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https://doi.org/10.32388/IBEAQ0

Quantum mechanics and symplectic topology

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Non-relativistic quantum mechanics is in this article built from the indeterminacy relation with the aid of symplectic topology. The concept of probability is re-interpreted and the Schrödinger equation is derived from its conservation. The superposition principle is reformulated.

Keywords: Non-squeezing theorem, Indeterminacy relation, Squeezed coherent states, Probability, Schrödinger equation, Superposition principle

Introduction

It is a curious fact that the indeterminacy relation, despite its fundamental significance for the characteristic behaviour of quantum systems, do not play a foundational role in the various mathematical formulations of non-relativistic quantum mechanics. In e.g. Dirac's transformation theory [1], where quantum states are represented as complex-valued vectors in a Hilbert space, and Feynman's integrals over space-time paths [2], it is the superposition principle which takes the center stage.

This paper intend to describe a new formulation which consider the indeterminacy relation as the foundational principle on which the quantitative theory should be built. This formulation was inspired by the concept of distinguishability between pairs of states as it appears in quantum information theory. There, a useful physical measure for distinguishability is given by the quantum fidelity, which is physically interpreted as the probability that the pair are mistaken for each other by an observer upon measurement.

The mathematical framework employed in the formulation is symplectic topology. This is due to the roles played by the Gromov non-squeezing theorem [3] and the concept of symplectic capacity [4] in representing the indeterminacy relation [5] [6]. The role of symplectic topology in the foundations of non-relativistic quantum mechanics is, in our opinion, under-developed and deserve more attention from the physics community. This paper, despite a significant drawback concerning the appearance of complex-valuedness on phase space, will hopefully add some valuable original ideas on the interpretation of probability and superposition in quantum mechanics.

The quantum fidelity between pairs of states is the absolute square of the complex-valued overlap of the symplectic capacities of the pair. The probability that the identity of the state preparation is mistaken for a member of its quantum ensemble, i.e. the set of states with which it cannot be completely distinguished, is given by the absolute square of the linear sum of complex-valued contributions, one from each alternative overlap of symplectic capacities. This is the superposition principle in the symplectic topological formulation.

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Finite distinguishability

In classical mechanics, it is assumed that the state of the system can be specified with infinite precision. There is no uncertainty in the state. An observer is infinitely able to specify the physical degrees of freedom for the state. Consider any given pair of classical systems. The states of the systems at some time $t = 0$ are given by ψ and ϕ. This define the initial condition for the pair of systems. Due to the infinite ability of the observer to distinguish between states, the pair of systems can either be identified to be identical, i.e. $\psi = \phi$, or, completely distinct, i.e. $\psi \neq \phi$. These are the only two possibilities. Since states in classical mechanics are represented as infinitesimal points on phase space, the systems are identical if the points coincide and distinct if there is a finite distance between them. The Liouville theorem state that the physical distinctions between the pair of systems is conserved in time [7]. In other words, if ψ and ϕ are initially distinguishable, and their distinctions conserved, then their evolutionary paths are not allowed to diverge or converge anywhere on phase space, such that they would become indistinguishable, see Fig.1. The Hamiltonian flow of a classical system is thus incompressible and the volume of a given set of states is conserved in time.

In statistical mechanics, the observer is not infinitely able to specify the state of the system. This is not due to an inherent property of the system. It is entirely due to the difficulty of the observer to keep perfect track of the large number of degrees of freedom. Due to the uncertainty in the state of the system, the ability of the observer to distinguish between states decrease exponentially over time, as stated by the second law of thermodynamics, until the system has reached statistical equilibrium where all states are indistinguishable.

In quantum mechanics, due to the indeterminacy relation, there exist a universal finite upper bound on the ability of the observer to distinguish between the states of any given pair of systems. To state the indeterminacy relation in the language of symplectic topology, it

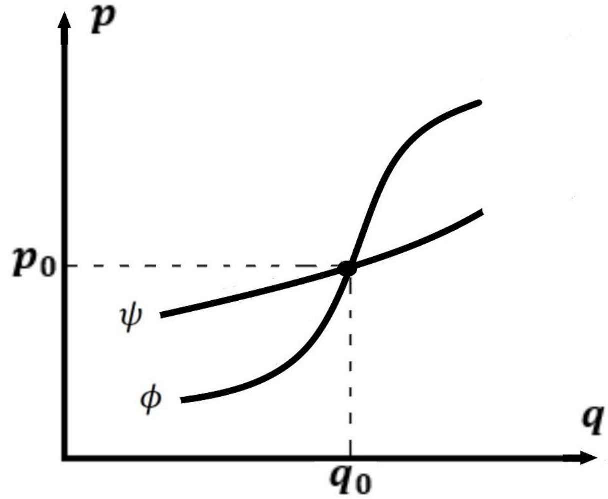

FIG. 1. The pair of initial conditions ψ and ϕ, defining the distinct states of a pair of classical systems, converge at the point (q_0, p_0), thus becoming indistinguishable. This violate the Liouville theorem and is therefore not allowed in classical mechanics.

is first necessary to mathematically define the notion of quantum state within symplectic topology.

Squeezed coherent states

Due to finite distinguishability, it is impossible to physically define, in the sense of observation, the notion of the state as given by an infinitesimal point. In other words, the geometry of phase space is pointless. To obtain a picture of the notion of state on a pointless phase space, consider an $N-$particle system, in d spatial dimensions, at some given time $t = 0$. Let it be assumed that the state of the system, denoted by ψ, is known, at this time, with maximum precision. Such a state is referred to as being saturated. It is further assumed that all conjugate pairs of degrees of freedom for the system, i.e. the coordinate and momenta pairs (q_k, p_k), with $k \in \{1, 2, ..., n\}$ where $n = d \cdot N$, are known to the same level of maximum precision. These symmetric states are the coherent states [8–10]. The state of the system at time $t = 0$, $\psi(t = 0)$, occupy the $2n$-dimensional ball $B(\epsilon)$ defined by

$$\sum_{k=1}^{n} \left\{(q_k - a_k)^2 + (p_k - b_k)^2\right\} = \epsilon^2, \tag{1}$$

with radius ϵ and origin (a_k, b_k). This define the initial condition of the system. Due to the spherical symmetry in the initial condition, the orthogonally projected area $A^k_\psi(t = 0)$ of the ball onto any given conjugate pair (q_k, p_k), see Fig.2, is given by

$$A^k_\psi(t = 0) = \pi\epsilon^2 \quad \forall k \in \{1, 2, ..., n\}. \tag{2}$$

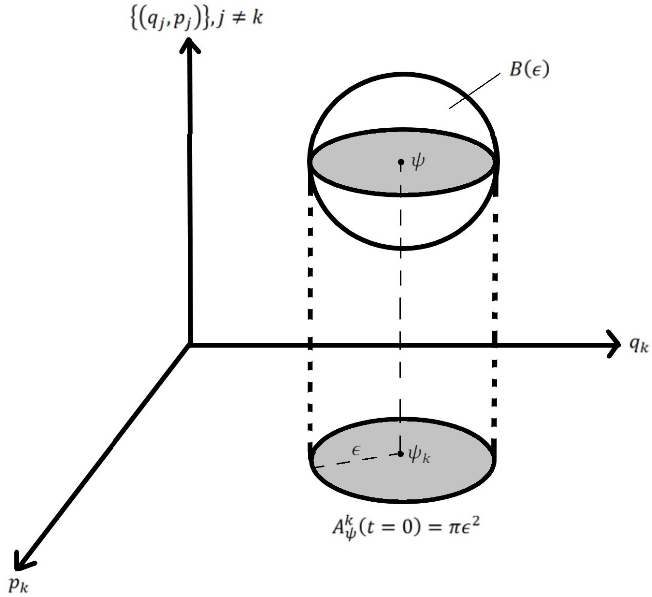

FIG. 2. The projected area A^k_ψ of the phase-space ball $B(\epsilon)$ onto the conjugate pair (q_k, p_k), at time $t = 0$, is given by the minimum uncertainty area $\pi\epsilon^2$.

The projected area $\pi\epsilon^2$ represent the maximum level of precision by which the state of the system can be known for each conjugate pair. In other words, the radius ϵ quantify the greatest resolution available to the observer. Upon the identification of the resolution ϵ with the Planck constant h according to

$$\epsilon \equiv \sqrt{\frac{h}{2\pi}}, \tag{3}$$

the minimum uncertainty area of the projection of the ball $B(\sqrt{h/2\pi})$ onto the conjugate plane k is given by

$$A^k_\psi(t = 0) = \frac{h}{2} \quad \forall k \in \{1, 2, ..., n\}. \tag{4}$$

The ball $B(\sqrt{h/2\pi})$ is thus a representation for the coherent state ψ. More generally, the saturated initial condition ψ can have its minimum uncertainty non-symmetrically distributed between the position and momenta. These states are the squeezed coherent states [10–13]. In the limit that $h \to 0$, the coherent, squeezed or not, state ψ collapse into an infinitesimal point. This is the classical approximation, valid at large scales relative to $h/2$.

The appearance of complex-valuedness in quantum mechanics is a major mystery. In the present article, it is later shown that in order to reproduce the Schrödinger equation it is necessary that the interior of the coherent state, as represented by the interior of the ball $B(\sqrt{h/2\pi})$, is complex-valued. This condition state that the position and momentum degrees of freedom cannot be considered as real-valued measurable quantities at scales lower than $h/2$. But what does it mean to say that the phase space of quantum systems is complex-valued below a certain scale? This condition appear strange and difficult to physically interpret. However, it seems to be that the very notion of state loses its physical meaning below this scale due to the impossibility of the observer to gain additional information about the physical distinctions characterizing the system.

Indeterminacy relation

The coherent states, squeezed or not, are the states which can be distinguished to greatest resolution. Therefore, the projected area $A_\xi^k(t)$ for an arbitrary state ξ, at any given time t, onto the conjugate plane (q_k, p_k), is either equal to, or greater than $h/2$, i.e.

$$A_\xi^k(t) \geq \frac{h}{2} \quad \forall k \in \{1, 2, ..., n\}. \tag{5}$$

This is the indeterminacy relation on phase space. It states that the shape of the state ξ cannot deform during its Hamiltonian flow in such a way that it breach the lower bound as defined by $h/2$. In the language of symplectic topology, the projected area A_ξ^k is referred to as the symplectic capacity c_ξ^k and its minimum value, i.e. $h/2$, as the Gromov width c_G [4]. The arbitrary state ξ is thus mathematically represented by the set of symplectic capacities $\left\{c_\xi^1, ..., c_\xi^k, ..., c_\xi^n\right\}$. The indeterminacy relation thus state that the symplectic capacities of an arbitrary state ξ cannot deform during its Hamiltonian flow in such a way that its value gets smaller than the Gromov width[1], i.e.

$$c_\xi^k(t) \geq c_G \quad \forall k \in \{1, 2, ..., n\}. \tag{6}$$

This indeterminacy relation is related to the Robertson-Schrödinger indeterminacy relation[2] [6]. The mathematical proof of the impossibility of squeezing the state ξ into a smaller symplectic capacity than $h/2$ at any given time, as the system experience an Hamiltonian flow, was given by Mikhail Gromov in 1985 [3] and is referred to as Gromov's non-squeezing theorem.

The key character of the quantum Hamiltonian flow, contrasting its classical approximation, is thus the constraint on the shape of the flow as encoded in the indeterminacy relation. This is in direct contradiction with the Liouville theorem. The Liouville theorem state that any initial region on phase space can deform continuously in any conceivable way as long as its volume do not change [14]. Thus, according to the Liouville theorem, it is possible to deform the arbitrary state ξ in such a way that the symplectic capacity onto some given subset of conjugate pairs is smaller than the Gromov width $h/2$, as long as it is balanced by an increase in the symplectic capacity of another subset of conjugate pairs, keeping the volume invariant. Thus, classical mechanics, whose dynamics on phase space is governed by the Liouville theorem, violate the indeterminacy relation and can only be considered a valid approximation when the system is observed at scales much larger than $h/2$.

State overlap

Consider any given pair of systems. The pair of saturated states, ψ and ϕ, at some time $t = 0$, define the initial conditions for the pair of systems. The finite size of the pair of states, as represented by their balls B_ψ and B_ϕ, allow for the possibility that they have a non-zero overlap Γ, see Fig.3. This imply that there exist a subset of symplectic capacities, e.g. c_ψ^k and c_ϕ^k, which have a non-zero overlap, $\Omega_k(\psi, \phi)$. In other words, there might be a non-zero degree of indistinguishability between the pair of states ψ and ϕ if they are sufficiently close to each other. Of course, if the pair of states have zero overlap, for all conjugate planes, then they are completely distinguishable. The total area of overlap, $\Omega(\psi, \phi)$, is

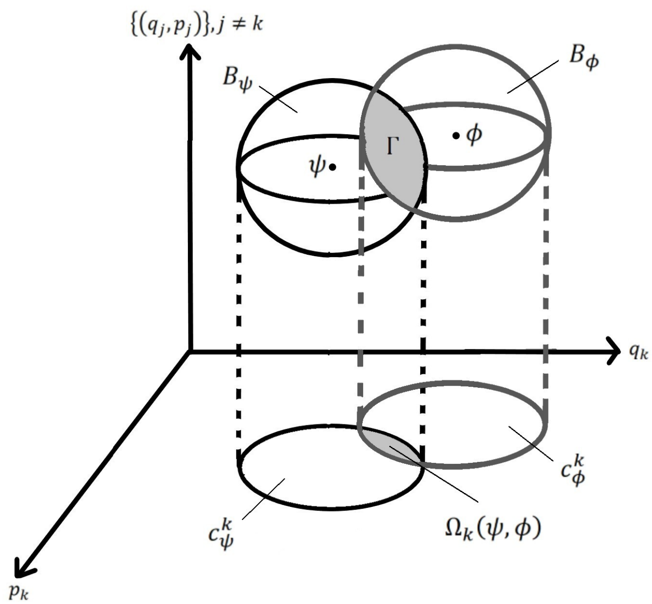

FIG. 3. Given that the state overlap Γ between the saturated balls B_ψ and B_ϕ is complex-valued, the overlap $\Omega_k(\psi, \phi)$ between the symplectic capacities c_ψ^k and c_ϕ^k must also be complex-valued.

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[1] It is important to emphasize that there is no restriction on the symplectic capacity of the state onto a non-conjugate pair of degrees of freedom, i.e. the symplectic capacities for (q_i, q_j), (p_i, p_j) or (q_i, p_j), $\forall i \neq j$, can have arbitrarily small sizes.

[2] The Robertson-Schrödinger indeterminacy relation [11, 15–18] generalize the Heisenberg indeterminacy relation [19] [20] due to its inclusion of the covariance between observables.

given by the linear sum of the contributions Ω_k, for all $k \in \{1, 2, ..., n\}$, i.e.

$$\Omega(\psi, \phi) = \sum_{k=1}^{n} \Omega_k(\psi, \phi). \tag{7}$$

The summation is linear since the $n-$dimensional set of conjugate planes are linearly independent.

Due to the complex-valuedness of the pair of saturated states ψ and ϕ, the overlap $\Omega_k(\psi, \phi)$ between the symplectic capacities c_ψ^k and c_ϕ^k is complex-valued. Furthermore, consider the arbitrary pair of non-saturated states ξ and η whose overlap is $\Omega_k(\xi, \eta)$, see Fig. 4. Independent on the size of their symplectic capacities, and their overlap, it is always possible to define a pair of saturated states ψ and ϕ whose overlap $\Omega_k(\psi, \phi)$ lie within the overlap $\Omega_k(\xi, \eta)$. Thus, given that the overlap $\Omega_k(\psi, \phi)$ is complex-valued, the overlap $\Omega_k(\xi, \eta)$, of which $\Omega_k(\psi, \phi)$ is a part, is also complex-valued.

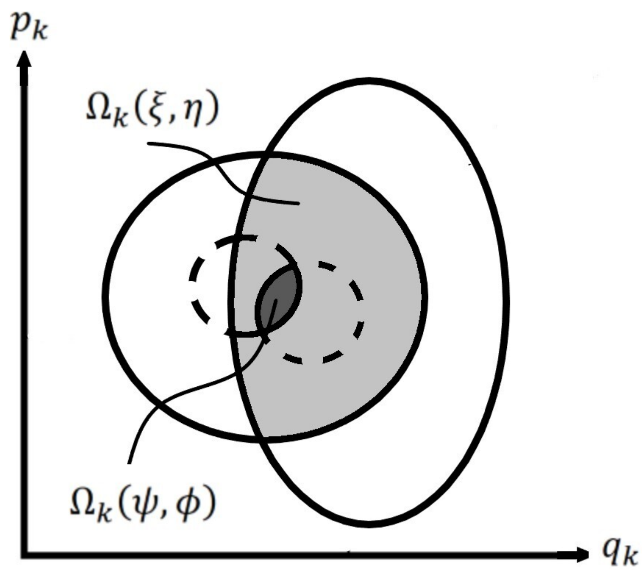

FIG. 4. The overlap $\Omega_k(\xi, \eta)$ between arbitrary pairs of non-saturated states ξ and η is complex-valued.

Fidelity and mistaken identity

Due to the complex-valuedness of the state overlap, it cannot serve as a physical measure for the degree of distinguishability between arbitrary pairs of systems, whose states are given by ξ and η, at some given time t. For the purpose of constructing a useful physical measure, the function $F(\Omega)$ is introduced, and required to satisfy the following set of conditions:

i It is real-valued.

ii It is non-negative, i.e. $F(\Omega) \geq 0$.

iii It is unitless.

iv $F(\Omega) = 0$ iff $\Omega = 0$. The pair ξ and η are completely distinguishable.

v $F(\Omega)$=1 iff $\Omega_k = c_\xi^k$ and/or[3] $\Omega_k = c_\eta^k$ for all $k \in \{1, 2, ..., n\}$. The pair ξ and η are completely indistinguishable.

The conditions (ii) and (v) correspond to the first and second, respectively, Kolmogorov axioms of a probability measure [21]. The physical interpretation[4] of $F(\Omega)$ is thus that it give the probability that the pair of systems, occupying states ξ and η, are mistaken for each other by the observer upon a measurement at the given time t. It is a quantitative measure for the belief of the observer about the state of the system, rather than a description of the state of the system itself. This point of view on the character of probability originate from the works of Cox [23] [24] and, when applied to statistical mechanics, Jaynes [25] [26]. The probability presented here is the symplectic topological representation of the quantity known in quantum information theory as the quantum fidelity between pairs of pure states [27] [28] [29].

The Born rule [30] [31], in the symplectic representation, give the most obvious candidate for the fidelity, satisfying all the imposed conditions, i.e.

$$F(\Omega) = |\Omega(\xi, \eta)|^2. \tag{8}$$

Conservation of probability

Considering that the Liouville theorem is a statement on the conservation of distinguishability between pairs of classical states, its generalization to the pointless geometry of phase space in quantum mechanics is proposed to be given by the following statement:

The distinguishability between an arbitrary pair of quantum states, as measured by quantum fidelity, is conserved in time.

Thus, the fidelities between the pair of states evaluated at arbitrary different times, e.g. t_0 and t, are equal, i.e.

$$F(\Omega)|_t = F(\Omega)|_{t_0}, \tag{9}$$

[3] For saturated states, ψ and ϕ, $\Omega_k = c_\psi^k = c_\phi^k = h/2$. For arbitrary non-saturated states, ξ and η, the possibilities are that $c_\xi^k = c_\eta^k$ or that one of the symplectic capacities are enclosed by the other.

[4] It also has been interpreted as the probability associated with the process that the states transition into each other, and thus referred to as the transition probability [22].

or, alternatively,

$$\frac{F(\Omega)|_t}{F(\Omega)|_{t_0}} = 1. \tag{10}$$

Due to the Born rule, the conservation of fidelity can equivalently be stated in terms of the overlaps as

$$\frac{\Omega^*\Omega|_t}{\Omega^*\Omega|_{t_0}} = 1. \tag{11}$$

The infinitesimal flow of the overlap, from the initial time t_0 to the final time $t = t_0 + \delta t$, to first order in the infinitesimal time step δt, is given by

$$\Omega|_{t_0} \rightarrow \Omega|_t = \Omega|_{t_0} - \delta\Omega|_{t_0,t} \cdot \Omega|_{t_0}, \tag{12}$$

or, alternatively,

$$\frac{\Omega|_t}{\Omega|_{t_0}} = 1 - \delta\Omega|_{t_0,t}, \tag{13}$$

where $\delta\Omega|_{t_0,t}$ represent the infinitesimal change in the overlap, during the time δt, relative to the initial overlap $\Omega|_{t_0}$. The flow of the complex-conjugated overlap is given by

$$\frac{\Omega^*|_t}{\Omega^*|_{t_0}} = 1 - \delta\Omega^*|_{t_0,t} \tag{14}$$

which thus gives that

$$\begin{aligned}\frac{\Omega^*\Omega|_t}{\Omega^*\Omega|_{t_0}} &= (1 - \delta\Omega^*|_{t_0,t})\,(1 - \delta\Omega|_{t_0,t}) \qquad (15)\\ &= 1 - \delta\Omega^*|_{t_0,t} - \delta\Omega|_{t_0,t} + \delta\Omega^*|_{t_0,t} \cdot \delta\Omega|_{t_0,t}\\ &\approx 1 - \delta\Omega^*|_{t_0,t} - \delta\Omega|_{t_0,t},\end{aligned}$$

where the second-order term has been dropped. If quantum fidelity is conserved, then it must be that

$$\delta\Omega^*|_{t_0,t} + \delta\Omega|_{t_0,t} = 0. \tag{16}$$

This is only possible if $\delta\Omega|_{t_0,t}$ is imaginary-valued. Furthermore, since the pair of systems is assumed to be closed, it has no explicit dependence on time, i.e.

$$\delta\Omega|_{t_0,t} \sim i\delta t \cdot \mathcal{H}, \tag{17}$$

where the phase-space function $\mathcal{H}$ is explicitly time-independent and real-valued with the units of energy. It is the Hamiltonian. Thus, in quantum mechanics, the Hamiltonian generate the flow in time of the overlap between pairs of systems. Furthermore, due to the indeterminacy relation, the Hamiltonian cannot quantify changes in the overlap with infinite precision. The Hamiltonian can therefore only be defined in units of the greatest possible resolution $\epsilon = \sqrt{h/2\pi}$. However, since the infinitesimal change in the overlap must be unitless, the measure of resolution that enter into its definition must be $\epsilon^2 = h/2\pi$. Thus, in conclusion, the infinitesimal flow of the overlap is given by

$$\frac{\Omega|_t}{\Omega|_{t_0}} = 1 - i\frac{\delta t \cdot \mathcal{H}}{h/2\pi}. \tag{18}$$

Extending over arbitrarily many time-steps m, such that $m \cdot \delta t = t - t_0$, the flow in time of the overlap is determined by

$$\begin{aligned}\frac{\Omega|_t}{\Omega|_{t_0}} &= \lim_{m\rightarrow\infty}\left(1 - i\frac{(t-t_0)}{m}\frac{\mathcal{H}}{h/2\pi}\right)^m \qquad (19)\\ &= e^{2\pi i\mathcal{H}\cdot(t-t_0)/h}. \qquad (20)\end{aligned}$$

The relation between overlaps at different times is commonly denoted by $U(t, t_0)$, i.e.

$$U(t, t_0) \equiv \frac{\Omega|_t}{\Omega|_{t_0}} = e^{2\pi i\mathcal{H}\cdot(t-t_0)/h}, \tag{21}$$

and referred to as the time-evolution operator. It is unitary, i.e.

$$U^*U = 1. \tag{22}$$

The notion of unitarity is thus just a restatement, by the application of the Born rule, of the conservation of quantum fidelity.

The Schrödinger equation

Eq. 18 can be rewritten as a differential equation, i.e.

$$i\frac{h}{2\pi}\frac{\Omega|_t - \Omega|_{t_0}}{\delta t} = \mathcal{H}\Omega|_{t_0}, \tag{23}$$

which becomes

$$i\frac{h}{2\pi}\frac{\partial\Omega(t)}{\partial t} = \mathcal{H}\Omega(t). \tag{24}$$

This is the Schrödinger equation for the overlap. It is a direct consequence of conservation of quantum fidelity. This is the analog of the situation in classical mechanics, where the Hamilton equations are the direct consequences of the Liouville theorem. Thus, the Schrödinger equation is a representation for the quantum generalization of the Liouville theorem.

The Schrödinger equation predict exactly the value of the overlap at some given time, if the initial condition on the overlap is known. This displays the key difference between the notion of determinism in classical and quantum mechanics. In classical mechanics, the exact state of the system is predictable at any given time, given the initial condition. In quantum mechanics, the state cannot be predicted with absolute certainty. It is only the overlap between the symplectic capacities of pairs of states which is exactly predictable, given the initial overlap.

Ensemble of similar states

Consider an ensemble of closed systems. Each member has been submitted to the same, arbitrary, state preparation ξ at the same time $t = 0$. This define the initial conditions for the members of the ensemble. Alternatively, a single system could be considered. The requirement is that it is observed in many successive trials and before each new measurement it is resubmitted to the same state preparation ξ[5]. In classical mechanics, the initial condition can, by assumption, be prepared with infinite precision. The members of the ensemble are thus identical copies of each other. Identical measurements on the identical members will yield the same experimental outcomes. In quantum mechanics, due to the non-zero overlap between ξ and η, this is no longer the situation. The state preparation ξ might be mistaken for the state η by the observer upon a measurement. By this, it is meant that eventhough the system is prepared in the state ξ, it might occupy the state η, due to their non-zero overlap. In other words, while the observer thought the system was prepared in ξ it might have been prepared in η. Therefore, when the measurement is performed, the system might be found in the state η rather than the state preparation ξ. In this sense, the two states are mistaken for each other, from the perspective of the observer. Thus, the members are not necessarily identical. Identical measurements on the ensemble will not necessarily yield the same results. More generally, consider the situation when the state preparation ξ has a non-zero overlap with each member of the $M-$ dimensional set of states $\{\eta_1, ..., \eta_j, ..., \eta_M\}$, i.e. $\Omega_k(\xi, \eta_j) \neq 0,\ \forall j \in \{1, 2, ..., M\}$, see Fig.5. Such a set will be referred to as the quantum ensemble associated with the state preparation ξ. Then, the initial condition ξ might be mistaken for any given state η_j in the quantum ensemble. The members of the ensemble of systems, all of which have been submitted to the same state preparation ξ at the same time, are thus not necessarily identical. However, they are similar, in the sense that they all have a non-zero overlap with ξ. Upon measurement, there will be a statistical distribution for the states in which the systems are found, depending on the size of the overlap between ξ and the members of the quantum ensemble. If e.g. the overlaps are all equal, i.e. $\Omega_k(\xi, \eta_j) = \Omega_k(\xi, \eta_i),\ \forall i \neq j \in \{1, 2, ..., M\}$, then it is expected, in the limit of a very large ensemble of systems, that all states in the quantum ensemble will appear an equal number of times.

[5] The typical single-system experiment is the double-slit experiment, where e.g. individual electrons are subsequently, and independently from each other, submitted to the same initial condition.

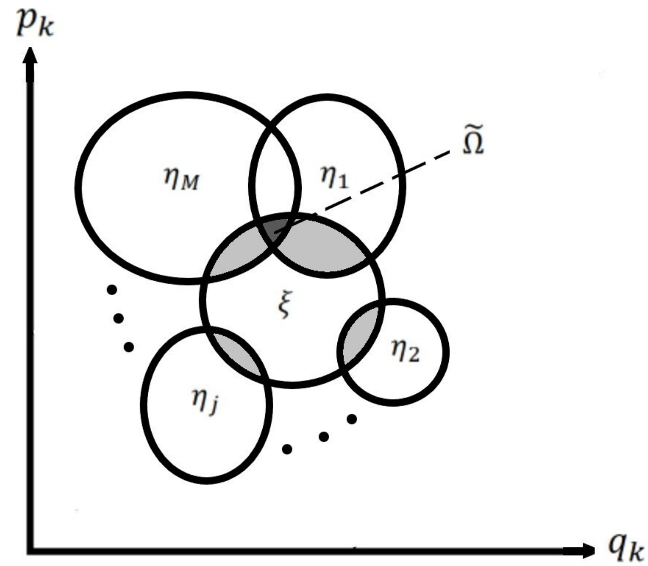

FIG. 5. The $M-$dimensional quantum ensemble associated with the state preparation ξ is defined by the set of all states $\{\eta_1, ..., \eta_j, ..., \eta_M\}$ which have a non-zero overlap with ξ. The black shaded area, denoted by $\tilde{\Omega}$, i.e. the mutual overlap between ξ, η_1 and η_M, is physically constrained to be zero.

In conclusion, the state preparation ξ, due to the possibility that it is mistaken for any other state in its quantum ensemble $\{\eta_1, ..., \eta_j, ..., \eta_M\}$, cannot be understood, from the observational point of view, to describe a unique and specific state of an individual system. It describes an ensemble of systems which are similar, in the sense that their states have non-zero overlaps with the state preparation. This is despite the fact that they have been prepared, from the viewpoint of the observer, in an identical manner. This is the quantum ensemble interpretation of the quantum state as advocated in this article. There have been many other variants of ensemble interpretations for the quantum state, see e.g. the excellent reviews [32] [33] and references therein.

Superposition of overlaps

Consider the linear combination, or, superposition, of overlaps between the state preparation and its quantum ensemble, denoted by $\omega(\xi|\eta_1, ..., \eta_M)$, i.e.

$$\omega(\xi|\eta_1, ..., \eta_M) \equiv \sum_{j=1}^{M} a_j \Omega(\xi, \eta_j), \tag{25}$$

where the coefficients a_j are in general complex-valued. Since the fidelity between the state preparation and any given member of the quantum ensemble is postulated to be conserved in time, any given overlap $\Omega\left(\xi,\eta_j\right)$ is a solution to the Schrödinger equation. Therefore, due to the linearity of the Schrödinger equation, the linear combination $\omega\left(\xi|\eta_1,...,\eta_M\right)$ is also a solution.

For the quantum ensemble depicted in Fig.5, the overlaps $\Omega(\xi,\eta_1)$ and $\Omega(\xi,\eta_M)$ have a part which is mutual, denoted by $\tilde{\Omega}$. Therefore, it is necessary to consider the subtraction $\omega-\tilde{\Omega}$ in order to not count the same area twice. The mutual overlap $\tilde{\Omega}$ do not represent a physical solution of the Schrödinger equation. This is because the Schrödinger equation originate from the postulate that the quantum fidelity between pairs of quantum states is conserved in time. The mutual overlap $\tilde{\Omega}$ is not an overlap between pairs of states and hence cannot be incorporated into the postulate. Thus, there is no physical reason which suggest that it should satisfy the Schrödinger equation. The mutual overlap should therefore be excluded from any physical discussions on the distinguishability between the state preparation and its quantum ensemble. The quantum ensemble is for this reason physically constrained by the requirement that there exist no mutual overlaps between the state preparation and two, or more, members of the quantum ensemble. Put differently, the set of overlaps between the state preparation and the members of the quantum ensemble are linearly independent from each other.

However, it should be noted that any given pair of members of the quantum ensemble are allowed to have non-zero overlaps with each other, i.e. $\Omega(\eta_i,\eta_j)\neq 0,\ \forall i\neq j\in\{1,2,...,M\}$, as long as this overlap do not coincide partially with the state preparation.

Ensemble fidelity

The superposition of overlaps, Eq.25, can be used to generalize the notion of quantum fidelity to measure the distinguishability between the state preparation and the quantum ensemble. Consider the situation when $M=2$. The fidelity $F\left(\omega\left(\xi|\eta_1,\eta_2\right)\right)$ for the linear combination $\omega\left(\xi|\eta_1,\eta_2\right)=a_1\Omega\left(\xi,\eta_1\right)+a_2\Omega\left(\xi,\eta_2\right)$ becomes, using the Born rule,

$$\begin{aligned} F &= \left|a_1\Omega(\xi,\eta_1)+a_2\Omega(\xi,\eta_2)\right|^2 \\ &= |a_1|^2F\left(\Omega(\xi,\eta_1)\right)+|a_2|^2F\left(\Omega(\xi,\eta_2)\right)+ \\ &+ a_1^*a_2\Omega^*(\xi,\eta_1)\Omega(\xi,\eta_2)+a_2^*a_1\Omega^*(\xi,\eta_2)\Omega(\xi,\eta_1) \\ &= \sum_{j=1}^{M=2}|a_j|^2F\left(\Omega(\xi,\eta_j)\right) \\ &+ \sum_{j=1}^{M=2}\sum_{i\neq j}^{M=2}a_j^*a_i\Omega^*(\xi,\eta_j)\Omega(\xi,\eta_i). \end{aligned} \tag{26}$$

The last two terms clearly illustrate the key difference between the notion of probability in statistical and quantum mechanics. In classical probability theory, any disjoint pair of events satisfy Kolmogorov's third axiom [21]. Thus, the classical prediction would be that if the state preparation ξ were mistaken for e.g. the state η_1, then that would exclude the possibility that ξ were mistaken for the state η_2, with the consequence that the fidelity for the linear combination would be given by

$$F\left(\omega\left(\xi|\eta_1,\eta_2\right)\right)=F\left(\Omega(\xi,\eta_1)\right)+F\left(\Omega(\xi,\eta_2)\right). \tag{27}$$

In quantum mechanics, on the other hand, there are additional terms which mix the states η_1 and η_2, despite the fact that the the members of the ensemble of systems are all supposed to be closed. The conclusion is thus that the mistaking of identity for the state preparation with the states η_1 and η_2 are not mutually exclusive[6]. This type of non-exclusivity between members of the quantum ensemble is referred to as quantum interference. It is the key distinction between the theories of statistical and quantum mechanics.

For an arbitrary M-dimensional quantum ensemble, the fidelity for the ensemble is given by

$$\begin{aligned} F\left(\omega(\xi|\eta_1,...,\eta_M)\right) &= \sum_{j=1}^{M}|a_j|^2F\left(\Omega(\xi,\eta_j)\right) \\ &+ \sum_{j=1}^{M}\sum_{i\neq j}^{M}a_j^*a_i\Omega^*(\xi,\eta_j)\Omega(\xi,\eta_i). \end{aligned} \tag{28}$$

The physical interpretation of the fidelity is that it give the probability associated with the event that the state preparation is mistaken for any given state in the quantum ensemble upon measurement by an observer. Given this interpretation, the ensemble fidelity is required to satisfy the condition $0\leq F\left(\omega\right)\leq 1$. Clearly, $F(\omega)=0$ when $\Omega(\xi,\eta_j)=0,\ \forall j\in\{1,...,M\}$, at which the state preparation is completely distinguishable from the quantum ensemble. It is furthermore real-valued for arbitrary non-zero overlaps, for all possible complex-valued coefficients. The requirement that the ensemble fidelity is bounded from above by unity, i.e. $F(\omega)\leq 1$, is the problem of normalization in quantum mechanics. It amounts to the statement that, in the limit $M\to\infty$, it is guaranteed that the state preparation will be mistaken by the observer upon measurement. In other words, the normalization condition is given by

$$\lim_{M\to\infty}F\left(\omega(\xi|\eta_1,...,\eta_M)\right)=1. \tag{29}$$

Conclusion

To summarize, we have attempted to formulate the theory of non-relativistic quantum mechanics within the

[6] Put differently, in the jargon of transition probability, the transitions $\xi\to\eta_1$ and $\xi\to\eta_2$ cannot be considered as mutually exclusive events.

language of symplectic topology, in which the following set of postulates are taken as its foundation:

i The state of a system is represented by its set of symplectic capacities on the complex-valued phase space.

ii The symplectic capacity of a state is constrained from below by the Gromov width $c_G = h/2$.

iii The probability F that the identity of a state ξ is mistaken for any given member of its $M-$dimensional quantum ensemble $\{\eta_1, ..., \eta_j, ..., \eta_M\}$ is given by

$$F = \sum_{j=1}^{M} |a_j|^2 \cdot |\Omega(\xi, \eta_j)|^2 + \sum_{j=1}^{M} \sum_{i \neq j}^{M} a_j^* a_i \Omega^*(\xi, \eta_j) \Omega(\xi, \eta_i) \tag{30}$$

where $\Omega(\xi, \eta_j)$ is the overlap between the symplectic capacities of the pair of states ξ and η_j.

iv For a closed Hamiltonian system, the probability is conserved in time.

The first postulate encode the requirement that the phase space must become complex-valued at scales below the Gromov width. Otherwise, the formulation will not be able to reproduce standard results in quantum mechanics. This condition seem ad-hoc but signify the fundamental mystery associated with the appearance of complex-valuedness in quantum mechanics which is present in all contemporary formulations. Consider e.g. the complex-valuedness appearing in the Hilbert space of state vectors and the probability amplitudes associated with space-time paths. They are not derived, nor explained from a more fundamental principle, but rather postulated by necessity to obtain theoretical predictions which agree with measurements. The problem here, however, is to precisely understand and physically interpret what it means to say that the phase space of quantum systems is complex-valued below a certain scale. We have not been able to bring any clarity to this issue, which is thus a significant drawback of the article. In any case, we believe that the article approach the foundations of non-relativistic quantum mechanics in an original manner by trying to connect it with the mathematics of symplectic topology and the non-squeezing theorem. Hopefully, the article will encourage the physics community to take a greater interest in this programme and strengthen the weaknesses presented in this article.

The second postulate is the indeterminacy relation, stating that there exist a universal finite limit to the precision by which the state can be distinguished. This give rise to the necessity to introduce the concept of probability, as defined in the third postulate. Due to the complex-valuedness of the phase space, probability in quantum mechanics has some special properties as compared to classical probability theory. In particular, disjoint pair of overlaps are not necessarily mutually exclusive, giving rise to quantum interference. The fourth postulate is the statement of unitary evolution as represented mathematically by the Schrödinger equation.

I thank Maurice A. de Gosson and Pontus Vikstl for valuable comments.

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