Finsler non–connectivity to Clifton–Pohl Torus for Geodesic Completeness

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https://doi.org/10.32388/H4ZAZW

Abstract: *Here I will use the Hopf–Rinow Theorem to check the difference in geodesic completeness and non–geodesic completeness for Finsler Manifold and Clifton–Pohl Torus.*

Keywords: *Differential Geometry, Riemannian Manifolds.*

MSC (2020): *53 – XX*

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PROOF

Considering a smooth manifold S having a metric signature (S,g) where this S is a differentiable manifold having a continuously differentiable curve γ of length $\mathcal{J}$ such that there exists the function $\mathcal{J}(\gamma)$ having the infimum of length between the continuously piecewise differentiable curves $[\bar{x},\bar{y}]$ which exists for $\gamma:[\bar{x},\bar{y}] \longrightarrow S$ where the distance $D(m,n)$ is measured by,

$$\begin{cases}\gamma(\bar{x}) = m \\ \gamma(\bar{y}) = n\end{cases}$$

There exists the length $\mathcal{J}(\gamma)$ and energy functional $\epsilon(\gamma)$ where for the Riemannian manifold $M \in S$ the length can be given by,

$$\mathcal{J}(\gamma) = \int_{\bar{x}}^{\bar{y}} \sqrt{g_{\gamma(t)}(\dot{\gamma}(t),\dot{\gamma}(t)\,)}\,dt$$

If one considers a gauge function for G of a complex vector space V then this can be determined for every seminorm s on $\tilde{G}$ there exists for $G \subseteq V$ for the interval $s\tilde{G}: G \to [0,\infty]$ then for the relation[1],

$$\bigcup_{s\in I\equiv G\equiv O\subseteq G_V} \tilde{G}_I$$

For the arbitrary subset O one can define the relation,

$$\{\tilde{G} \in G: s(G) < 1\} \subseteq V \subseteq \{\tilde{G} \in G: s(G) \leq 1\}$$

Thus, defining the Finsler metric $(F,\tilde{g})$ such that $F \in S$ one can modify the length of the curve of Riemannian manifold M where the analogy holds for F for the same $\gamma:[\bar{x},\bar{y}] \longrightarrow S$ such that,

$$F \approx \left(g_{\gamma(t)}\right)^{\frac{1}{2}}$$

$$\mathcal{J}_F(\gamma) \equiv \int_{\bar{x}}^{\bar{y}} F\left(\left(\int \dot{\gamma}(t)\right)^2 dt, \dot{\gamma}(t)\right) dt$$

Where for $\mathcal{J}(\gamma)$ the energy functional in case of $M \in S$ is given by,

$$\epsilon(\gamma) = \frac{1}{2}\int_{\bar{x}}^{\bar{y}} g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)\ )\ dt$$

Whereas previously shown,

$$F \approx \left(g_{\gamma(t)}\right)^{\frac{1}{2}} \text{ in } \mathcal{J}(\gamma)$$

Thus, this can reduce to,

$$F^2 \approx g_{\gamma(t)} \text{ in } \epsilon(\gamma)$$

Therefore, for $\gamma: [\bar{x}, \bar{y}] \longrightarrow S$ having endpoints $\begin{cases} \gamma(\bar{x}) = m \\ \gamma(\bar{y}) = n \end{cases}$ one can easily define the energy functional for Finsler manifold in case of $F \in S$ is,

$$\epsilon_{F[\gamma]} = \frac{1}{2}\int_{\bar{x}}^{\bar{y}} F^2\left(\left(\int \dot{\gamma}(t)\, dt\right), \dot{\gamma}(t)\right)\, dt$$

The representations of the curve being positive and the length minimising nature with homogeneity assumes the parameterization of the manifold geometry where the length of the curve takes the form for $\mathcal{J}_F[\gamma]$ yielding the relation[2-4],

$$\mathcal{J}_F(\gamma) \equiv \int_{\bar{x}}^{\bar{y}} F\left(\left(\int \dot{\gamma}(t)\right)^2 dt, \dot{\gamma}(t)\right) dt$$

$$\xrightarrow{\gamma|_{[l,m]} \longrightarrow S:\gamma[l] \rightarrow \gamma[m]} \mathcal{J}_F[\gamma] \equiv \int_{\bar{x}}^{\bar{y}} F\left(\left(\int \dot{\gamma}(t)\right)^2 dt, \dot{\gamma}(t)\right) dt$$

The length minimising equations of motions for functional $\epsilon(\gamma)$ as the geodesic completeness in a local coordinate can be provided via the Christoffel symbol $\Gamma_{a,b}^{\curlywedge}$,

$$\frac{d^2 x^{\curlywedge}}{dt^2} + \Gamma_{a,b}^{\curlywedge} \frac{dx^a}{dt}\frac{dx^b}{dt} = 0$$

Where for the manifold M in the C^{∞} atlas one can get smooth functions Δ for two limits β_1 and β_2 for $C^{\infty}[(\beta_1, \beta_2)]$ for k^{th} limit on Δ_i dependent on n^{th} variable on x_j there exists,

$$\bigvee_{i=1}^{k} \Delta_i \xrightarrow{dependable\ on} \bigvee_{j=1}^{n} x_j$$

For a coefficient of span δ_j such that this satisfies,

$$f \cong \bigvee_{j=1}^{n} \delta_j$$

To determine the equation,

$$\sum_{j=0}^{n} \sum_{\delta_1 \leq, \ldots, \leq \delta_j} (-1)^j \partial_f^j \left(\frac{\partial \mathcal{L}}{\Delta_i, f} \right)$$

Where for $g_{\overline{mn}}$ and the function $\bar{\Delta}$ where if we define for the strongly convex parameter $c > 0$: this can be justified for all the x such that the equation satisfies,

$$x \mapsto \Delta x - \frac{c}{2} \|x\|^2 \ \exists \Delta''(x) \geq c > 0$$

To satisfy the equation,

$$g_{\overline{mn}}(x, z) := g_z \left(\left. \frac{\partial}{\partial x^{\overline{m}}} \right|_x, \left. \frac{\partial}{\partial x^{\overline{n}}} \right|_x \right)$$

Then for the curve $\gamma: [\bar{x}, \bar{y}] \longrightarrow S$ the geodesic parameter satisfies through a tangent curve $\gamma': [\bar{x}, \bar{y}] \longrightarrow TS \backslash \{0\}$ the geodesic equation can be given for the manifold F in $[F, S]$ via,

$$\bar{S}[\Delta] = \int_{\beta_1}^{\beta_2} \left(\mathcal{J}_F \circ \frac{\Delta}{\Delta t} \right) (t) \, dt \ \exists \forall t$$

$$\in [\beta_1, \beta_2] \text{ in neighbourhood} \frac{\Delta}{\Delta t}(t)$$

$$\forall \text{ frame } \left(x^i, X^i \right) \exists i: \left. \frac{d}{dt} \frac{\partial \mathcal{J}_F}{X^i} \right|_{\frac{\Delta}{\Delta t}(t)} = \left. \frac{\partial \mathcal{J}_F}{x^i} \right|_{\frac{\Delta}{\Delta t}(t)}$$

For the length minimising curve as taken in $[F, S]$ the geodesic will satisfy the Euler–Langrage motion on curves $\epsilon_{F[\gamma]}$ where $\gamma: [\bar{x}, \bar{y}] \longrightarrow S$ where for the norm $x \mapsto \Delta x - \frac{c}{2}\|x\|^2$ $\exists\Delta''(x) \geq c > 0$ as taken in strongly convex form for the metric $g_{\overline{mn}}$ which can be defined by F^2 in $F^2 \approx g_{\gamma(t)}$ in $\epsilon(\gamma)$ formalism as described in the above equations; then for the space of the symmetric tensors on h^{th} order of a vector space $V^{\otimes h}$ one can define the bilinear form for the symmetric tensor $T \in V^{\otimes h}$ where a bilinear mapping parameter ∇ can give a map of $V \times V$ to the field H such that if one defined the dimension of the symmetric form σ one can define a direct sum taking its dimensions,

$$deg_{\binom{\sigma \quad +h \quad -1}{h}} = \bigoplus_{h=0}^{\infty} Sym(T)$$

Where the geodesic complete manifold $T^\star$ having the geodesic $\cap$ for which the tangent space can be defined taking the exponential map at point $\bar{\bar{\beta}}$ it is easy to show that for tangent $\tilde{T}$ and the tangent manifold $\tilde{T}_{\bar{\bar{\beta}}}T^\star$ such that on the Riemannian manifold M for the metric (M, g) the bilinear symmetric form of tensor can be defined over the summation as[5,6],

$$\bigoplus_{h=0}^{\infty} Sym\left(\bigwedge^2 T^{\wedge}M\right)$$

The geodesic over the manifold is complete for,

$$\cap: \ (\rho_k \longrightarrow M)$$

Where,

$$\bigvee_{k=-\infty}^{\infty} \rho_k$$

Where ρ defines the coefficient of a specific operand that defines the Euler–Lagrange equations in a matrix form with the operand having the parameter as,

$$\rho\left[\bigvee_{i=1}^{k}\Delta_i\right]\cong\begin{pmatrix}\dfrac{\partial\mathcal{L}}{\partial\Delta_1} & - & \sum_{j=1}^{n}\dfrac{\partial}{\partial x_j} & \left(\dfrac{\partial\mathcal{L}}{\partial\Delta_{1,j}}\right) & = & 0_1\\ \dfrac{\partial\mathcal{L}}{\partial\Delta_2} & - & \sum_{j=1}^{n}\dfrac{\partial}{\partial x_j} & \left(\dfrac{\partial\mathcal{L}}{\partial\Delta_{2,j}}\right) & = & 0_2\\ \vdots & & & \vdots & & \vdots\\ \dfrac{\partial\mathcal{L}}{\partial\Delta_k} & & \sum_{j=1}^{n}\dfrac{\partial}{\partial x_j} & \left(\dfrac{\partial\mathcal{L}}{\partial\Delta_{k,j}}\right) & & 0_k\end{pmatrix}$$

Where if we take the scheme $\cap_{\gamma(\bar{\bar{\beta}})}$ then for the curves along γ which when converge to the initial point $\gamma(i)$ having a conjugate form $\gamma(\bar{\imath})$ one can define shorter curves such that $i>\bar{\bar{\imath}}$ to proceed towards $\gamma(\bar{\bar{\imath}})$ from $\gamma(\bar{\imath})$ in the F^2 norms there exists the geodesics $[F,M]$ taking the tangent vector $V^{\otimes h}$ provides a scheme of being symmetric bilinear for the Hessian taking the projection,

$$\bigcup_{\substack{\nu\in V^{\otimes h}\\ \nu\neq 0}}\mathcal{P}_{V^{\otimes h}}(X,Y)\equiv\mathcal{P}^{\circ}$$

Giving,

$$2^{-1}\frac{\partial^2}{\partial i\partial\bar{\bar{\imath}}}(F\nu^2+(iX)^2+(\bar{\bar{\imath}}Y)^2)\;\forall i=\bar{\bar{\imath}}=0$$

Considering the Riemannian manifold M having a metric form (M,g) where, as we have considered taking two points in the geodesic curve γ for the geodesic completeness scheme $\cap_{\gamma(\bar{\bar{\beta}})}$ such that $i,\bar{\bar{\imath}}\in\bar{\bar{\bar{\beta}}}$ when those two points are the length minimising geodesic for a complete metric space having closed and bounded subsets of that metric space such that for a notion of $i,\bar{\bar{\imath}}\in M$ for the metric (M,g) apart from the entire space being a tangent space $T_i(M,g)\;\forall\; i\in M$ there exists a converging Cauchy series to make it geodetically complete. This is the essence of the Hopf–Rinow Theorem that can't be determined in case of the Clifton–Pohl Torus for the fact that if we determine a Cauchy series c_γ taking the points i_k points and i_n points for the metric space $(\bar{\bar{\bar{M}}},\bar{\bar{g}})$ where one if determines the distance between i_k and i_n

where for a positive real number of points in the Cauchy series the parameterization should be η_{c_γ} where $k, n \in \eta_{c_\gamma}$ giving the relations[7,8],

$$\bigvee_{k=1}^{>\infty} i_k$$

$$\bigvee_{n=1}^{>\infty} i_n$$

If we determine the Cauchy sequence for $\eta_{c_\gamma} > 0$ having the distance $\bar{\bar{g}}(i_k, i_n) < \eta_{c_\gamma}$ for $k, n > \overline{\eta_{c_\gamma}}$ $\exists \overline{\eta_{c_\gamma}}$ denotes positive integer while η_{c_γ} is positive real number. This, thus defines the completeness of the manifold $\left(\bar{\bar{M}}, \bar{\bar{g}}\right)$ which is the case of Finsler Manifold.

Thereby getting,

$$\left.\begin{array}{ccc} \text{Geodesic complete} & \xrightarrow{\text{Hopf–Rinow Theorem} \cong} & \text{Finsler Manifold} \\ \\ \text{Non} - \text{Geodesic Complete} & \xrightarrow{\text{Hopf–Rinow Theorem} \ncong} & \text{Clifton} - \text{Pohl Torus} \end{array}\right|$$

This can easily be defined for $\mathbb{R}^n$ while for the punctured place the concept of geodesic completeness like $\mathbb{R}^n$ doesn't hold giving it as $\mathbb{R}^n \backslash \{0\}$ where in the former case for the concerned manifold having metric norm $\left(\bar{\bar{M}}, \bar{\bar{g}}\right)$ for $\eta_{c_\gamma} > 0$ and a specific point $\bar{p} \in \bar{\bar{M}}$ a function can be defined being a semicontinuous functional on $\bar{\bar{M}}$ and not identically equal to $+\infty$,

$$\tilde{\Delta}: \bar{\bar{M}} \longrightarrow \mathbb{R} \bigcup \{+\infty\}$$

Where, if one denote $\xi := \sqrt{\eta_{c_\gamma}}$ then for $\xi > 0$ and an arbitrary point $p \in \bar{\bar{M}}$ for the existence of,

$$\tilde{\Delta}(p) \leq \tilde{\Delta}(\bar{p}) \text{ for } d(\bar{p}, p) \leq \xi$$

Where in the manifold $\bar{\bar{M}}$ for a discontinuous group action for a group ϖ acts giving the structure a quotient space which is a torus $T \equiv \bar{\bar{M}}/\varpi$ when the isometry group ϖ action referred the quotient as Clifton–Pohl Torus provided the parent group is ξ and this suffice,

$$\varpi \subset \xi$$

Thus, the curve $\gamma(l)$ for the manifold $\bar{\bar{M}}$ forms a null geodesic that is incomplete for $\gamma(l) := (+\infty, 0)$ if $l = 1$ in the equation,

$$\gamma(l) := (1-l)^{-1}, 0$$

REFERENCE

1. Hall, B. C. (2015). *Lie groups, lie algebras and representations: An elementary introduction*. Springer.
2. Bhattacharjee. D. Eliminating Hyperbolic Curvature of 9-Dimensions in Quotient Space for Elliptic Generators, 02 May 2023, PREPRINT (Version 1) available at Research Square [https://doi.org/10.21203/rs.3.rs-2879172/v1]
3. Davvaz, B. (2021). *A first course in group theory*. Springer Singapore.
4. Bhattacharjee, D. (2022): An outlined tour of geometry and topology as perceived through physics and mathematics emphasizing geometrization, elliptization, uniformization, and projectivization for Thruston's 8-geometries covering Riemann over Teichmuller spaces. TechRxiv. Preprint. https://doi.org/10.36227/techrxiv.20134382.v1
5. Hall, M. (2018). *The theory of groups*. Dover Publications, Inc.
6. Bhattacharjee, D. (2022, October 31). Establishing Equivariant Class [O] for Hyperbolic Groups. *Asian Research Journal of Mathematics*, 362–369. https://doi.org/10.9734/arjom/2022/v18i11615
7. Bhattacharjee, D. (2023). Algebraic properties of $\boldsymbol{PGL2}$ (ℂ) for long exact fibration sequence with sporadic extensions. *International Journal of Research in Science and Technology*, *13*(02), 13–02. https://doi.org/10.37648/ijrst.v13i02.001
8. Tu, L. W. (n.d.). *Differential geometry: Connections, curvature and characteristic classes*. Springer.

The author holds no conflicting interests related to this paper.