Research Article

On Complex Dynamics and Primordial Gravity

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We recently pointed out that, under suitably defined conditions, the Schrödinger equation represents a limit case of the *complex Ginzburg-Landau equation* (CGLE). As generic prototype of complex dynamics, CGLE is naturally tied to *dimensional fluctuations* conjectured to develop far above the electroweak scale. The goal of this work is to uncover an unforeseen connection between CGLE and the equation of *geodesic deviation* in General Relativity (GR). This connection is likely to come into play in primordial cosmology, where strongly fluctuating gravitational fields evolve in far-from-equilibrium conditions. Our findings unveil the duality between primordial gravitation and Kolmogorov entropy and suggest a potential gateway towards field unification outside Lagrangian theory.

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Cautionary remarks

We caution from the outset that the sole intent of this paper is to lay the groundwork for further analysis and exploration. Independent work is needed to develop, validate, or reject the ideas presented here.

1. Introduction

This contribution is a sequel to[1], which argues that Schrödinger equation is a particular embodiment of the *complex Ginzburg-Landau equation* (CGLE). It has been known for quite some time that CGLE is a prototype model of far-from-equilibrium phase transitions and complex phenomena, particularly helpful in describing systems exhibiting *wave patterns, spatial structures, and solitons*[2]. Following[1][3], CGLE can be naturally tied to the onset of *dimensional fluctuations* far above the electroweak scale set by the vacuum expectation value of the Higgs sector, namely $v = 246GeV$. The goal of this work is to uncover an unexpected connection between CGLE and the equation of *geodesic deviation* in General Relativity (GR). This connection is likely to come into play in primordial cosmology, where strongly fluctuating gravitational fields evolve in far-from-equilibrium conditions. Here we highlight the dual nature of primordial gravitation and Kolmogorov (K) entropy and suggest a potential gateway towards field unification outside Lagrangian theory.

The paper is organized as follows: elaborating from[1] or[3], next section recalls the route from dimensional fluctuations to CGLE; section 3 and 4 establish the link between CGLE, the Jacobi equation of geodesic deviations and the K-entropy. As K-entropy naturally ties in with the regime of dimensional fluctuations above the electroweak scale, the chain of connections discussed in sections 3 and 4 bridge the gap between *dimensional fluctuations* and *primordial manifestations of classical gravity*. Concluding remarks are detailed in the last section, followed by a list of abbreviations and a couple of Appendix sections.

2. CGLE from dimensional fluctuations above the electroweak scale

Reaction-Diffusion (RD) processes are a subset of complex phenomena defined within the framework of Nonequilibrium Statistical Physics. These models are typically formulated in $d+1$ dimensions, where d is the dimension of the Euclidean manifold representing the physical space and t is the time coordinate. Ref.[3] develops a toy RD model acting on a two-dimensional lattice ($d = 2$), whose local variables are time-varying *dimensional fluctuations* $\delta\varepsilon(t) = \delta[2 - d(t)]$. The model includes a *scattering* event at rate D, a *clustering* event at rate u and a *decay* (or *percolation*) event at rate $\kappa = \lambda - \lambda_c$, with λ being a control parameter nearing its critical value λ_c. Up to a leading order approximation, the macroscopic properties of RD processes may be encoded in a *mean-field* (MF) equation, which quantifies the competition between losses and gains in a generic density parameter $\rho(t)$. In particular, the decay/percolation process occurs with a rate proportional to $\kappa\rho(t)$ and leads to a gain in density. By contrast, the clustering process drops the density with a rate proportional to $u\rho^2(t)$. Ignoring diffusion, the resulting MF equation takes the form

$$\frac{\partial \rho(t)}{\partial t} = \kappa\rho(t) - u\rho^2(t) \tag{1}$$

In the context of[1][3] the control parameter $\lambda(t) = \lambda[\delta\varepsilon(t)]$ represents the *density of dimensional fluctuations* $\delta\varepsilon(t) << 1$ while $\rho(t)$ denotes the *density of active (or unstable) lattice sites.* A

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straightforward extrapolation of (1) is given by the system of coupled partial differential equations

$$\frac{\partial\rho_1(x,t)}{\partial t} = D_1\Delta\rho_1(x,t) + f(\rho_1,\rho_2,\kappa) \tag{2a}$$

$$\frac{\partial\rho_2(x,t)}{\partial t} = D_2\Delta\rho_2(x,t) + g(\rho_1,\rho_2,\kappa) \tag{2b}$$

According to[1] and references therein, an arbitrary solution of (2) lying near the bifurcation point at $\kappa > \kappa_0$ can be expressed through a *complex-valued function* $W(r,\tau) = U(r,\tau) + iV(r,\tau)$ obeying the CGLE

$$\frac{\partial W}{\partial \tau} = W + (1+ic_1)\frac{\partial^2 W}{\partial r^2} - (1+ic_2)W|W|^2 \tag{3}$$

Here, the set of new coordinates is given by[1][3]

$$r = \eta x \tag{4a}$$

$$\tau = \eta^2 t \tag{4b}$$

where,

$$\eta = (\kappa-\kappa_0)^{\frac{1}{2}} \propto (\lambda-\lambda_c)^{\frac{1}{2}} << 1 \tag{5}$$

3. From CGLE to the Jacobi equation of geodesic deviation

In what follows, we use an alternative form of CGLE presented as[4]

$$\frac{\partial W}{\partial \tau} = \alpha W + \beta\frac{\partial^2 W}{\partial r^2} - \gamma W|W|^{2s} \tag{6}$$

The choice α, β real, $\beta = 1$,$\gamma = 0$ leads to

$$\frac{\partial W}{\partial \tau} = \frac{\partial^2 W}{\partial r^2} + \alpha W \tag{7}$$

As stated in the Introduction, since complex dynamics is expected to arise in the far-from-equilibrium regime of GR (see e.g.,[5] or[6]), it is instructive to investigate the relationship between (7) and the geometry of primordial spacetime. To this end and with reference to Appendix A, we introduce the following assumptions:

A1) The space coordinate r is taken to represent the analogue of the *metric parameter* s and the time coordinate the analogue of *proper time*, which means that

$$dr \Rightarrow ds \tag{8a}$$

$$ds = d\tau \ (c = 1, g_{00} = O(1)) \tag{8b}$$

A2) To further streamline the derivation, we assume that α is independent of s, i.e., $\alpha \neq \alpha(s)$.

A3) By analogy with (A5) and the study of two-dimensional surfaces in Riemannian geometry, we write the geodesic deviation as *complex-valued entity.* In line with the conjectured onset of complex dynamics far above the electroweak scale, geodesic deviation is interpreted here as a *fluctuating vector field.*

The solution of (7) takes the form[7]

$$W(s) = C_1 \exp(p_1 s) + C_2 \exp(p_2 s) \tag{9}$$

If $\alpha \neq \frac{1}{4}$, $p_{1,2}$ are given by

$$p_{1,2} = \frac{1 \pm \sqrt{1-4\alpha}}{2} \tag{10}$$

Furthermore, if α is real and $\alpha > \frac{1}{4}$, $p_{1,2}$ become complex-valued and the solution to (7) turns into[7]

$$W(s) = \exp(s/2)\left[A\cos(\omega_N s) + B\sin(\omega_N s)\right] \tag{11}$$

in which the characteristic frequency is

$$\omega_N = \frac{1}{2}\sqrt{4\alpha - 1} \tag{12}$$

and the constants $C_{1,2}, A, B$ are fixed by the boundary conditions. If α is reasonable small, developing the square root in (12) yields the approximation

$$W(s) \propto \exp(\pm i\alpha s) \tag{13}$$

A glance at (13) and (A8) shows that $W(s)$ may be interpreted as *complex-valued analogue* of geodesic separation $\zeta(s)$, while α mirrors the role of *Gaussian curvature* K, i.e.

$$W(s) \Leftrightarrow \zeta(s) = \zeta_1(s) + i\zeta_2(s) \tag{14a}$$

$$\alpha \Leftrightarrow K \tag{14b}$$

4. From the Jacobi equation to the K-entropy

In nonlinear dynamics theory, K-entropy is a representative measure of chaotic behavior in phase space. By[8] and Appendix A, the unavoidable *sensitivity to initial conditions* in the evolution of geodesics can be characterized by the divergence of the affine parameter $\zeta(s)$ along s. Specifically, the local Gaussian curvature takes on the role of a *Lyapunov exponent*

$$K(s) \Leftrightarrow \lambda(s) \tag{15}$$

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In general, K-entropy relates to the spectrum of Lyapunov exponents λ_i of a dynamical system and quantifies the amount of information lost or gained during its evolution. It is given by the sum of the log of all Lyapunov exponents $|\lambda_i| > 1$ averaged over a given region of the phase space Π. The K-entropy associated with the system of nearby geodesics Γ_0 and Γ can be computed as

$$S_K = \int_\Pi \sum_{|\lambda_i|>1} \Phi\left(|\lambda_i|\right) d\eta \tag{16a}$$

where

$$\Phi\left(|\lambda_i|\right) = \log|\lambda_i| \tag{16b}$$

and $d\eta$ stands for the differential measure of Π.

Taken together, (14), (15) and (16) set the stage for a probabilistic interpretation of the metric g_{ij} in primordial gravity, matching the chaotic behavior of geodesics.

With reference to[9][10][11], K-entropy can be alternatively defined using the concept of *information dimension* $D_1(\mu)$, i.e.

$$D_1(\mu) = -\frac{S_K(\mu)}{\log \varepsilon(\mu)} \tag{17}$$

Here, μ is the sliding observation scale associated with a "time-like" parameter as in

$$dt = d(\log \mu) = \frac{d\mu}{\mu} \tag{18}$$

Relation (16) and (17) imply,

$$\sum_{|\lambda_i|>1} \Phi\left(|\lambda_i|\right) \propto \frac{dS_K}{d\eta} \tag{19}$$

and

$$\frac{dS_K(\mu)}{d\eta} = -\frac{D_1(\mu)}{\varepsilon(\mu)} \frac{\beta_\varepsilon(\varepsilon)}{\beta_\eta(\eta)} \tag{20}$$

in which the beta-functions of dimensional deviation and phase-space measure are respectively given by

$$\beta_\varepsilon(\varepsilon) = \frac{d\varepsilon}{d\mu} \tag{21}$$

$$\beta_\eta(\eta) = \frac{d\eta}{d\mu} \tag{22}$$

It can be shown that the combined use of (20), (21) and (22) yields the following constraint[11]

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$$\beta_\varepsilon(\varepsilon) \approx -\frac{\varepsilon^{\frac{3}{2}}}{D_1}\beta_\eta(\eta) \tag{23}$$

In summary and on account of (14) to (23), the Gaussian curvature of primordial gravity can be interpreted as dual to dimensional deviations, i.e.,

$$K[g_{ij}(\mu)] \Leftrightarrow \varepsilon(\mu) \tag{24}$$

Appendix B shows that the same conclusion can be arrived at by using the duality between *fractional dynamics on flat spacetime* and *classical gravitation*.

5. Concluding remarks

5.1) Relations (14) to (24) indicate that dimensional deviation $\varepsilon(\mu)$ represents the *information content* of the K-entropy[11], a conclusion that supports Wheeler's "*it from bit*" philosophy[12]

5.2) The evolving regime of dimensional deviations well above the electroweak scale is consistent with chaotic behavior described by the K-entropy. This observation justifies the analogy between the curvature fluctuations of primordial gravity, K-entropy, and dimensional fluctuations having the form $\delta\varepsilon(\mu) \propto \varepsilon = 4 - D(\mu)$, as illustrated by the diagram below:

$$\boxed{\textbf{Dimensional Fluctuations}} \rightarrow \boxed{\textbf{CGLE}} \rightarrow \boxed{\textbf{Primordial Gravity}}$$

Moreover, combining these results with[1], suggests an intriguing gateway towards field unification outside the boundaries of Lagrangian field theory:

$$\boxed{\textbf{Dimensional Fluctuations}} \rightarrow \boxed{\textbf{CGLE}} \rightarrow \boxed{\left\{\begin{array}{c}\textbf{Quantum Physics} \\ \textbf{Primordial Gravity}\end{array}\right.}$$

List of abbreviations

- CGLE = complex Ginzburg-Landau equation
- GR = General Relativity
- RD = Reaction-Diffusion
- K-entropy = Kolmogorov entropy.

Appendix A: The Jacobi equation[8]

The Jacobi equation is a second-order ordinary differential equation that describes how geodesics behave under variations in their initial conditions, particularly regarding *nearby geodesics.* This Appendix is a brief account on the construction and geometric interpretation of the Jacobi equation.

As it is well-known, characterization of four-dimensional Riemannian spacetime is done in terms of coordinates x_i $(i = 0, 1, 2, 3)$ and local metric ds defined by the quadratic function

$$ds^2 = g_{jk} dx_j dx_k \tag{A1}$$

The Hamiltonian of a non-relativistic freely moving particle on Riemannian spacetime is

$$H(p, x) = \frac{1}{2m} g^{jk} p_j p_k \tag{A2}$$

where,

$$p_j = g_{jl} \dot{x}_l \tag{A3}$$

$$g_{jl} g^{lk} = 1 \tag{A4}$$

The curved trajectory Γ in Riemannian spacetime spanned by solutions of (A2) is called a *geodesic.* Let Γ_0 represent a fixed geodesic whose coordinates are function of the distance s measured along it. Denote a nearby geodesic by Γ. Let the geodesic normal to Γ_0 be called Γ_1 and assume that Γ_1 intersects Γ at point P. Let the distance between Γ_0 and Γ measured along Γ_1 at P be denoted as $\zeta(s)$ (see fig. 1 below). It can be shown that $\zeta(s)$ satisfies the *Jacobi equation*[8][13]

$$\frac{d^2 \zeta(s)}{ds^2} + K(s) \zeta(s) = 0 \tag{A5}$$

in which $K(s)$ is the Gaussian curvature at P.

The Jacobi equation reflects the behavior of the Gaussian curvature at P. Specifically, the neighboring geodesic Γ is pulled back towards Γ_0 if $K > 0$, or pushed away from Γ_0 if $K < 0$. It follows that the Gaussian curvature represents a *local measure of geodesic instability*. On a spherical surface, $K > 0$ means stability whereas $K < 0$ on hyperbolic surfaces describes instability.

Using[7], the solution of (A5) is given by

$$\zeta(s) = C_1 \exp(q_1 s) + C_2 \exp(q_2 s) \tag{A6}$$

where,

$$q_{1,2} = \pm i \sqrt{K} \tag{A7}$$

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and so,

$$\zeta(s) \propto \exp(\pm i\sqrt{K}s) \tag{A8}$$

which is formally identical to (13).

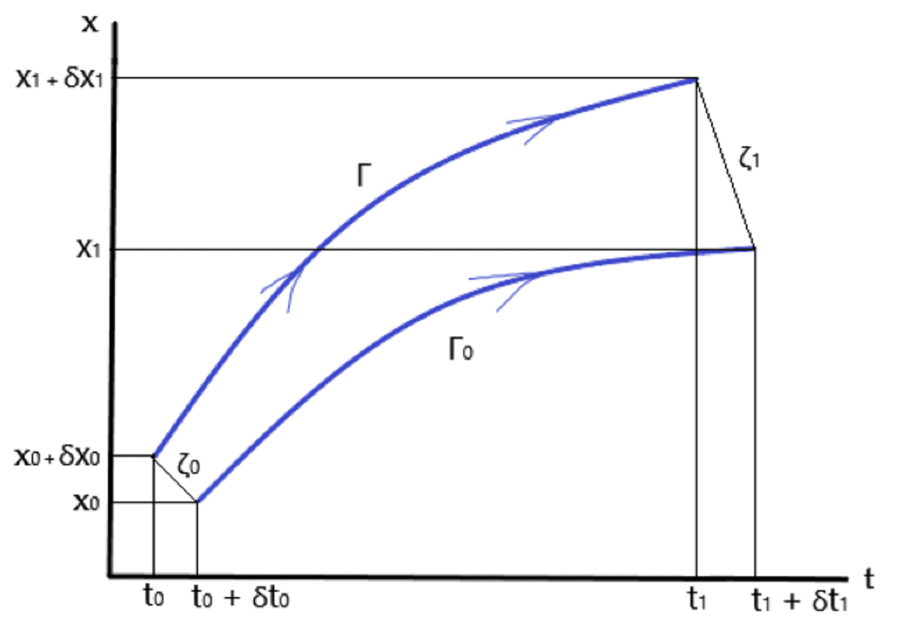

Figure 1. Generic geodesic deviation in a two-dimensional (x, t) plane

Appendix B: Duality of fractal spacetime and classical gravitation

Following[14], consider the fractional analog of a free particle Hamiltonian in flat spacetime (A2)

$$H = \frac{1}{2m} p^{2\kappa} \tag{B1}$$

in which κ denotes the order of fractional integration. If $\kappa = 1 - \varepsilon$, with $\varepsilon << 1$ (B1) approximates the classical non-relativistic Hamiltonian in the limit $\varepsilon = 0$ namely,

$$H = \frac{1}{2m} p^{2(1-\varepsilon)} \approx \frac{1}{2m} p^2 \tag{B2}$$

Refer now to the action of a free non-relativistic particle in a weak gravitational field,

$$S = \frac{1}{2m} \int dx \sqrt{-g} g^{ij} p^2 \tag{B3a}$$

where η^{ij} is the Minkowski metric, $g = \det(g_{ij})$ and

$$g^{ij} = \eta^{ij} + h^{ij}; \ \left|h^{ij}\right| << 1 \tag{B3b}$$

Side-by-side comparison of (B2) and (B3a) suggests the following analogy between the dimensional deviation ε and gravitational metric,

$$p^{2(1-\varepsilon)} \Leftrightarrow \sqrt{-g} g^{ij} p^2 \tag{B4}$$

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