerical algorithm for solving partial integro-di erential equations

Reviewer comments:

The topic of this paper is interesting to me, the paper requires some minor revision before it can be considered for publication in this reputable journal, and my comments are listed as follows:

1. The abstract should be improved.
2. Page No#1, In the Introduction, line 2 has some grammatical errors that must be corrected.
3. The work should be checked for spelling, spacing, and grammatical errors.
4. A few equations are not well aligned according to page alignment and must be properly aligned.
5. To improve the section “conclusion", please add some suggestions for future works in this area to the end of this section.
6. More discussion of physical mechanisms is needed in result and discussions.
7. The quality of the graphs should be improved.
8. Where is the benefit of this problem in real life?

Overall, the manuscript is well-written, and the methodology and results are clearly presented. Introduction section can be supported with some more recent related literature:

- https://doi.org/10.1007/s40819-023-01593-5

See more

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GW

Guanqian Wang posted a **Review**

December 5, 2023

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https://doi.org/10.32388/JZR9JM

In this paper, a numerical method based on orthonormal Bernoulli polynomials is proposed to solve a class of PIDEs (partial integro-differential equations). The paper has four main parts. In the first section, the type of the PIDEs is given with some background. The approximation method (Gauss Legendre quadrature) for the integral part in the PIDEs is also provided in this section. In sections 2 and 3, the setup and some derivations of the numerical methods are given. In section 4, the analysis of the error is provided, and the boundedness of the error is proved. In section 5, several numerical tests are provided; the comparison between the newly proposed method and other methods is given; it is stated that the new method has higher accuracy.

There are a few issues and suggestions regarding this paper:
(1) The lemma 2 does not have a proof or reference. The reference or proof needs to be added.
(2) In the conclusion, the author declares the method can be applied to higher dimensions; there should be an example to showcase the application for higher-dimensional cases and compare with other well-established methods.
(3) The author compared the new method with other methods without any analysis or discussion. If possible, there should be some analysis or at least some discussion regarding the reason why the new method is faster than other established methods.
(4) It has some expression issues. For instance, the author used OBP to denote orthonormal Bernoulli polynomials without reporting it.
(5) It has some display errors. For instance, at the beginning of proof 1, it says "32 reduces to 29"; it should be (32) reduces to (29). Similar errors occurred many times in the following proof. In proof 2, after equation (36), it should be lemma 2, not lemma 31.
(6) It has some grammatical errors. For instance, at the beginning of section 5, it says "considered to shown," which should be "to show."

See more

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TN

Tahir Naseem posted a **Review**

December 5, 2023

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https://doi.org/10.32388/8O91QU

In my opinion, the paper seems to be scientifically correct and interesting, and I recommend it be accepted for publication after taking the following important points during the preparation of the revised paper:

1.The Introduction should make a compelling case for why the study is useful and a clear statement of its novelty or originality by providing relevant information and answers to basic questions such as: What is already known in the open literature? What is missing (i.e., research gaps)? What needs to be done, why, and how? Clear statements about the novelty of the work should also appear briefly in the introduction section.

2. mention software used to solve the modeled problems in the paper.

3. List the advantages of the used scheme in an order of convergence and accuracy.

See more

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NV

Nathalie Verdière posted a **Review**

December 5, 2023

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https://doi.org/10.32388/5HOK1K

The article is well-written, clear and the calculations well mastered. It aims to propose a new adaptative numerical algorithms to solve PIDEs.

I have only some minor remarks:

1. page 2: SDE must be explicited

2. page 3:

- same remark for OBP (line 2)
- Eq. (7) and afterwards, there is a problem of indexes with \Phi
- part 3, first line "we describe our numerical techniques t to solve PIDEs": remove t

3. page 5, eq. (20): k -> K

4. page 6, line 1 after r1, remove the dot

5. page 7,

- line 2, u should be substituted by \Phi
- the line after equ. (26): system of orthonormal -> orthonormal system in L^2
- The line after: Could you precise the notation x^(j) and a bracket is missing

6. page 8: brackets are missing around the equation number.

7. page 9, section 5: shown ->show, time ->times

8. page 18 [5], MAathematics -> Mathematics

See more

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MP

Marzieh Pourbabaee posted a **Review**

December 5, 2023

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https://doi.org/10.32388/L7NWYN

Review of the manuscript:

“New adaptive numerical algorithm for solving partial integro differential equations”

For “Qeios”

In this paper, the author introduces an accurate numerical approach based on orthonormal Bernoulli polynomials for solving parabolic partial integro-differential equations (PIDEs). By this approach, the problem is transformed into a nonlinear algebraic system. Also, the author has a discussion about convergence for the proposed scheme, and finally, some numerical examples are presented to test the validity of the established difference scheme.

I think that the results obtained in this paper are sound and interesting. This paper can be considered for publication after the authors address the following comments and suggestions:

1. There are some grammatical errors. Please check the whole of the paper, carefully.
2. Please check the in Eq. (6). In the rest of the paper, you have used the symbol .
3. In Eq. (10) u(x) and x) should be replace to u(x,t) and t) respectively.
4. Please replace with in Eq. (20).
5. The coefficients and are missing in the above relation (24).
6. Please check the Lemma 2. I ask the author to cite or proof for this lemma.
7. Please cite the Theorem 2 and check the proof.
8. At the end of proof 2, Lemma 31 should be replaced with Lemma 2 and I ask the author that check this proof carefully. What does happen for integration??
9. In the whole of the paper caption of figures should be set in the below figures.
10. The author says that the obtained error for example 6 presented in Table 5. But the caption of this table is for example 5. Please check it.
11. The style of references is different. Please check them.
12. Please cite doi: 10.11948/20230039 in introduce Bernoulli polynomials.

See more

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KI

Kais Ismail Ibraheem posted a **Review**

December 5, 2023

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https://doi.org/10.32388/9LL6EM

- Please add summarized finding of this paper at the end of the abstract.

- Add some discussion for Figure 4 and 5.

- explain some founding from Table 2 and 3.

- add clear future work in the conclusion.

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MA

M. H.T. Alshbool posted a **Review**

December 4, 2023

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https://doi.org/10.32388/K1F8YS

I agree to publish this article, nice writing , interested results, manage with perfect formatting

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1 comment

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TG

Tao Guo posted a **Review**

December 3, 2023

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https://doi.org/10.32388/UQUQ7C

In this paper, a numerical method based on orthogonal Bernoulli polynomials (OBP) is proposed to solve parabolic partial integro-differential equations. Through OBP matrix, the approximate solution is obtained, and the given problem is transformed into a nonlinear algebraic system. Through numerical experiments and comparison with other methods, the advantages and accuracy of this method are highlighted. But on the whole, there are still many defects that need major revision.

1. The author should scrutinize manuscripts for typos and/or grammatical error, e.g., on page 5, the word “We” should be changed to “we” in the line 4. In addition, the English structure of the article, including punctuation, semicolons and other structures, must be scrutinized.
2. Please show the innovation of the article. In fact, there has been a lot of work.
3. Please provide a nonlinear numerical example. In addition, reasonable, accurate and detailed charts should be provided so that readers can better understand the meaning of the paper.
4. In the process of writing a manuscript, the author should pay attention to some writing standards, and symbols and equations should be consistent. For example, some parts of the article use () to introduce equations, while others do not.
5. Please note that the up-to-date of references will contribute to the up-to-date of your manuscript.
https://doi.org/10.1002/mma.9788,
https://doi.org/10.1007/s10444-023-10050-2.

See more

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FO

Francisco Ortegón Gallego posted a **Review**

December 2, 2023

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https://doi.org/10.32388/ATUM0E

This can be an interesting paper on the approximation of certain integro-differential equations by means of orthonormal Bernoulli polynomials. The manuscript has many minor typos, and at least one major math mistake. Indeed, the estimate given in (36) cannot be deduced from the previous estimates by means of just the Lipschitz character of the functions K_1 and K_2 (maybe, a higher regularity for these functions should be needed). The estimate (36) holds true for r=0 and this fits with the computed errors shown in the examples.

Consequently, the authors should explain how the estimate (36) can be obtained for r>1 before this work may be considered for publication. Also, many typos should be corrected and review the written English (see the attached file).

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WQ

Wenlin Qiu posted a **Review**

December 2, 2023

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https://doi.org/10.32388/4BO4JT

In this manuscript, the author attempted to propose a numerical method based on orthogonal Bernoulli polynomials (OBP) for solving parabolic partial integro-differential equations. The matrices of OBP were derived and utilized to obtain the approximate solution. With the proposed method, the given problem was transformed into a nonlinear algebraic system. On the one hand, he did some experimental tests to validate the good accuracy of the numerical algorithm. On the other hand, by comparing the proposed technique with some other methods, the advantage of this approach was highlighted.
However, overall, there are significant flaws with this article. Therefore, major revision has to be done.

Major comments:

$\bullet$ The article lacks innovative points. In fact, there should already be a lot of related work in place.

$\bullet$ There are too many typos in the entire text, which need to be carefully checked. Meanwhile, regarding the use of grammar, it is hoped that the author can carefully consider it repeatedly.

$\bullet$ In terms of scientific rigor, the description of space in the Section 4 is not clear and the expression is confusing. Meanwhile, the characterization of mathematical models is not well presented.

$\bullet$ When writing a paper, please pay attention to different language environments, as there are many places in the article where italicized and nonitalicized fonts are used interchangeably for certain symbols and equations. Even in some parts of this manuscript, () is used when referencing equations, but there are others that are not. Please maintain consistency.

$\bullet$ Please cite some relatively laster articles and pay more attention to authoritative journals for references, such as:
https://doi.org/10.1007/s13540-023-00198-5,
https://doi.org/10.1016/j.cam.2023.115287,
https://doi.org/10.1007/s10092-023-00533-5,
https://doi.org/10.1007/s10444-023-10050-2.

$\bullet$ The effect of the figures and tables is not sufficiently highlighted. Please provide a reasonable, accurate, and detailed description in each example, combined with corresponding graphs and tables, so that readers can better understand the meaning paper want to express.

Minor comments:

$\bullet$ In page 2, the line above equation (3), the word ``oppeared" should be ``appeared". The same problem appears in the line above equation (22). Furthermore, it would be better to replace the word ``integral" with the ``integral term".

$\bullet$ In Page 3, the line above equation (5), what is the meaning of ``OBPs"? Does this word refer to ``orthogonal Bernoulli polynomial"? If that's the case, before abbreviations appear here, a prompt should be provided in the previous text.

$\bullet$ In page 3, equation (5), ``$i=0,1,\dots$" should be replaced by ``$i=0,1,\dots,N$". This type error in this manuscript needs to be thoroughly checked.

$\bullet$ In page 3, equation (6), the definition of $\Phi_{N,b}$ is given. However, all the symbols used in the following text are $\Phi_{b,N}$. Please maintain strict consistency.

$\bullet$ In page 3, Section 3, first line, remove the charecter ``t" between ``technique'' and ``to''.

$\bullet$ In page 4, the equation between (15) and (16), check and correct errors in punctuation usage. The same problem appears in the last equation in page 6 and the equation below (32).

$\bullet$ In page 6, top of the page, ``$k=0,1,2,\dots,r_{1}.$ are Gauss ...'', what does using a period here mean?

$\bullet$ In page 6, the sentence above Section 4, ``Gaus'' should be replaced by ``Gauss''.

$\bullet$ In page 7, top of the page, ``$k=(k_{1},k_{2};\dots,k_{d})$'' should be replaced by ``$k=(k_{1},k_{2},\dots,k_{d})$''.

$\bullet$ In page 7, bottom of the page, ``The following theorems gives'' should be replaced by ``The following theorems give''. Furthermore, why didn't you give a period at the end of the sentence?

$\bullet$ In page 8, Proof 1, ``32 reduces to 29'', do 32 and 29 here refer to equations? During the process of writing the manuscript, please pay attention to accurate expression.

$\bullet$ In page 8, the line above equation (34), review and correct grammar error. The same problem appears at the bottom of page 9 and the fourth line in page 15.

$\bullet$ In page 11, the equations below (40), pay attention to maintaining consistent spacing.

$\bullet$ In page 11, the sentence above Example 4, remove the period before ``better accuracy than ...''.

$\bullet$ In page 13, Table 3, please come up with a suitable name for Table 3 to highlight its intended expression. Similar problems arise in other graphs and tables.

$\bullet$ In page 15, top of the page, ``Le given'' should be replaced by ``Consider''.

$\bullet$ In page 18, reference [5], ``MAathematics of Computation'' should be replaced by ``Mathematics of Computation''.

See more

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ET

Elham Taghizadeh posted a **Review**

December 2, 2023

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https://doi.org/10.32388/8QZNDA

Review

Nov 30, 2023

Qeios ID: NT39BW

Open Access

https://doi.org/10.32388/NT39BW

Review of: New adaptative numerical algorithm for solving partial integro-differential equations

The reviewer(s) rated it 3/5

Dr. Srinivasarao Thota1

Declarations

In this paper, authors present a numerical approach based on orthonormal Bernoulli polynomials to solve the parabolic PIDEs. Using the proposed approach, the authors transformed the given problem into a nonlinear algebraic system. They also discussed the convergence analysis with some examples for the good accuracy of the numerical algorithm, and also the proposed technique was compared with some other well known methods to justify the proposed method.

The paper is written well. I think that it is good to publish in this journal.

See more

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AS

Abhishek Kumar Singh posted a **Review**

December 1, 2023

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https://doi.org/10.32388/U71SI2

A report on the paper titled New adaptative numerical algorithm for
solving partial integro-dierential equations
The authors in this manuscript propose a spectral method based on Bernoulli polynomials. This type of construction is not new to readers of the journal. Therefore, before a decision is made, the authors should answer the following questions and make changes to the manuscript:
 Remove all the typos.
 Authors should not introduce short form in the abstract.
 In the abstract, the author somehow used a numerical approach. I would suggest the author use a numerical algorithm everywhere in the manuscripts.
 In Paragraph 1 of the Introduction section, the author cites all references to the application of integral equations in one place. I would suggest the author use a reference in the appropriate place.
 Introduce short form correctly.
 In the Equation 1, what is the nature of K1 and K2, whether they are non-liner or linear.
 In Example 4, the authors said that their numerical results are better than the Legendre-wavelet collocation method. This is not true in general.
 The conclusion looks like an abstract. The authors should rewrite the conclusion in terms of novelty.
1

See more

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DT

Dr. Srinivasarao Thota posted a **Review**

November 30, 2023

...

https://doi.org/10.32388/NT39BW

In this paper, authors try to present a numerical approach based on orthonormal Bernoulli polynomials to solve the parabolic PIDEs. Using the proposed approach, the authors transformed the given problem into a nonlinear algebraic system. They also discussed the convergence analysis with some experiment tests for the good accuracy of the numerical algorithm, and also the proposed technique was compared with some other well known methods to justify the proposed method.

The paper is written well. However, there are comments to improve the quality of the paper.

Comments:

1. There are many typos. For example: in abstract, misspelled: acurrate , appraoch , etc.
2. Some symbols/equations are in italic and some are in non-italic format. These should be in uniform.
3. Whole paper needs to check for spelling and grammar corrections.

Over all, this paper is well organized and it can be acceptable for publication after revision.

See more

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1 comment

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MA

Mohammad A. Alqudah posted a **Review**

November 29, 2023

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https://doi.org/10.32388/4DK70Z

In this paper, a numerical approach based on orthonormal Bernoulli polynomials is developed for solving parabolic partial integro-differential equations. Using this approach, the problem is transformed into a nonlinear algebraic system. Error estimates for the orthogonal systems are then given.

1. The abstract is short; expand it with more details.
2. Add the motivation of the paper and a literature review.
3. The paper must provide an explanation of the innovation of the topic.

4. The proposed method is intended to solve nonlinear problems, so provide more nonlinear examples.
5. Provide a more in-depth discussion of the resulting algebraic systems.
6. Include a detailed algorithm for the proposed method so that readers can reproduce the results.
7. Additional numerical results must be included. Clearly state the numerical method used to solve the linear algebraic system.
8. Must provide a comparison between different methods to support the research of this article.
9. Include recent references to strengthen the literature review and show awareness of recent developments in the field.
10. Highlight the results obtained in the Conclusions section.
11. Carefully proofread the entire manuscript for English and grammatical errors to improve readability.

See more

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TA

T.Asadi Asadi posted a **Review**

November 29, 2023

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https://doi.org/10.32388/DIRLX8

Dear Editor

I have carefully read this article.

Overall, this article has an interesting topic and is well written. Before the final acceptance, it is suggested that authors apply the reviewer's comments to the article:

1- What is the advantage of using this method?

2- Discuss the accuracy and convergence of this method.

3- Abbreviations should be first comprehensively introduced in first uses.

4- All symbols are not defined in the text. Symbols must be presented in full in the article or “Nomenclature” section.

Best regards,

See more

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ZC

Z C posted a **Review**

November 29, 2023

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https://doi.org/10.32388/ARHFQ1

In this paper, a numerical approach based on orthonormal Bernoulli polynomials is developed for solving parabolic partial integro-differential equations. By using this approach the problem is transformed into a nonlinear algebraic system. Then the error estimates for the orthogonal systems are given.

I think there is a necessary explanation for the innovation in the paper. And the paper also needs to provide a comparison between different methods to support the research of this article.

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BB

Brahim Benhammouda posted a **Review**

November 28, 2023

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https://doi.org/10.32388/GISNHX

Comments:

1. Carefully review the entire manuscript for English language and grammatical errors to enhance its readability.
2. Show the novelty in the contributions of the paper. What distinguishes this work from existing literature?
3. Most references are old. Include more recent references to strengthen the literature review and show awareness of the latest developments in the field.
4. Include a detailed algorithm for the proposed method so that readers can replicate the results.

5. Give more discussion about the numerical results.

6. Provide a more in-depth discussion on the resulting algebraic systems.
7. Clearly state the numerical method used to solve the linear algebraic system.
8. In Table 3, include information on computing time along with the accuracy comparison.
9. The proposed method is supposed to solve nonlinear problems; however, all examples are linear except one. Use a wider range of examples, including nonlinearities like 1/(1+u^2) and cos(u), to showcase the method's applicability to different types of problems.
10. At the end of the introduction, briefly state the content of each section to provide readers with an overview of the paper's structure.

Other comments:

1. Abstract:
    - Check and correct spelling errors such as "acurrate" to "accurate" and "appraoch" to "approach." Also replace the word “matrix” by “matrices”
2. Page 3, section 3, first line:
    - Review and correct any errors in the first line of section 3.
3. Page 5, bottom:
    - Correct errors such as "noeuds" to "nodes" and "oppeared" to "appeared."
4. Page 7, top of the page, first sentence:
    - Correct the word "mutiple" to "multiple".
5. Page 9, last line of proof of Theorem 3:
    - Review and correct the last line of the proof of Theorem 3.
6. Page 9, section 5, first sentence:
    - Replace "to shown" with "to show".
7. Consistency:
    - Use the same notation for names of functions K_1 and K_2 consistently throughout the paper.
8. Page 6, line below equation (24):
    - Replace "boundary conditions given in (6)" with "boundary conditions given in (2)."
9. Page 8, Theorem 2:

- Include equation numbers in brackets for Theorem 2 and consistently throughout the paper.

10. Theorem 3:

- Use absolute values for Lambdas in Theorem 3. Check the reference to Lemma31 in the proof.

11. Figure 2 and figure 3:

- Ensure that graphs of exact and numerical solutions are plotted using similar frames. Also check the graphs.

12. Example 2:

- Provide a reference for Example 2 if available.

13. Example 3:

o Provide a reference for Example 3 if available. Also check the expression of g(x,t)

See more

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DJ

Davron Aslonqulovich Juraev posted a **Review**

November 28, 2023

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https://doi.org/10.32388/57KLS6

Review of the article “New adaptive numerical algorithm for solving partial integro-differential equations”

This article deals with new adaptive numerical algorithm for solving partial integro - differential equations.

The paper introduces an accurate numerical approach based on orthonormal Bernoulli polynomials for solving parabolic partial integro - differential equations (PIDEs). This type of equations arises in physics and engineering. Some operational matrixes are given for these polynomials and are also used to obtain the numerical solution. By this approach, the problem is transformed into a nonlinear algebraic system. Convergence analysis is given and some experiment tests are studied to examine the good accuracy of the numerical algorithm, the proposed technique is compared with some other well-known methods.

Various approximations by orthonormal family of functions have been investigated in physical sciences, engineering, etc. This type of numerical approximations can be also used in optimal control problems and in general to approximate solutions of dynamical systems. Integral equations arise in many physical problems, diffusion problems, concrete problem of physics and mechanics and some others problems of engineering, different applications of potential theory, synthesis problem, mathematical modelling of economics, population, geophysics, antenna, genetics, communication theory, radiation problems, concerning transport of particles, etc.

I can say that the work is very well written, the results are new. The theorems are fully proven and correct. Several examples are given that fully reveal the meaning of this study. The results are presented in a table and also shown in drawings.

I would like to make the following remarks regarding the article:

1) The grammar of the article should be corrected, for example, the title of the article contains the words “adaptative”, this word should be written as “adaptive”; in the abstract section the word “acurrate” should be written as “accurate”; the word "appraoch" should be written as "approach" and then.

2) Formula (35) has gone beyond the page, I advise you to write it on a new line.

3) In some formulas it is correct to put punctuation marks, for example in formula (13) you need to put a period, not a comma, in the formula after the formula you need to put punctuation marks (15).

These comments do not affect the meaning and quality of the article; it can be easily corrected.

With the exception of minor corrections, I recommend this article for publication.

Best regards, Reviewer

Head of Department of Scientific research, Innovation and Training of Scientific and Pedagogical Staff of University of Economics and Pedagogy, Ph.D. of Physics and Mathematics Sciences,

Associate Professor Dr. Davron Aslonqulovich Juraev

28.11.2023

See more

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HK

Hisham Mohammed Khudhur posted a **Review**

November 27, 2023

•••

https://doi.org/10.32388/VFWUAC

Comments

1- The extract is small; add more details to it.

2- Write the objective of the paper and the motivations for the work in the introduction.

3- What is the stopping measure used in the numerical results? Add it to the numerical results.

4- Highlighting the results reachedThe paper is well organized.

I recommend accepting the paper after making minor revisions to it.

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Best Regards,

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HM

Hemanta Mandal posted a **Review**

November 27, 2023

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https://doi.org/10.32388/3U0CZ8

Dear editor,

I have had the privilege of reviewing the manuscript titled "New adaptative numerical algorithm for solving partial integro-differential equations". I wish to express my appreciation for the scholarly contribution made by the authors, as well as to provide a positive recommendation for the acceptance of this manuscript.

The authors have adeptly introduced a numerical approach utilizing orthonormal Bernoulli polynomials to address parabolic partial integro-differential equations. The methodology employed is robust, meticulously articulated, and the results are presented with a commendable level of precision. The manuscript effectively establishes the convergence of the proposed method, supported by detailed figures and a comprehensive comparison with existing methodologies.

In my assessment, the quality of the research, coupled with the clarity of presentation, aligns with the high standards set by our journal. The significance of the findings and the methodological rigor demonstrated throughout the manuscript contribute meaningfully to the field.

Based on the aforementioned considerations, I wholeheartedly recommend the acceptance of this research paper for publication. I believe it will make a valuable addition to the scholarly discourse in the field.

Thank you for considering my evaluation, and I look forward to the successful publication of this noteworthy contribution.

Sincerely,

Hemanta Mandal

See more

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EK

Elham Keshavarz posted a **Review**

November 27, 2023

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https://doi.org/10.32388/FWCLZP

Comments on the paper

New adaptative numerical algorithm for solving partial integro-differential

Equations

In this paper, a new numerical method based on orthonormal Bernoulli polynomials has been applied to give the approximate solution of the parabolic partial integro-differential equations. The proposed method is applied to reduce the problem to a nonlinear algebraic system.

1. Check the manuscript carefully for typos and grammatical errors. For instance,

- on page 2, after Eq. (2), and on page 5, in Eq. (20) "k_1" and "k_2" should be corrected as "K_1" and "K_2".
- on page 3, in Eq. (10), "P_{j-1,T}(x) "should be replaced by "P_{j-1,T}(t) ".
- on page 3, section 3, letter "t" should be deleted.
- on page 4, after Eq. (15), and, on page 6, in section 4, "with" should be replaced by "where".

- on page 4, before Eq. (17), "and" should be replaced by "Also".
- on page 5, in Eq. (20) "Lambda_2" should be corrected as "Lambda_3".
- on page 5, before Eq. (21), "defined by (3)" should be deleted.
- on page 6, before Eq. (24), "Lambda_2" and "Lambda_3" should be added before Sigma symbols.
- on page 6, after Eq. (24), "Using relation (9), initial … given in (6)" should be corrected as " Using relation (9) and initial … given in (2)".
- on page 7, before Eq. (27), "(x^(j) "should be replaced by "(x^(j)) ."
- on page 7, after Eq. (29), "u-P_N " should be replaced by "u-P_Nu".
- on page 8, in proof 1, "r=0, 32 reduce to 29. Now suppose that 32" and "by using 31" and "the results 32" should be replaced by " r=0, relation (32) reduce to (29). Now suppose that (32)" and "by using (31) and "the results (32)", respectively.
- "Proof 1" and "Proof 2" should be replaced by "Proof".
- In Proof 2, "equation 1" and "Lemma 31" should be replaced by " equation (1)" and "relation (31)", respectively .
- On page 9, before section 5, "is" should be deleted.

1. On page 8, in proof of Theorem 2, why is the last inequality true?
2. On page 4, what is the used reference for "T_{b,N}" ?
3. What is the used reference for Lemma 2 , and Examples 1 and 2?
4. Authors must present arguments in the conclusions demonstrating the advantages of using the proposed method.

See more

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E. M. Elsayed posted a **Review**

November 27, 2023

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https://doi.org/10.32388/IH3AGU

n this paper, the author proposed a new numerical approach. He utilized this approach to solve partial integro-differential equations with Volterra and Fredholm types. The matrices of orthonormal Bernoulli polynomials were derived and used to obtain the approximate solution of PIDEs. The author applied this approach to obtain numerical solutions of some test problems.

This paper is interesting and well written. It contains correct and new results.
In summary, I strongly recommend the acceptance of this paper for publication

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ZZ

Zhe Zhang posted a **Review**

November 26, 2023

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https://doi.org/10.32388/B1HPPW

The significance of the problem studied in this article have not been clearly stated. This makes readers unaware of why such research should be conducted. I think this will undoubtedly reduce the attractiveness of this article.

In addition, the method used in this article is undoubtedly a very common method similar to polynomial approximation. Using this very common method to solve a problem without clear meaning is not worth advocating and encouraging.

To sum up, I believe that the author should first declare the innovation and research value of this article, and rethink it. Therefore, I cannot recommend that this article be accepted for publication

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RB

Rafid Habib Buti posted a **Review**

November 24, 2023

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https://doi.org/10.32388/3TO6VZ

The paper is very good ,The paper contains new ideas, as well as the research language is good. The author touched on new equations on the subject of differential equations. The researcher also discussed a new idea related to polynomials, and good and modern analogies were also used.

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1 comment

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HA

Hadis Azin posted a **Review**

November 23, 2023

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https://doi.org/10.32388/WTQ3ND

The paper aims to propose the solution of the partial integro-differential equations using operational matrices built based on shifted Bernoulli polynomials.

The proposed approach is by no means new. Approximating the solution of differential equations by some family of polynomials, and hence solving the original differential equation after building the operational matrices, is a technique used in several and several works. The application of this widely known technique to a specific problem does not lead to a new method.

In addition, the problem investigated in the paper is academic and there is no discussion supporting the need to solve this problem. There is no discussion about the choice of the particular family of polynomials and this choice, indeed, seems to have been made by chance.

Finally, the manuscript is written in a very poor way. There are several grammatical errors and lots of sentences have inconsistencies; for instance, in Abstract, " acurrate numerical appraoch" is wrong.

For the above reasons I do not feel to support the publication of this manuscript and I can just suggest its rejection.

See more

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YY

Youssri Hassan Youssri posted a **Review**

November 23, 2023

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https://doi.org/10.32388/6CP314

Positive Points:

1. The abstract and introduction clearly outline the paper's objective of introducing a numerical approach based on orthonormal Bernoulli polynomials for solving parabolic partial integro-differential equations (PIDEs). The paper contributes to the field by providing operational matrices and a transformation strategy for solving PIDEs, enhancing the understanding of numerical techniques in this domain.

2. The paper establishes a solid theoretical foundation by introducing operational matrices for orthonormal Bernoulli polynomials. This adds mathematical rigor to the proposed numerical approach, contributing to the credibility of the presented method.

3. The approach's transformation of the PIDE problem into a nonlinear algebraic system is well-explained, simplifying the computational process and providing clarity in the methodology.

4. The inclusion of a convergence analysis is commendable, providing insights into the reliability and accuracy of the proposed numerical algorithm. This analysis enhances the paper's scientific rigor.

5. The paper goes beyond presenting the proposed technique by including a comparison with other well-known methods. This comparative analysis adds value by benchmarking the new approach against existing ones, aiding readers in assessing its effectiveness.

6. The conclusion appropriately highlights the versatility of the proposed algorithm, indicating its potential application to various types of PIDEs and differential equations. This broad scope enhances the practical utility of the presented methodology.

7. The inclusion of numerical results and test problems, along with comparisons with other algorithms, is a positive aspect. It provides practical evidence of the proposed approach's efficiency and effectiveness.

8. The paper emphasizes that the presented method is easily implementable and simple, making it accessible to a broader audience and promoting its practical usability.

9. The conclusion encourages the extension of the proposed algorithm to more dimensions, showcasing a forward-thinking

approach and suggesting potential avenues for future research.

10. The introduction provides a comprehensive review of existing literature, establishing a context for the paper and showcasing the authors' awareness of relevant research in the field.

Points of improvement:

1. While the abstract and introduction provide a clear overview, some terminology, such as "pseudo-spectral method," could benefit from additional clarification to ensure a more accessible understanding for readers.

2. The paper could benefit from a more detailed explanation of the proposed methodology, providing step-by-step procedures for readers to follow. This would enhance the paper's educational value.

3. While the paper compares the proposed technique with other methods, a more extensive comparative analysis, including a broader range of existing approaches, could strengthen the paper's argument regarding the novelty and superiority of the introduced method.

4. The paper does not explicitly discuss any limitations or potential drawbacks of the proposed approach. Including a discussion of limitations would provide a more balanced perspective and guide future research directions.

5. While the paper mentions the application of the approach to "some test problems," providing specific examples and their detailed solutions would enhance the practical understanding of the proposed methodology.

6. The paper lacks a discussion on the computational efficiency of the proposed method. Including details on computation time or resource requirements would provide insights into the practical feasibility of the approach.

7. Incorporating visual representations, such as graphs or figures, to illustrate the convergence analysis or numerical results would enhance the paper's clarity and make it more accessible to a broader audience.

8. The paper could benefit from a more detailed explanation of key parameters used in the methodology, such as the choice of Gauss-Legendre nodes, to provide readers with a deeper understanding of the method's intricacies.

9. Some in-text citations lack specific details, making it challenging for readers to locate the referenced works. Ensuring comprehensive citations would enhance the paper's transparency and facilitate further exploration by readers.

10. The conclusion could benefit from a brief recapitulation of the main findings and contributions, reinforcing the significance of the presented approach and leaving a lasting impression on the reader.

Maryam Arabameri posted a **Review**

November 22, 2023

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https://doi.org/10.32388/8P69F1

Comments on the manuscript *Qeios* R4546K entitled

"**New adaptative numerical algorithm for solving partial integro-differential equations**"

This manuscript uses an approach based on orthonormal Bernoulli polynomials to solve the parabolic partial integro-differential equations numerically. The Convergence issue is analyzed, also, in order to investigate the efficiency and accuracy of the introduced method, some test problems have been solved.

Some comments are given in below.

1. The word of "convergence" should be added to keywords.
2. With the appropriate grammatical convention and punctuation, sentences introducing equations should, with the inclusion of the equation, constitute a complete sentence. More editing for writing is needed. For example, on page 3, after Equation 6, "contains" can be replaced by "containing".
3. Moderate editing English language required. Only some of them are given here.

- On page 2, before Equation 3, "oppeared" should be replaced by " appeared".
- On page 3, section 3, what is the letter "t" after "technique"?
- On page 5, before Equation 21, "noeuds" should be replaced by "nodes".
- On page 5, before Equation 22, "oppeared" should be replaced by " appeared".
- On page 17, conclusion, what the meaning of "after" in the following sentence?

"After we take Gauss-Legendre nodes in the intervals [0; b] and [0; T] as collocation points."

1. You have claimed that your presented method is easily to implement and simple. What is the criterion for this claim? It is suggested to report the CPU run times in the given tables for all of the examples. Also, write the configuration of your computer at the beginning of section corresponding to numerical examples.
2. The paper's layout usually comes at the end of the introduction.
3. All acronyms should be defined for the first time. For example, on page 3, you have used "OBP" but you haven't defined it.
4. On page 5, what is the used reference for invertibility of matrix "T_b,N". Give it before Equation 16.
5. What is the reference of Lemma 2?
6. On page 8, proof of Theorem 2, why is the first inequality true?

7. It is suggested to confirm the theoretical results for convergence of the proposed approach numerically in tables.

See more

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1 comment

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MA

Maryam Arabameri posted a **Review**

November 22, 2023

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https://doi.org/10.32388/3VXXWB

Comments on the manuscript *Qeios* R4546K entitled

"**New adaptative numerical algorithm for solving partial integro-differential equations**"

This manuscript uses an approach based on orthonormal Bernoulli polynomials to solve the parabolic partial integro-differential equations numerically. The Convergence issue is analyzed, also, in order to investigate the efficiency and accuracy of the introduced method, some test problems have been solved.

Some comments are given in below.

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2. With the appropriate grammatical convention and punctuation, sentences introducing equations should, with the inclusion of the equation, constitute a complete sentence. More editing for writing is needed. For example, on page 3, after Equation 6, "contains" can be replaced by "containing".
3. Moderate editing English language required. Only some of them are given here.

- On page 2, before Equation 3, "oppeared" should be replaced by " appeared".
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"After we take Gauss-Legendre nodes in the intervals [0; b] and [0; T] as collocation points."

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6. On page 8, proof of Theorem 2, why is the first inequality true?
7. It is suggested to confirm the theoretical results for convergence of the proposed approach numerically in tables.

See more

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ST

Stepan A. Tersian posted a **Review**

November 21, 2023

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https://doi.org/10.32388/BYS0B3

REFEREE'S REPORT

on the paper R4546K.pdf

New adaptative numerical algorithm for solving partial

integro-differential equations

by Rebiha Zeghdanea

The paper deals with. a numerical appraoch based on orthonormal Bernoulli polynomials for solving parabolic partial integro-differential equations (1). Convergence analysis, numerical tests and figures are given.

Detailed Introduction is given.

Othonormal Bernoulli polynomials and approximation are presented in Section 2. The main result are Theorem 2 and

Theorem 3, presented and proved in Section 3. In last section six examples are given. How are they connected with the main results ?

I can recommend the paper for publication in the journal Qeios after a major revision. My recommendations are as follows:

1. In the Abstract cancel second sentence.

2. Section 2 is short and can be included in Section 3. The revised Section 2 can be “Othonormal Bernoulli polynomials and Pseudo-spectral method for Solving PIDE”

3. It should be proved Lemma 2 in Section 4 (3). On pages 8 and 9 Proof 1 and Proof 2 can be only Proof or Proof of Theorem 2/3.

4.On page 9 should say By Equation (1), Lemma 1, Theorem 2. Where is used Lemma 2 ?

See more

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MA

Mohamed Ait Ichou posted a **Review**

November 20, 2023

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https://doi.org/10.32388/QRLKGT

The method presented in this article stands out when compared to existing approaches used for solving partial integro-differential equations.
Could you please provide more context regarding "the order of convergence of this approximation"?

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IW

Ireneusz Winnicki posted a **Review**

November 20, 2023

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https://doi.org/10.32388/1085RA

In this paper Rebiha Zeghdanea proved two new theorems and described six numerical examples. I didn't find any methodological errors. The quality of presentation and scientific soundness are on high level.

I recommend the Author to discuss more recent paper on the parabolic partial integro-differential equation: Rostami, Yaser: An Effective Computational Approach Based on Hermite Wavelet Galerkin for Solving Parabolic Volterra Partial Integro Differential Equations and its Convergence Analysis, 2023, DOI:10.3846/mma.2023.15690. I suggest to cite this paper.

From the mathematical point of view this reviewed paper is prepared correctly. In my opinion it is very interesting and it is a great contribution to the development of the existence theory.

1.Does the introduction provide sufficient background - YES and include all relevant references? - YES

2.Did you detect inappropriate self-citations by authors? - NO

3.Is the research design appropriate? - YES

4.Is the method adequately described? - YES

5.Are the results clearly presented? - YES

6.Are the conclusions supported by the results? - YES

7.Originality / Novelty and Significance of Content and Scientific Soundness - HIGH

See more

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1 comment

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HL

Hojatollah Laeli Dastjerdi posted a **Review**

November 19, 2023

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https://doi.org/10.32388/YKLH3X

This is a good paper for publication. This is the first time this method use for this equation. So I recommended it for publication. If the author can give an example that its exact solution is unknown, then the paper will be improved.

I recommended it for publication.

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YR

Y. N. Reddy posted a **Review**

November 19, 2023

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https://doi.org/10.32388/RX9BK9

This is a new approach hence recommended to accept.

It is tested and applied will be useful to many engineers and scientists

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HM

Hamid Reza Marzban posted a **Review**

November 18, 2023

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https://doi.org/10.32388/2SI1VE

The current paper contains minor significant information regarding the theoretical and numerical solution of integro-differential equations. The author should clarify the main contribution of the paper and describe the novelty of the proposed method. Additionally, the following relevant references are suggested:

1: Marzban, H.R., Rostami Ashani, M. A class of nonlinear optimal control problems governed by Fredholm integro-differential equations with delay. ***International Journal of Control***, 2020, 93(9), pp. 2199–2211.

2: Marzban, H.R., Numerical solution of optimal control problems governed by integro-differential equations. ***Asian Journal of Control***, 2020, 22(3), pp. 1138–1146.

3: Marzban, H.R., Hajiabdolrahmani, S., Numerical Solution of Piecewise Constant Delay Systems Based on a Hybrid Framework. ***nternational Journal of Differential Equations***, 2016, 2016, 9754906

4: Marzban, H.R., Nezami, A., Analysis of nonlinear fractional optimal control systems described by delay Volterra–Fredholm integral equations via a new spectral collocation method. ***Chaos, Solitons and Fractals***, 2022, 162, 112499

5:Tabrizidooz, H.R., Marzban, H.R., Pourbabaee, M., Hedayati, M. A composite pseudospectral method for optimal control problems with piecewise smooth solutions

Journal of the Franklin Institute, 2017, 354(5), pp. 2393–2414.

See more

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HO

Hameeda Oda Al-Humedi posted a **Review**

November 18, 2023

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https://doi.org/10.32388/IWPGB9

New adaptative numerical algorthim for Solving Partial Integro-Differential Equations

The author's present the solution of the partial integro-differential equations using OBPs. I recommend revised the search to check the following reasons:

- The language of the manuscript requires considerable attention.
- The novelty of this research is not new.
- The presented results in the article are stated without validation.
- The conclusion section of the article is written similarly to the abstract. No conclusions are listed.

The paper is missing the following:
1. Rewrite the abstract of paper in which explains the purpose of this
paper and the important results in 5-10 lines?.

2. Historic introduction presents more details but is not explain the relation previous works and this paper motivated by references by putting in the end of the paragraph.
3. The important behind this work is not stated in introduction section.
4. The purpose of this article is not explained in the introduction.
5. The outline of this paper is not stated?
6. Highlights of the research.

- Scientific and regularity remarks:

1- The space of parameter is not clear??

2- The mathematical model is not well presented.

3-Put remark contains … Form figure 1 and table we can deduce that …..

What is the obtained result? For all figures and corresponding tables?

See more

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DT

Dagnachew Mengstie Tefera posted a **Review**

November 17, 2023

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https://doi.org/10.32388/48OMY9

Reviewer personal information

Name: Dr. Dagnachew Mengstie Tefera

Title: Assistant Professor in Mathematics

Organization: Debre Tabor University, Debre Tabor, Ethiopia

E-mail: dagnmeng@yahoo.com

Review on the paper **"New adaptative numerical algorithm for solving partial integro-differential equations**"

The manuscript reported the theoretical and numerical results solving initial-boundary value problem for singularly perturbed parabolic integro-differential equations.

The content and design of the material meet the requirements of a scientific publication with minor correction.

Therefore I would recommend the publication of this manuscript.

I only have a few questions and comments:

1. Please show the gap and state the novelty of your work at introduction section.
2. Does your method work for singularly perturbed parabolic integro- differential equations?
3. You have stated the present method is good efficiency method. Is it in terms of accuracy or computational time?
4. I haven't seen the order of convergence of your proposed method at any section, so what is the order of convergence of your proposed method?
5. Compute the rate of convergence?

Additional comment

Authors should include the two references for the completeness of the literature on singularly perturbed differential equations:

1. D.M., Tefera, A.A., Tiruneh. G.A., Derese, Numerical Treatment on Parabolic Singularly Perturbed Differential Difference Equation via Fitted Operator Scheme, Abstr. Appl. Anal., 2021 1 - 12, 2021.
2. D.M., Tefera, A.A., Tiruneh. G.A., Derese, Fitted Operator Method over Gaussian Quadrature Formula for Parabolic Singularly Perturbed Convection-Diffusion Problem. Numer. Analys. Appl. 15,256–269 (2022).
3. D.M., Tefera, A.A., Tiruneh. G.A., Derese, Fitted Operator Method Using Multiple Fitting Factors for Two Parameters Singularly Perturbed Parabolic Problems, Mathematical Problems in Engineering, Hindawi, vol. 2022, pages 1-10.

Mudavath Lalu posted a **Review**

November 16, 2023

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https://doi.org/10.32388/B1YKY8

Reviewer #5. The general structure of the study is good. Here are some comments to improve the quality of this manuscript:

1. Please emphasize the novelty of your research.

2. Why is it important to know the research on this topic?

3. Please emphasize the applicability of your model in a real-life engineering setting; give examples. What benefits would your paper bring to a company? How easy is it to implement it in practice? Please add a case study section to your paper.

3. which tools you use to get this results and Please share the code of your research in git-hub share git-hub account.

New adaptative numerical algorithm for solving partial integro-differential equations

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12. a, b *R. Zeghdane, Numerical solution of stochastic integral equations by using Bernoulli operational matrix. Mathematics and Computers in Simulation 165, (2019) 238-254.*
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https://doi.org/10.1155/2017/2853679.

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