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NEGATIVITY, ZEROS, AND EXTREME VALUES OF SEVERAL SINGLE-VARIABLE POLYNOMIALS

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Abstract. In the paper, by Descartes' rule of signs and other techniques, the authors present the negativity, zeros, and extreme values of the single-variable polynomials

$$\begin{aligned} G(t) &= 5t^{43} - 218t^{30} + 720t^{17} - 455t^{13} - 52, \\ H(t) &= 5t^{29}\sum_{\ell=0}^{12}(13-\ell)t^{\ell} - t^{16}\sum_{\ell=0}^{12}(2704-213\ell)t^{\ell} \\ &\quad - 169t^{13}\sum_{\ell=0}^{2}(7+3\ell)t^{\ell} - 52\sum_{\ell=0}^{12}(\ell+1)t^{\ell}, \\ J(t) &= 43t^{30} - 1308t^{17} + 2448t^{4} - 1183, \\ K(t) &= 43t^{17}\sum_{k=0}^{12}t^{k} - 1265t^{4}\sum_{k=0}^{12}t^{k} + 1183\sum_{k=0}^{3}t^{k}. \end{aligned}$$

1. Motivations and main results

On 21 March 2023, via the Tencent QQ, Professor Chao-Ping Chen (Henan Polytechnic University, China) claimed that the polynomial

$$52 + 455t^{13} - 720t^{17} + 218t^{30} - 5t^{43}$$

is positive on $[0,1)$.

In this paper, we prove the following propositions.

Proposition 1. *The polynomial*

$$G(t) = 5t^{43} - 218t^{30} + 720t^{17} - 455t^{13} - 52$$

is negative on the interval $[0,1)$ *and* $G(1) = 0$.

Proposition 2. *The polynomial*

$$J(t) = 43t^{30} - 1308t^{17} + 2448t^{4} - 1183$$

is decreasing on $(-\infty, 0)$, *totally has four real zeros on* $(-\infty,\infty)$: *a negative zero on* $\left(-1, -\frac{1}{2}\right)$, *a positive zero on* $\left(\frac{1}{2}, \frac{9}{10}\right)$, *the zero* 1, *and another positive zero on*

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$\left(\frac{6}{5},\frac{3}{2}\right)$, *while it totally has two minimums* $J(0)=-1183$ *and*

$$
\begin{aligned}
J\left(\sqrt[13]{\frac{1853+13\sqrt{18241}}{215}}\right)&=-\frac{338\bigl(109\sqrt{18241}\,+8789\bigr)}{1075}\left(\frac{1853+13\sqrt{18241}}{215}\right)^{4/13}\\
&\quad-1183\\
&=-18789.29\dots
\end{aligned}
$$

and a maximum

$$
\begin{aligned}
J\left(\sqrt[13]{\frac{1853-13\sqrt{18241}}{215}}\right)&=\frac{338\bigl(109\sqrt{18241}\,-8789\bigr)}{1075}\left(\frac{1853-13\sqrt{18241}}{215}\right)^{4/13}\\
&\quad-1183\\
&=278.16\dots
\end{aligned}
$$

on $(-\infty,\infty)$.

The polynomial $G(t)$ *totally has two real zeros on* $(-\infty,\infty)$*: a double zero* 1 *and a single zero on* $\left(\frac{6}{5},\frac{3}{2}\right)$*, while it totally has four extreme values on* $(-\infty,\infty)$*: a maximum on* $\left(-1,-\frac{1}{2}\right)$*, two minimums on* $\left(\frac{1}{2},\frac{9}{10}\right)$ *and* $\left(\frac{6}{5},\frac{3}{2}\right)$ *respectively, and a maximum* $G(1)=0$.

The polynomial

$$
\begin{aligned}
H(t)&=5t^{29}\sum_{\ell=0}^{12}(13-\ell)t^\ell-t^{16}\sum_{\ell=0}^{12}(2704-213\ell)t^\ell\\
&\quad-169t^{13}\sum_{\ell=0}^{2}(7+3\ell)t^\ell-52\sum_{\ell=0}^{12}(\ell+1)t^\ell
\end{aligned}
$$

has only one real zero on $(-\infty,\infty)$*, which locates on the open interval* $\left(\frac{6}{5},\frac{3}{2}\right)$.

The polynomial

$$
K(t)=43t^{17}\sum_{k=0}^{12}t^k-1265t^4\sum_{k=0}^{12}t^k+1183\sum_{k=0}^{3}t^k
$$

totally has three single zeros on $\left(-1,-\frac{1}{2}\right)$*,* $\left(\frac{1}{2},\frac{9}{10}\right)$*, and* $\left(\frac{6}{5},\frac{3}{2}\right)$ *respectively.*

2. Proofs of Propositions 1 and 2

In this section, we give two proofs of Proposition 1 and a proof of Proposition 2.

First proof of Proposition 1. The polynomial $G(t)$ can be factorized as $G(t)=(t-1)^2H(t)$. The derivatives of $H^{(k)}(t)$ for $1\le k\le 28$ are

$$
\begin{aligned}
H^{(k)}(t)&=5\sum_{\ell=0}^{12}(13-\ell)\langle 29+\ell\rangle_k t^{29+\ell-k}-\sum_{\ell=0}^{12}(2704-213\ell)\langle 16+\ell\rangle_k t^{16+\ell-k}\\
&\quad-169\sum_{\ell=0}^{2}(7+3\ell)\langle 13+\ell\rangle_k t^{13+\ell-k}-52\sum_{\ell=0}^{12}(\ell+1)\langle \ell\rangle_k t^{\ell-k}
\end{aligned}
$$

and

$$
H^{(29)}(t)=5\sum_{\ell=0}^{12}(13-\ell)\langle 29+\ell\rangle_{29}t^\ell>65\times 29!,\quad t\ge 0,
$$

where an empty sum is understood as 0 and the falling factorial $\langle\alpha\rangle_n$ for $n \ge 0$ and $\alpha \in \mathbb{C}$ is defined by

$$\langle\alpha\rangle_n = \prod_{k=0}^{n-1}(\alpha-k) = \begin{cases}1, & n=0;\\ \alpha(\alpha-1)\cdots(\alpha-n+1), & n \in \mathbb{N}.\end{cases}$$

The values at $t = 0, 1$ of these derivatives are

$$H'(0) = -104, \quad H''(0) = -312, \quad H'''(0) = -1248, \quad H^{(4)}(0) = -6240,$$
$$H^{(5)}(0) = -37440, \quad H^{(6)}(0) = -262080, \quad H^{(7)}(0) = -2096640,$$
$$H^{(8)}(0) = -18869760, \quad H^{(9)}(0) = -188697600,$$
$$H^{(10)}(0) = -2075673600, \quad H^{(11)}(0) = -24908083200,$$
$$H^{(12)}(0) = -323805081600, \quad H^{(13)}(0) = -7366565606400,$$
$$H^{(14)}(0) = -147331312128000, \quad H^{(15)}(0) = -2872960586496000,$$
$$H^{(16)}(0) = -56575223857152000, \quad H^{(17)}(0) = -886017383387136000,$$
$$H^{(18)}(0) = -14584607301648384000,$$
$$H^{(19)}(0) = -251197132344238080000,$$
$$H^{(20)}(0) = -4505734519143137280000,$$
$$H^{(21)}(0) = -83738054219431772160000,$$
$$H^{(22)}(0) = -1602825037810868551680000,$$
$$H^{(23)}(0) = -31358496304267476664320000,$$
$$H^{(24)}(0) = -620448401733239439360000000,$$
$$H^{(25)}(0) = -12207322304101485969408000000,$$
$$H^{(26)}(0) = -231489298686671634825216000000,$$
$$H^{(27)}(0) = -3930881871601025130037248000000,$$
$$H^{(28)}(0) = -45123475002533651354222592000000,$$

and

$$H'(1) = -463905, \quad H''(1) = -7937930, \quad H'''(1) = -134301258,$$
$$H^{(4)}(1) = -2179937760, \quad H^{(5)}(1) = -32335446000,$$
$$H^{(6)}(1) = -384463778400, \quad H^{(7)}(1) = -1422141084000,$$
$$H^{(8)}(1) = 123005557584000, \quad H^{(9)}(1) = 6118209626256000,$$
$$H^{(10)}(1) = 209898202524192000, \quad H^{(11)}(1) = 6258088312283808000,$$
$$H^{(12)}(1) = 172533094320787200000,$$
$$H^{(13)}(1) = 4507415070256530432000,$$
$$H^{(14)}(1) = 112826895527710780416000,$$
$$H^{(15)}(1) = 2719636547804209313280000,$$
$$H^{(16)}(1) = 63248332563385515786240000,$$

$$
\begin{aligned}
H^{(17)}(1) &= 1419354109575085036953600000,\\
H^{(18)}(1) &= 3070760754676225427841024000 0,\\
H^{(19)}(1) &= 63950391823536439871569920000 0,\\
H^{(20)}(1) &= 127945622543209447939522560000 00,\\
H^{(21)}(1) &= 24535866996923990404541644800000 0,\\
H^{(22)}(1) &= 449851248585655164744214118400000 0,\\
H^{(23)}(1) &= 7863573836065379073585384849408000 0,\\
H^{(24)}(1) &= 130657267489759000696316362752000000 0,\\
H^{(25)}(1) &= 205664663526499037034716980838400000 00,\\
H^{(26)}(1) &= 305559094392501547207633344921600000 000,\\
H^{(27)}(1) &= 426729126388456884139075396435968000000 0,\\
H^{(28)}(1) &= 557592733788119348237695991288954880000 00.
\end{aligned}
$$

These long computations imply that,

(1) the derivative $H^{(28)}(t)$ is increasing on $(-\infty,\infty)$ and only has one real zero which locates on the unit interval $(0,1)$,
(2) the derivatives $H^{(k)}(t)$ for $8\le k\le 27$ only have one minimum and only have one real zero on $(0,1)$,
(3) the polynomials $H^{(k)}(t)$ for $1\le k\le 7$ are all negative on $[0,1]$,
(4) the polynomial $H(t)$ is decreasing on $[0,1]$.

From $H(0)=-52$ and $H(1)=-27885$, it follows that $H(t)$ is negative on $[0,1]$. Hence, the polynomial $G(t)=(t-1)^2H(t)$ is negative on $[0,1)$ and $G(1)=0$ clearly. The first proof of Proposition 1 is complete. □

Second proof of Proposition 1. Descartes' rule of signs [6, p. 22] states that,

(1) if the nonzero terms of a single-variable polynomial with real coefficients are ordered by descending variable exponent, then the number of positive zeros of the polynomial is either equal to the number of sign changes between consecutive (nonzero) coefficients, or is less than it by an even number. A zero of multiplicity k is counted as k zeros.
(2) the number of negative zeros is the number of sign changes after multiplying the coefficients of odd-power terms by -1, or fewer than it by an even number.

Consequently, the polynomial $H(t)$ has at most one positive zero. From

$$
H(0)=-52,\quad H(1)=-27885,\quad H(2)=43746480037836,
$$

we easily see that there exists a real zero on $(1,2)$. Therefore, the polynomial $H(t)$ is negative on $[0,1]$. Accordingly, the polynomial $G(t)=(t-1)^2H(t)$ is negative on $[0,1)$ and $G(1)=0$. The second proof of Proposition 1 is complete. □

Remark 1. The first proof of Proposition 1 is long, but it is elementary. The second proof of Proposition 1 is short, but it uses advanced knowledge.

Remark 2. The negativity and its second proof of Proposition 1 were recited in [9, Lemma 2.4] to refine the Shafer–Fink type inequalities for $\arcsin x$, $\arctan x$, and arctanh x. This type of inequalities have been investigated in the papers [2, 3, 4, 7, 8, 10], for example.

Remark 3. Since

$$G(-t) = -5t^{43} - 218t^{30} - 720t^{17} + 455t^{13} - 52,$$

by Descartes' rule of signs, it follows that the polynomials $G(t)$ has either two negative single zeros, or a unique double zero, or no negative zero. Since $G(t) = (t-1)^2 H(t)$, the polynomial $H(t)$ has either two negative single-zeros, or a unique double zero, or no negative zero. As done in [1, Remark 4], if $G(t)$ and $H(t)$ had any negative zero(s), then it (they) must locate between

$$-\min\left\{\max\left\{1, \frac{218}{5} + 144 + 91 + \frac{52}{5}\right\}, 1 + \max\left\{\frac{218}{5}, 144, 91, \frac{52}{5}\right\}\right\} = -145$$

and 0.

The graph of $G(t)$ on $[-1, 0]$, see Figure 1 plotted by Wolfram Mathematica

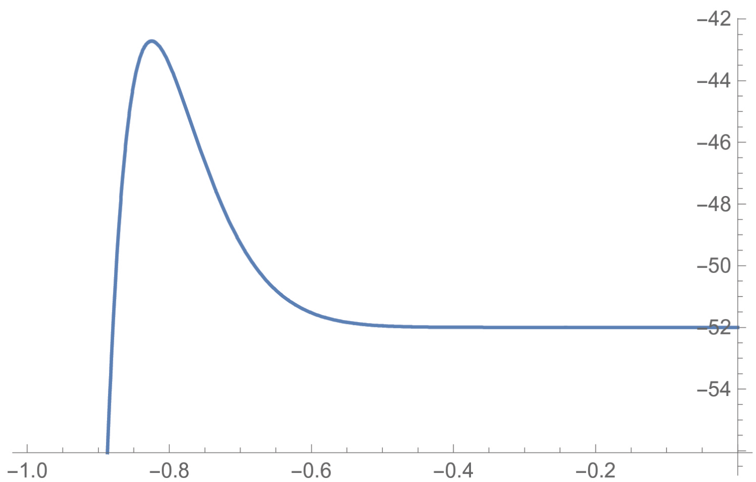

Figure 1. The graph of $G(t)$ on $[-1, 0]$

12, demonstrates that, the unique maximum of $G(t)$ on $(-\infty, 0]$ is negative and the maximum point locates on $(-1, 0)$. This means that,

(1) the polynomials $G(t)$ and $H(t)$ have no negative zero,
(2) the polynomial $K(t)$ is increasing on $(-\infty, 0]$ and has a unique negative zero.

In what follows, we will analytically consider the functions $G(t)$, $H(t)$, $J(t)$, $K(t)$, their zeros, and extreme values in details.

Proof of Proposition 2. By Descartes' rule of signs, the polynomial $J(t)$ has at most one negative zero and at most three positive zeros. Since

$$J(-1) = 2616, \quad J\left(-\frac{1}{2}\right) = -\frac{1105943363541}{1073741824}, \quad J\left(\frac{1}{2}\right) = -\frac{1105964793813}{1073741824},$$

$$J\left(\frac{9}{10}\right) = \frac{2068180831741955742311146606276 43}{10000000000000000000000000000000}, \quad J(1) = 0$$

$$J\left(\frac{6}{5}\right)=-\frac{1389447504912951419429400 7}{931322574615478515625},\quad J\left(\frac{3}{2}\right)=\frac{7481600916309779}{1073741824},$$

we easily see that the polynomial $J(t)$ totally has four zeros which locate on the intervals $\left(-1,-\frac{1}{2}\right)$, $\left(\frac{1}{2},\frac{9}{10}\right)$, $\left(\frac{9}{10},\frac{6}{5}\right)$, and $\left(\frac{6}{5},\frac{3}{2}\right)$ respectively.

Due to that the derivative

$$J'(t)=6t^3\left[215\left(t^{13}\right)^2-3706t^{13}+1632\right]$$

has three real zeros

$$0,\quad \sqrt[13]{\frac{1853-13\sqrt{18241}}{215}}=0.94\dots,\quad \sqrt[13]{\frac{1853+13\sqrt{18241}}{215}}=1.24\dots$$

on the whole $(-\infty,\infty)$, we can immediately write out three extreme values of $J(t)$ on $(-\infty,\infty)$, which are listed in Proposition 2.

It is easy to see that $J'(t)<0$ and $J(t)$ is decreasing on $(-\infty,0)$. By Descartes' rule of signs, the polynomial $J(t)$ has at most one negative zero. Since $J(0)=-1183$ and $J(t)\to\infty$ as $t\to-\infty$, the polynomial $J(t)$ has a unique negative zero. Therefore, the polynomial $G'(t)$ has a unique zero on $(-\infty,0)$ and $G(t)$ has a unique maximum on $(-\infty,0)$.

Directly differentiating and factorizing yield $G'(t)=5t^{12}J(t)$. This means that the derivative $G'(t)$ has five real single zeros 0, 1, and another three ones which locate on $\left(-1,-\frac{1}{2}\right)$, $\left(\frac{1}{2},\frac{9}{10}\right)$, and $\left(\frac{6}{5},\frac{3}{2}\right)$. Since

$$G'(-1)=13080,\quad G'\left(-\frac{1}{2}\right)=-\frac{5529716817705}{4398046511104},\quad G'(0)=0,$$
$$G'\left(\frac{1}{2}\right)=-\frac{5529823969065}{4398046511104},\quad G'\left(\frac{3}{2}\right)=\frac{19880147362822926307695}{4398046511104},$$
$$G'\left(\frac{9}{10}\right)=\frac{58411535366776961210135341569028833025544283}{200000000000000000000000000000000000000000},$$
$$G'(1)=0,\quad G'\left(\frac{6}{5}\right)=-\frac{30245247854937858674400466428260352}{4547473508864641189575195312 5},$$

the polynomial $G(t)$ has a maximum on $\left(-1,-\frac{1}{2}\right)$, two minimums on $\left(\frac{1}{2},\frac{9}{10}\right)$ and $\left(\frac{6}{5},\frac{3}{2}\right)$ respectively, and a maximum $G(1)=0$.

From the relation $G(t)=(t-1)^2H(t)$ and the second proof of Proposition 1, it follows readily that the polynomial $G(t)$ totally has two positive zeros: a double zero 1, and a single zero which locates on $\left(\frac{6}{5},\frac{3}{2}\right)$, where we used the computation

$$G\left(\frac{6}{5}\right)=-\frac{63654172066238835450977047768720 84}{227373675443232059478759765625}$$

and

$$G\left(\frac{3}{2}\right)=\frac{127905339918037458365 5}{8796093022208}.$$

Let

$$q(t)=720t^{17}-455t^{13}-52.$$

Then the derivative $q'(t)=5t^{12}\left(2448t^4-1183\right)$ has only one negative zero

$$-\frac{1}{2}\sqrt[4]{\frac{7}{17}}\sqrt{\frac{13}{3}}=-0.83\dots$$

which is a maximum point such that

$$q\left(-\frac{1}{2}\sqrt[4]{\frac{7}{17}}\sqrt{\frac{13}{3}}\right)=\frac{753295946585}{124696184832}\sqrt[4]{\frac{7}{17}}\sqrt{\frac{13}{3}}-52=-41.92\dots.$$

Hence, the polynomials $q(t)$ and $G(t)=5t^{43}-218t^{30}+q(t)$ are negative on $(-\infty,0)$. Accordingly, the polynomial $G(t)$ and $H(t)$ has no negative zero.

Since $G(t)=(t-1)^2H(t)$, the polynomial $H(t)$ has a unique real zero on $(-\infty,\infty)$, which locates on the open interval $\left(\frac{6}{5},\frac{3}{2}\right)$.

Since $G'(t)=5t^{12}(t-1)K(t)$, considering extreme values of $G(t)$, the polynomial $K(t)$ totally has three single zeros on $\left(-1,-\frac{1}{2}\right)$, $\left(\frac{1}{2},\frac{9}{10}\right)$, and $\left(\frac{6}{5},\frac{3}{2}\right)$ respectively. □

Remark 4. Simple numerical computation by the Wolfram Mathematica 12 shows that the unique real zero of $H(t)$ on $(-\infty,\infty)$ is $1.3300988040778609\dots$. Can one write out an accurate closed-form expression of the unique positive zero of the polynomial $H(t)$ on $(-\infty,\infty)$?

Remark 5. This is a revised version of the electronic preprint [5].

3. Declarations

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