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Fermat Surfaces and Hypercubes

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Abstract

When observed from a natural vector space viewpoint, Fermat's last theorem appears not as a unique property of natural numbers, but as the bottom line of extended possible issues involving larger dimensions and powers. The fabric of this general Fermat's theorem structure consists of a well-defined set of vectors associated with N-dimensional vector spaces and the Minkowski norms one can define there. Here, this special vector set is studied and named a Fermat surface. The connection between Fermat surfaces and hypercubes is unveiled.

Keywords

Fermat Surfaces · Fermat Last Theorem · Whole Vectors · Perfect Vectors · Vector Semispaces · Fermat Vectors · Unit Shell · Fermat Extended Theorem · Natural Vector Spaces · Minkowski Norms ·

1. Introduction

Fermat's last theorem demonstration by Wiles [1] in 1995 was a step toward unlocking a centuries-unsolved demonstration. But it might be accepted, besides a great mathematical step, as the starting path of many related subjects with the original Fermat's idea. In references, one can consult

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several recent studies about Fermat's last theorem [2-6]. Also, in our laboratory, several papers dealing with the extension of the theorem in larger dimensions have been published; for instance, references [7-10]. Even more recent publications deal with a simple demonstration of the theorem [11] and discuss the nature of empirical proofs of extending the theorem in larger dimensional spaces [12]. In this last reference, the possibility to study the structure of possible Fermat surfaces has been issued. The present paper tries to deal with this task.

2. Whole Vectors

Given any N-dimensional vector space $V_N(\mathsf{F})$ constructed over a field F, one could define a *whole* vector[3] $\langle \mathbf{w}| \in V_N(\mathsf{F})$ as one with non-null components. The whole vectors form a vector set $W_N(\mathsf{F})$, which one can structure in turn as:

$$\forall \langle \mathbf{w}| \in W_N(\mathsf{F}) \subset V_N(\mathsf{F}) \rightarrow \langle \mathbf{w}| = (w_1, w_2, w_3, \dots w_N) : \{ w_I \neq 0 | I = 1, N \}. \quad (1)$$

The set $W_N(\mathsf{F})$ is the most relevant structure of a vector set within the vector space $V_N(\mathsf{F})$. Because the vectors possessing some null components correspond to elements of lesser dimension subspaces of $V_N(\mathsf{F})$, as will be commented on next.

2.1. Un-whole vectors.

The possible classes and structure of *un-whole* vectors in a vector space $V_N(\mathsf{F})$, that is, vectors possessing from 1 to $N-1$ zeros as components, are given by the number and nature of the vertices of a Boolean hypercube of the same dimension as $V_N(\mathsf{F})$, and bearing the same number of zeros, see for more information reference [12-15].

Adopting this kind of vector pattern, the unit components of the Boolean hypercube vertices become connected with the non-zero un-whole vector components.

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[3] Along this paper, the bra symbol $\langle \mathbf{w}|$ will describe row vectors. It must be noted that all the equations where row vectors are present can be considered and also be changed in a column or ket vector frame. The practical use of bra vectors to avoid waste of print space has been chosen here.

Admitting the null vector: $\langle \mathbf{0}| = (0,0,0,...,0) \in V_N(\mathsf{F})$, as a zero-pattern class by itself, the number of possible un-whole vector patterns in the vectors of a $V_N(\mathsf{F})$ space is $2^N - 1$.

It is also interesting to realize that this number of un-whole vector classes in a N-dimensional vector space coincides with the N-th Mersenne number.

In a vector space with a class pattern made by whole and un-whole vectors, the whole vectors can lie in the class associated with the unity Boolean hypercube vertex: $\langle \mathbf{1}| = (1,1,1,...,1)$, the vertex of the corresponding Boolean hypercube, which is the bit representation of the Mersenne number, connected with the associated hypercube and vector space dimensions.

3. Perfect Vectors

When considering the whole vectors of some vector space $V_N(\mathsf{F})$, in general, one might name as *perfect* vectors $\langle \mathbf{p}|$ the ones that have their component modules ordered in a canonical increasing sequence, that is:

$$\forall \langle \mathbf{p}| = (|p_1|,|p_2|,|p_3|,...|p_N|) \in V_N(\mathsf{F}) \rightarrow \{0 < |p_1| < |p_2| < |p_3| < ...|p_N|\}. \quad (2)$$

3.1. Perfect vectors in vector semispaces

Then, perfect vectors defined according to the equation (2) can be considered a subset of the whole vectors. Moreover, perfect vectors are defined even simply in a *vector semispace*[4] environment.

In vector semispaces, only the non-negative definite part: F^+ of the involved field is relevant; then one can write: $V_N(\mathsf{F}^+)$. In semispaces, the vector addition acquires the structure of a semigroup, which furnishes the name semispace. This property also applies when the natural number set substitutes the field: $V_N(\bullet)$ as occurs in natural spaces[5].

In both of these more restricted cases, semispaces and natural spaces, one can define perfect vectors simply than in the previous equation (2), that is:

[4] Semispaces are also known as orthants.
[5] The name natural space corresponds to some vector space defined over the natural numbers set. Under some conditions, they are also called a lattice.

$$\forall\langle\mathbf{p}| = (p_1, p_2, p_3, \ldots p_N) \in V_N(\mathsf{F}^+) \rightarrow \{0 < p_1 < p_2 < p_3 < \ldots p_N\}. \quad (3)$$

3.2. Perfect vectors as generators of vector spaces

Perfect vectors correspond to vectors that can generate a set of related whole vectors, which can be associated with the permutations of all the components of a given perfect vector.

Thus, one can attach a collection of $N!$ vectors to every perfect vector by permuting its original components.

More than this, the N circular permutations of the components of a perfect vector allow the construction of a set of N linearly independent vectors, a basis set of the vector space or semispace.

4. Fermat Surfaces

Knowing the preliminary definitions of whole and perfect vectors and semispaces also makes it possible to find the structure of the vector sets, which one might call Fermat surfaces.

Given a $(N+1)$-dimensional vector space $V_{(N+1)}(\mathsf{F})$ constructed over a field F, one can define a Fermat surface: $F_N^p(\mathsf{F}|r)$, of dimension N, order p, and radius r as a set of perfect $(N+1)$-dimensional vectors, where the last and larger component r is a common positive definite real, rational, or natural number, called the *radius* of the Fermat surface, that is:

$$\exists r \in \mathsf{F}^+ : \langle\mathbf{f}| = (f_1, f_2, f_3, \ldots f_N, r) \in F_N^p(\mathsf{F}|r) \subset V_{(N+1)}(\mathsf{F}), \quad (4)$$

this equation above determines the dimension and radius of the surface.

4.1. Minkowski and Euclidean norms in Fermat Surfaces

To account for the order p of a Fermat surface, every vector element $\langle\mathbf{f}|$ of the surface, as constructed in the equation (4), has to be associated with a zero p-th order Minkowski norm, that is: $M_p(\langle\mathbf{f}|) = 0$, defined by the algorithm:

$$\forall\langle\mathbf{f}| = (f_1, f_2, f_3, \dots f_N, r) \in F_N^p(\mathsf{F}|r) \rightarrow M_p(\langle\mathbf{f}|) = \sum_{I=1}^{N} |f_I|^p - r^p = 0. \qquad (5)$$

Alternatively, one can consider such a Fermat surface $F_N^p(\mathsf{F}|r)$ definition as a set of N-dimensional vectors bearing a common p-th order Euclidean norm: $E_p(\langle\mathbf{f}|) = r^p$, defined now as:

$$\forall\langle\mathbf{f}| = (f_1, f_2, f_3, \dots f_N, r) \in F_N^p(\mathsf{F}|r) \rightarrow E_p(\langle\mathbf{f}|) = \sum_{I=1}^{N} |f_I|^p = r^p. \qquad (6)$$

5. Fermat surfaces and Fermat natural vectors

One might define a Fermat natural vector as an element of a Fermat surface with components made by natural numbers. Therefore, a Fermat vector possesses a dimension N, order p, and a natural number acting as radius: r, which fulfills a corresponding Minkowski zero norm. According to this, one can write for Fermat's vectors the equivalent expression connected with the equations (4) and (5):

$$\begin{aligned} &\langle\mathbf{f}| = (f_1, f_2, f_3, \dots f_N, r) \in F_N^p(\bullet\,|r) \subset V_{(N+1)}(\bullet\,) \\ &\rightarrow M_p(\langle\mathbf{f}|) = \sum_{I=1}^{N} |f_I|^p - r^p = 0 \end{aligned} \qquad (7)$$

Thus, Fermat vectors belonging to a natural vector space are also elements of a Fermat surface. Fermat vectors correspond to natural vectors with a null Minkowski norm. Then, one can consider them as sets of vectors submitted to Fermat's last theorem in the case of a vector space of dimension (2+1) [11]. For higher dimensions, they are subject to the empirical properties already described in previous research, for example, in references [7-10,12].

In a recent study [12], several computational exhaustive tests have been performed, showing the existence of different natural Fermat vectors but bearing the same radius, order, and dimension, indicating that Fermat surfaces might contain several natural Fermat vectors as points.

5.1. *Some remarks on natural Fermat vectors*

a. Within the set of Fermat surfaces with orders greater than 2, that is, the set that one can describe as: $F_2^{p>2}(\mathsf{Q})$, natural Fermat vectors do not exist as elements of such a surface. Natural vectors associated with powers greater than 2 in these 2-dimensional surfaces cannot exist because of the Fermat last theorem. One might describe this situation as: $F_2^{p>2}(\bullet\ ) = \varnothing$.

b. Calling as $S_N(r)$ any N- dimensional sphere of radius r, one can easily realize that: $F_N^2(\llbracket\ |r) = S_N(r)$. Thus, $F_2^2(\llbracket\ |r) = S_2(r)$ so it corresponds with a circle. Also, $F_3^2(\llbracket\ |r) = S_3(r)$ and it belongs to a 3-dimensional sphere.

c. Even bearing simple structures, though, the Fermat surfaces $F_2^3(\llbracket\ |r)$ and $F_3^3(\llbracket\ |r)$ pose challenging problems.

6. Shells in vector spaces

The concept of a shell in a vector space has been useful in rationalizing the vector structures and allowing the construction of sets and subsets of vectors with some add-on property [17].

Essentially, shells were employed to study quantum mechanical density functions developed in references [18-21].

The previous definition of Fermat surfaces in the present paper corresponds to a similar construct obtained from another perspective. The main idea is to elaborate some mathematical tools to build all the vectors of a given vector space from a subset of them only. Such a procedure uses homothecies of the vector elements belonging to a shell, constituting a well-defined vector set, which one shall associate to some Euclidian norm in the same way one constructs Fermat surfaces.

In this sense, Fermat surfaces constitute a general point of view as the involved norms in their definition hold the use of possible larger powers and the associated Minkowski norms.

6.1. Fermat's surface vectors and probability distributions

The vectors of a Fermat surface possess the modules of their components such that their powers: $\left\{ |f_I|^p \,|\, I = 1, N \right\} \subset \mathsf{F}^+$ belong to the non-negative part of the field elements.

In this manner, one could consider the Fermat surface vectors as able to generate a N-dimensional discrete probability distribution by forming the homothecy:

$$\forall\langle\mathbf{p}| = r^{-1}\langle\mathbf{f}| = \left(r^{-1}f_1, r^{-1}f_2, r^{-1}f_3, \ldots, r^{-1}f_N, 1\right) \in F_N^p\left(\mathsf{F}|1\right)$$
$$\rightarrow E_p\left(\langle\mathbf{p}|\right) = \sum_{I=1}^{N}|p_I|^p = r^{-p}\sum_{I=1}^{N}|f_I|^p = 1 \tag{8}$$

The equation (8) above shows that one can transform a vector lying on a Fermat surface into a unit shell element, which is also closely related to discrete probability distributions.

6.2. The shape of Fermat's surfaces

It is instructive to glimpse the shape of Fermat's surfaces. In the first step, one can remember the discussion about the connection of Fermat's surfaces of second-order and N-dimensional spheres, or in short: $F_N^2\left(\mathsf{F}|r\right) \equiv S_N\left(r\right)$.

Such an equivalence includes second-order natural Fermat vectors in these surfaces of any dimension, as it has been obtained empirically on several occasions [8,10].

The equivalence between second-order Fermat surfaces and spheres seems to preclude one might imagine the Fermat surfaces of superior order as spheroids, distorted spheres. However, simple tests seem to predict a completely different landscape. Fermat's surfaces of higher orders and dimensions, that is: $F_N^{p>2}\left(\llbracket\ |1\right)$, which can be straightforwardly defined via the attached Minkowski norm:

$$\forall p > 2 : M_N^p\left(\langle\mathbf{f}|\right) = \sum_{I=1}^{N}|f_I|^p - 1 = 0 \tag{9}$$

provide drawings, which become in the limit of infinite order to N-dimensional hypercubes. In the third order, drawings look like smooth or blunt-like hypercubes, which tend to structure corners with right angles as the power order grows.

Figure 1 corresponds to the plots of two-dimensional Fermat's surfaces starting at the third order (Fig. 1A), followed by orders 4, 7, 9, and 11 (Figs. 1B, 1C, 1D, 1E).

Of course, second-order surfaces are a circumference, and the 3-dimensional ones are a sphere. Therefore, they are not shown in the following figures.

Figure 1 shows a trend of the surfaces when the order grows: the smooth two-dimensional surface square tends to transform into a sharp square.

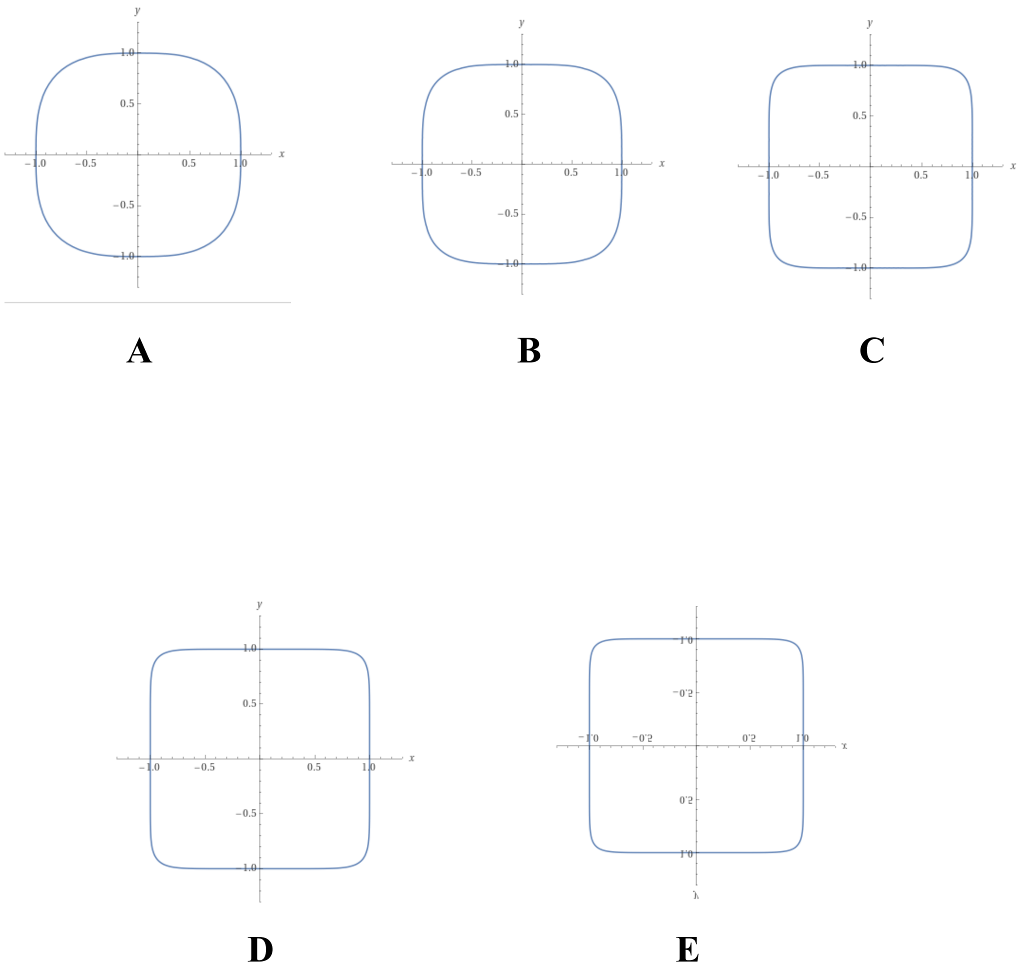

Figure 1. Shapes of the Fermat 2-dimensional surfaces of different orders: A) p=3 B) p=4 C) p=7 D) p=9 E) p=11

However, a third-order 3-dimensional Fermat's surface corresponds to a completely different object, resembling an edge and vertex smoothed or blunt-like 3-dimensional cube, as Figure 2 shows. This time, to evidence the surface trend with increasing order, Figure 2 shows orders 3, 7, 9, 47, 127 (Figure 1A, 2B, 2C, 2D, 2E).

In this sequence, the transformation from a sphere to a 3-dimensional smoothed cube is clear for order 3, and at the same time, the transformation of the smoothed $p=3$ cube towards a sharp structure appears evident as large order $p=41$ and $p=127$ surfaces show.

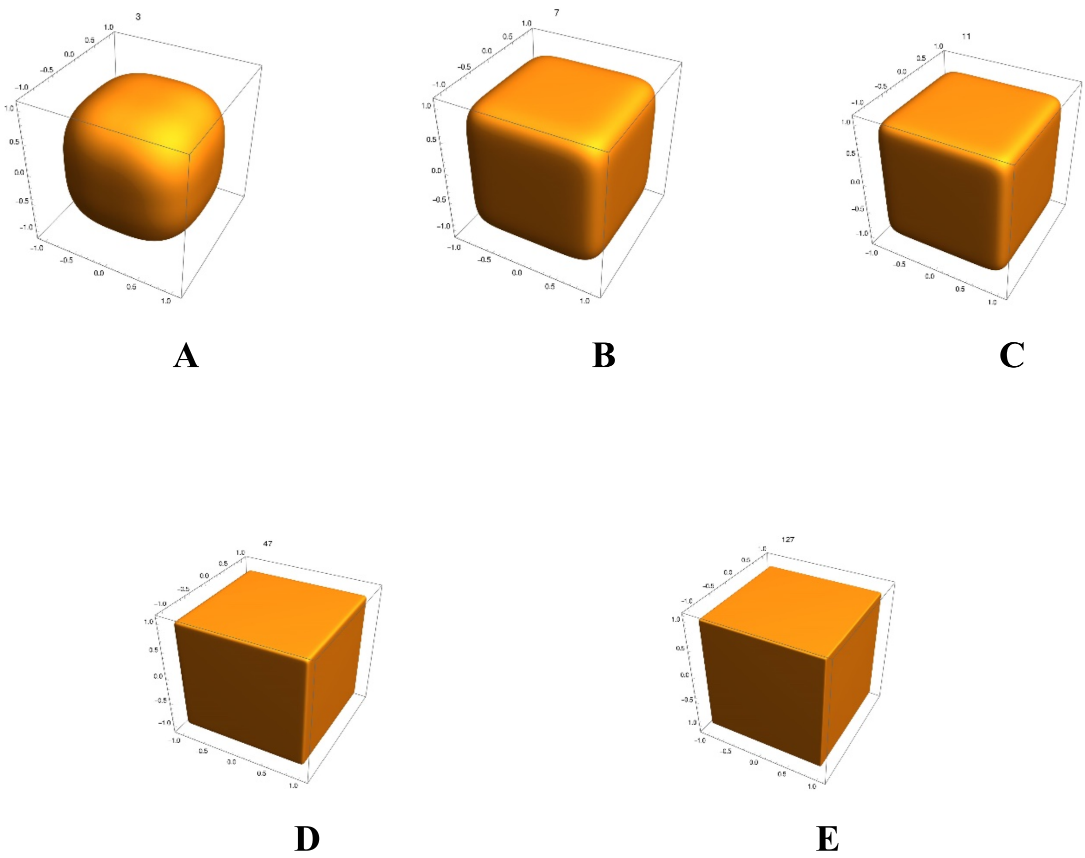

Figure 2. Shapes of three-dimensional Fermat's surfaces for diverse orders: A) p=3 B) p=7 C) p=11 D) p=47 E) p=127

Perhaps, in the light of the results, Figures 1 and 2, one is allowed to write, being H_N an *N*-dimensional hypercube, that:

$$\lim_{p\to\infty} F_N^p\left(\mathsf{F}|1\right) = H_N . \tag{10}$$

This final result is easy to accept when realizing that the structure of hypercubes is such that the construction of the hypercube H_N can be done with the concatenation of two hypercubes of one lesser dimension, so formally, one can write in general:

$$H_{N+1} = H_N \oplus H_N, \tag{11}$$

a feature indicating that the structure of a higher-dimension hypercube will be like the one of a lesser dimension.

So, the edge and vertex smoothness one can observe, say, in Figure 2A, which one can consider as a 3-dimensional cube but generated as a 3-dimensional Fermat surface of order 3, can be imagined it will be the same in the 4-dimensional surface of order 3, which one can construct via concatenation of two 3-dimensional surfaces of order 3. Unit radius Fermat's surfaces can follow a similar concatenation as hypercubes permit.

7. Conclusions

This paper discusses the nature of the surfaces generated when developing mathematical and computational tools to study the extension of Fermat's last theorem in vector spaces of arbitrary dimension.

The main trait that one can notice about Fermat's surfaces is the association of these surfaces with Minkowski spaces and vectors with zero Minkowski norms.

One of the deduced characteristics is the connection of Fermat's surfaces of unit radius, first with discrete probability distributions and second with a general unit shell structure.

Finally, the shapes of Fermat's surfaces have been observed as a transformation of *N*-dimensional hyperspheres into smoothed hypercubes, which tend to become N-dimensional hypercubes as the surface orders increase.

One must admit that extending Fermat's theorem to arbitrary dimensions is highly connected with transforming hyperspheres into smoothed hypercubes and finally to hypercubes of (in)finite dimensions.

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