

Whole Perfect Vectors and Fermat's Last Theorem

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Abstract

A naïve discussion of Fermat's last theorem conundrum is described. The present theorem's proof is grounded on the well-known properties of sums of powers of the sine and cosine functions, the Minkowski norm definition, and some vector-specific structures.

Keywords

Fermat's Last Theorem; Whole Perfect Vectors; Sine and Cosine functions; Natural and Rational Vectors; Fermat Vectors

1. Introduction

Time has passed since the Wiles 100-page demonstration of Fermat's last theorem [1]. Meanwhile, our laboratory has been working on several computational aspects of this mentioned theorem. These studies were focused on extending Fermat's theorem to larger dimensions and involving powers of higher natural numbers [2-5].

Still, as far as the present author knows, it seems that no new alternative proofs of the theorem exist, conforming to the handwritten note left by Fermat about a supposedly straightforward proof of the famous initial self-formulated theorem.

The present paper describes a simple proof of Fermat's last theorem.

2. Whole perfect vectors

In three-dimensional vector semispaces, see references [6-10], constructed on the non-negative real set: $V_3(\mathbb{R}^+)$, one can define a (*whole*) *perfect* vector: $\langle \mathbf{p} \rangle = (a, b, r)$, see for more information references [10,12], when the vector elements meet the following property:

$$\langle \mathbf{p} | \in V_3(\mathbb{R}^+) \wedge \{a, b, r\} \subset \mathbb{R}^+ \wedge 0 < a < b < r. \quad (1)$$

In such a semispace, the perfect vectors can be collected into a subset \mathbf{P} of the space: $V_3(\mathbb{R}^+)$, that is: $\forall \langle \mathbf{p} | \in \mathbf{P} \subset V_3(\mathbb{R}^+)$.

3. Minkowski n-th order norms

In the perfect vector subset \mathbf{P} , one can define a Minkowski norm of n -th order as follows:

$$\forall \langle \mathbf{p} | = (a, b, r) \in \mathbf{P}: M_n(\langle \mathbf{p} |) = a^n + b^n - r^n. \quad (2)$$

Thus, in the subset \mathbf{P} , one can suppose contained a vector set with a *Banach-Minkowski metric* associated with a *metric vector*: $\langle \mathbf{m} | = (1, 1, -1)$.

A recent general study of Minkowski metric spaces discussed such vector space structure; references [12-14].

For the sake of coherence, a 3-dimensional space with a defined Minkowski norm can be named as (2+1)-dimensional, for example: $V_{(2+1)}(\mathbb{N})$, notes a natural semispace where one has defined a Minkowski metric, like in the equation (2).

4. Homothecy and original vector

The homotheties of an *original* perfect vector $\langle \mathbf{p} |$ are defined as:

$$\forall \langle \mathbf{p} | \in \mathbf{P} \wedge \forall \lambda \in \mathbb{R}^+: \langle \mathbf{h} | = \lambda \langle \mathbf{p} | = (\lambda a, \lambda b, \lambda r) \in \mathbf{P}. \quad (3)$$

The Minkowski norms of the homotheties $\langle \mathbf{h} |$ of perfect vectors are easily related to the ones associated with a perfect origin vector:

$$M_n(\langle \mathbf{h} |) = (\lambda a)^n + (\lambda b)^n - (\lambda r)^n = \lambda^n (a^n + b^n - r^n) = \lambda^n M_n(\langle \mathbf{p} |). \quad (4)$$

This above equation corresponds to the fact that a vector with a Minkowski norm-specific value has the homothetic vector norms as the original vector one multiplied by a factor equivalent to the homothecy parameter, powered to the order of the norm.

5. Extended Fermat vectors

A WP vector can be named as an *extended* Fermat vector when its second-order Minkowski norm is null, that is:

$$\forall \langle \mathbf{f} | \in \mathbf{F} \subset \mathbf{P}: M_2(\langle \mathbf{f} |) = 0 \Rightarrow a^2 + b^2 - r^2 = 0 \Leftrightarrow a^2 + b^2 = r^2. \quad (5)$$

6. Natural Fermat vectors

As natural numbers are a subset of non-negative real numbers: $\mathbb{N} \subset \mathbb{R}^+$, there can exist within the extended Fermat vectors subset, some *natural* Fermat vectors with elements belonging entirely to the natural number set. If this is the case, they can be called shortly *true natural* Fermat vectors (of second order or order 2) and symbolize their subset with \mathbf{T} , that is:

$$\forall \langle \mathbf{t} | = (a, b, r) \in \mathbf{T} \subset \mathbf{F} \wedge \{a, b, r\} \subset \mathbb{N}: M_2(\langle \mathbf{t} |) = 0. \quad (6)$$

The so-called Pythagorean triples are a nickname for true Fermat vectors (of second order).

7. Rational Fermat vectors

Any true Fermat vector can be transformed into a vector with elements defined within the non-negative rational number set: $\mathbb{Q}^+ \subset \mathbb{R}^+$. Such a possibility is easy to consider, as it can be written:

$$\forall \langle \mathbf{t} | = (a, b, r) \in \mathbf{T}: M_2(\langle \mathbf{t} |) = 0 \Rightarrow a^2 + b^2 = r^2 \Rightarrow \left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = 1 \Rightarrow \left\{\left(\frac{a}{r}\right), \left(\frac{b}{r}\right)\right\} \subset \mathbb{Q}^+. \quad (7)$$

Therefore, the vectors defined over the non-negative rational set have the form:

$$\langle \mathbf{k} | = \left(\left(\frac{a}{r}\right), \left(\frac{b}{r}\right), 1\right) \in \mathbf{K} \rightarrow M_2(\langle \mathbf{k} |) = 0, \quad (8)$$

and could be considered as *extended rational* Fermat vectors, with elements constructed over the set \mathbb{Q}^+ , whenever equations (7) and (8) hold.

Therefore, by construction, one can write:

$$0 < a < b < r \leftrightarrow 0 < \frac{a}{r} < \frac{b}{r} < 1, \quad (9)$$

and thus, one can also consider the vectors of the subset \mathbf{K} as possessing elements defined within the 0,1 unit interval.

8. Isomorphism between natural and rational Fermat vectors

In fact, true natural Fermat vectors and rational Fermat vector sets are isomorphic via a homothety, which can also be accepted as acting like an operator, that is:

$$\forall \langle \mathbf{t} | \in \mathbf{T} : \exists r^{-1} \langle \mathbf{t} | = \langle \mathbf{k} | \in \mathbf{K} \Leftrightarrow \forall \langle \mathbf{k} | \in \mathbf{K} : \exists r \langle \mathbf{k} | = \langle \mathbf{t} | \in \mathbf{T}. \quad (10)$$

One can also symbolically write:

$$r^{-1}(\mathbf{T}) = \mathbf{K} \Leftrightarrow r(\mathbf{K}) = \mathbf{T}. \quad (11)$$

Therefore, proving Fermat's last theorem in the set \mathbf{T} is the same as proving it in the set \mathbf{K} , and vice versa.

9. Trigonometric Fermat vectors

Such an isomorphism between true natural and rational Fermat vectors is essential because the vectors in \mathbf{K} can be rewritten with trigonometric functions.

First, note that as the true Fermat vectors are whole perfect vectors, one can suppose that the relations of the equation (9) hold. Second, because one initially deals with natural Fermat vectors, one can also write:

$$a^2 + b^2 - r^2 = 0 \leftrightarrow \left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 - 1 = 0, \quad (12)$$

Then, taking angles in the interval $\alpha \in (0, \frac{\pi}{4}]$, due to the symmetrical nature of the sine and cosine functions, one can write the true natural Fermat vectors as vectors possessing trigonometric functions as elements instead of divisions of two natural numbers, that is:

$$\forall \alpha \in (0, \frac{\pi}{4}] \wedge C = \cos(\alpha); S = \sin(\alpha): \\ \langle \mathbf{u} | = (S, C, 1) \in \mathbf{U} \rightarrow M_2(\langle \mathbf{u} |) = S^2 + C^2 - 1 = 0 \quad (13)$$

9.1. Some remarks on trigonometric Fermat vectors

Not all the trigonometric vectors $\langle \mathbf{u} | \in \mathbf{U}$ written as in equation (13) could be associated to the rational Fermat vectors $\langle \mathbf{k} |$. Trigonometric vectors of type $\langle \mathbf{u} |$ can be seen as extended Fermat vectors. But because all true natural Fermat vectors $\langle \mathbf{t} |$ can generate rational Fermat vectors $\langle \mathbf{k} |$, one can undoubtedly write that:

$$\mathbf{K} \subset \mathbf{U} \Rightarrow r^{-1}(\mathbf{T}) \in \mathbf{U} \rightarrow r(\mathbf{K}) = \mathbf{T}. \quad (14)$$

As we have seen, one can transform all true natural Fermat vectors into rational Fermat vectors, which can also be expressed as trigonometric Fermat vectors.

10. Minkowski norms of trigonometric vectors

Then, due to the nature of the expressions of the powers of the sine and cosine functions, and in compliance with Fermat's theorem, one can write that:

$$\begin{aligned} & \forall \langle \mathbf{u} | = (S, C, 1) \in \mathbf{U} \rightarrow \\ & \forall n \in \mathbb{N} \wedge n \neq 2: M_n(\langle \mathbf{u} |) = S^n + C^n - 1 \neq 0 \end{aligned} \quad (15)$$

Such inequality can be easily proven upon knowing the sum of natural powers expressions of both sine and cosine functions. The most revealing source can be found in reference [15]. Thus, the formulation will not be explicitly repeated here, except for the 1st, 3rd, and 4th powers, given as a short illustrative example below:

$$\begin{aligned} S + C & \neq 1 \\ S^3 + C^3 & = \frac{1}{4}(3(S + C) + \cos(3\alpha) - \sin(3\alpha)) \neq 1 \\ S^4 + C^4 & = \frac{1}{4}(3 + \cos(4\alpha)) \neq 1. \end{aligned} \quad (16)$$

Inequality expressions like those shown in the equation (16) can be easily seen as different from unity *in general*. Sums of larger powers are readily available, yielding terms for the sums of powers, which also clearly differ from the unity.

Therefore, the natural true natural Fermat vectors are isomorphic to some trigonometric vectors $\langle \mathbf{u} |$, and their Minkowski norms satisfy the above equation (15) inequality.

Hence, a Fermat last theorem holds for *all* rational Fermat vectors and thus has to hold for the true natural Fermat vectors because of the isomorphism early discussed between the sets **T** and **K**.

11. Discarding the existence of true natural Fermat vectors of order greater than 2

In previous sections, one has implicitly shown Fermat's last theorem. One can suppose such a demonstration included within the definition of true natural Fermat (2+1)-dimensional vectors by an attached Minkowski norm. This fact makes the existence of true natural Fermat vectors of order higher than two impossible.

To complete Fermat's theorem proof, a discussion follows of whether one might construct natural (2+1)-dimensional vectors as true natural Fermat vectors of orders higher than the second.

Suppose one wants to demonstrate that vectors in any (2+1)-dimensional natural vector space with a well-defined Minkowski norm cannot be true natural Fermat vectors of order higher than 2.

That is:

$$\forall n \in \mathbb{N} \wedge n > 2: \forall \langle \mathbf{p} | = (a, b, r) \in V_{(2+1)}(\mathbb{N}) \Rightarrow M_n(\langle \mathbf{p} |) \neq 0, \quad (17)$$

then one can continue, trying to follow a *reductio ad absurdum* procedure leading to the demonstration.

One can start admitting that the equation (17) is false, so one can write the following property for some natural vector and Minkowski norm order:

$$\exists p \in \mathbb{N} \wedge p > 2: \exists \langle \mathbf{p} | = (a, b, r) \in V_{(2+1)}(\mathbb{N}) \Rightarrow M_p(\langle \mathbf{p} |) = 0, \quad (18)$$

therefore, if the expression (18) is true, then one can also write:

$$\begin{aligned} M_p(\langle \mathbf{p} |) = 0 &\rightarrow a^p + b^p - r^p = 0 \rightarrow a^p + b^p = r^p \\ \Rightarrow \left(\frac{a}{r}\right)^p + \left(\frac{b}{r}\right)^p &= 1 \end{aligned} \quad (19)$$

in the same manner, one can use the above equalities also to write:

$$x = \frac{a}{r} \wedge y = \frac{b}{r} \rightarrow x^p + y^p = 1. \quad (20)$$

However, the pair of rational numbers: $\{x, y\} \subset \mathbb{Q}^+$, corresponds to the Cartesian coordinates of a point situated into a circumference of unit radius.

One can admit such a previous affirmation following a similar reasoning as in section 9. Choosing the appropriate angle in trigonometric coordinates, one can write:

$$\{x, y\} \rightarrow \{\sin(\alpha), \cos(\alpha)\} \equiv \{S, C\}. \quad (21)$$

Then, owing to the equation (21), one can also write:

$$S^2 + C^2 = 1 \rightarrow x^2 + y^2 = 1 \Rightarrow \left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = 1 \rightarrow M_2(\langle \mathbf{p} |) = 0, \quad (22)$$

therefore, the vector $\langle \mathbf{p} |$ of the equation (17) is a rational Fermat vector. Such a result contradicts the existence of Minkowski norms higher than 2, as expressed in the equation (18).

Therefore, one cannot obtain natural vectors fulfilling the equation (18) providing Minkowski norms of order larger than two; thus, the equation (17) must be true.

As a result, one can say that true natural Fermat vectors of order higher than two cannot exist.

Moreover, this implies that in the context of the present study, only true natural Fermat vectors of dimension (2+1) and order two are relevant.

12. Discussion

One can write that some whole perfect natural vectors fulfill the equation concerning the nullity of the second-order Minkowski norm:

$$\exists \langle \mathbf{t} | = (a, b, r) \in \mathbf{T}: M_2(\langle \mathbf{t} |) = 0 \rightarrow \langle \mathbf{t} | = \langle \mathbf{f} |, \quad (23)$$

defining in this way true Fermat vectors of second order.

Here, in general, one has deduced, via the isomorphism between natural and rational Fermat vectors, under a trigonometric representation, that true natural Fermat vectors cannot possess Minkowski null norms other than the second-order ones:

$$\forall n \in \mathbb{N} \wedge n > 2 \wedge \forall \langle \mathbf{v} | = (a, b, r) \in V_{(2+1)}(\mathbb{N}) \Rightarrow M_n(\langle \mathbf{v} |) \neq 0. \quad (24)$$

A result corresponding to how one can formulate Fermat's last theorem.

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References

1. A. Wiles "Modular Elliptic-Curves and Fermat's Last Theorem". *Ann Math* 141 (1995) 443-551.
2. R. Carbó-Dorca; "Natural Vector Spaces, (Inward Power and Minkowski Norm of a Natural Vector, Natural Boolean Hypercubes) and Fermat's Last Theorem". *J. Math. Chem.* 55 (2017) 914-940.
3. R. Carbó-Dorca, C. Muñoz-Caro, A. Niño, S. Reyes; "Refinement of a generalized Fermat's Last Theorem Conjecture in Natural Vector Spaces". *J. Math. Chem.* 55 (2017) 1869-1877.
4. A. Niño, S. Reyes, R. Carbó-Dorca; "An HPC hybrid parallel approach to the experimental analysis of Fermat's theorem extension to arbitrary dimensions on heterogeneous computer systems". *The Journal of Supercomputing* 77 (2021) 11328-11352.
5. R. Carbó-Dorca, S. Reyes, A. Niño; "Extension of Fermat's Last Theorem in Minkowski Natural Spaces". *J. Math. Chem.* 59 (2021) 1851-1863.
6. R. Carbó-Dorca; "Fuzzy sets and Boolean tagged sets, vector semispaces and convex sets, QSM and ASA density functions, diagonal vector spaces and quantum Chemistry". *Adv. Molec. Simil.* Vol. 2 pg. 43-72. JAI Press, (1998).
7. R. Carbó-Dorca; "Shell partition and metric semispaces: Minkowski norms, root scalar products, distances and cosines of arbitrary order". *J. Math. Chem.* 32 (2002) 201-223.
8. P. Bultinck, R. Carbó-Dorca; "A mathematical discussion on density and shape functions, vector semispaces and related questions". *J. Math. Chem.* 36 (2004) 191-200.
9. R. Carbó-Dorca; "Molecular Quantum Similarity Measures in Minkowski Metric Vector Semispaces". *J. Math. Chem.* 44 (2008) 628-636.
10. R. Carbó-Dorca; "Role of the Structure of Boolean Hypercubes when Used as Vectors in Natural (Boolean) Vector Semispaces". *J. Math. Chem.* 57 (2019) 697-

700.

11. R. Carbó-Dorca; “Boolean Hypercubes and the Structure of Vector Spaces”. *J. Math. Sci. Mod.* 1 (2018) 1-14.
12. R. Carbó-Dorca; “Shadows’ Hypercube, Vector Spaces, and Non-Linear Optimization of QSPR Procedures”. *J. Math. Chem* 60 (2022) 283–310.
13. R. Carbó-Dorca, T. Chakraborty; “Extended Minkowski spaces, Zero norms, and Minkowski Hypersurfaces”. *J. Math. Chem.* 59 (2021) 1875-1879.
14. R. Carbó-Dorca; “Generalized Scalar Products in Minkowski Metric Spaces”. *J. Math. Chem.* 59 (2021) 1029-1045.
15. E. W. Weisstein, “Trigonometric Power Formulas”. From Math World: A Wolfram Web Resource.
<https://mathworld.wolfram.com/TrigonometricPowerFormulas.html>