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The new partitional approach to (literally) interpreting quantum mechanics

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Abstract

This paper presents a new 'partitional' approach to understanding or interpreting standard quantum mechanics (QM). The thesis is that the mathematics (not the physics) of QM is the Hilbert space version of the math of partitions on a set and, conversely, the math of partitions is a skeletonized set level version of the math of QM. Since at the set level, partitions are the mathematical tool to represent distinctions and indistinctions (or definiteness and indefiniteness), this approach shows how to interpret the key non-classical QM notion of superposition in terms of (objective) indefiniteness between definite alternatives (as opposed to seeing it as the sum of 'waves'). Hence this partitional approach substantiates what might be called the Objective Indefiniteness Interpretation or what Abner Shimony called the Literal Interpretation of QM.

1 Introduction: The basic thesis

The purpose of this paper is to expound a new way to interpret quantum mechanics (QM), or, to be more precise, to interpret the mathematics (not the physics[1]) of QM. The key mathematical, indeed logical, concept is the notion of a partition on a set–or equivalently, the notion of a quotient set or equivalence relation. The basic thesis is that the math of QM is the Hilbert space version of the math of partitions. Partitions are the basic logical concept to describe distinctions versus indistinctions, definiteness versus indefiniteness, distinguishability versus indistinguishability, or difference versus identity. The key non-classical notion in QM is that of *superposition*–with entanglement being a particularly unintuitive special case. The result of the thesis is to give a partitional explication of superposition in terms of objective indefiniteness between definite alternatives–so that this approach to QM could be called the *Objective Indefiniteness or Literal Interpretation* of QM.

> From these two basic ideas alone – indefiniteness and the superposition principle – it should be clear already that quantum mechanics conflicts sharply with common sense. If the quantum state of a system is a complete description of the system, then a quantity that has an indefinite value in that quantum state is objectively indefinite; its value is not merely unknown by the scientist who seeks to describe the system. Furthermore, since the outcome of a measurement of an objectively indefinite quantity is not determined by the quantum state, and yet the quantum state is the complete bearer of information about the system, the outcome is strictly a matter of objective chance – not just a matter of chance in the sense of unpredictability by the scientist. Finally, the probability of each possible outcome of the measurement is an objective probability. Classical physics did not conflict with common sense in these fundamental ways. [34, p. 47]

[1]The physics of QM is obtained by the quantization of classical physics. Our focus is on the specific nature of the *mathematical* framework of standard von Neumann/Dirac QM.

Abner Shimony suggested calling this interpretation of the math or "formalism of quantum mechanics" as the Literal Interpretation.

> These statements ... may collectively be called "the Literal Interpretation" of quantum mechanics. This is the interpretation resulting from taking the formalism of quantum mechanics literally, as giving a representation of physical properties themselves, rather than of human knowledge of them, and by taking this representation to be complete. [36, pp. 6-7]

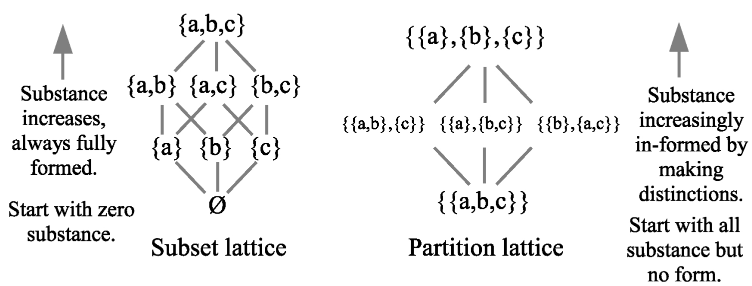
2 The lattice of partitions

A *partition* π on a set $U = \{u_1, ..., u_n\}$ is a set of nonempty subsets or blocks $\pi = \{B_1, ..., B_m\}$ that are mutually disjoint and jointly exhaustive (their union is U).[2] A equivalent definition, that prefigures the Hilbert space notion of a "direct-sum decomposition" of the space in terms of the eigenspaces of a Hermitian operator) is a set of nonempty subsets $\pi = \{B_1, ..., B_m\}$ such that every non-empty subset $S \subseteq U$ can be uniquely represented as the union of a set of nonempty subsets of the B_j–in particular $S = \cup \{S \cap B_j \neq \emptyset : j = 1, ..., m\}$.

As the mathematical tool to describe distinctions versus indistinctions, a *distinction* or *dit* of π is an ordered pair of elements of U in different blocks of the partition, and the *ditset* $\operatorname{dit}(\pi)$ is the set of all the distinctions of π (also called an "apartness relation"). An *indistinction* or *indit* of π is an ordered pair of elements in the same block of the partition, and the *indit set* $\operatorname{indit}(\pi) = \cup_{j=1}^{m} B_j \times B_j$ is the set of all indits of π–which is the equivalence relation associated with π whose equivalence classes are the blocks of π.

The *partial order* on partition is usually defined as $\sigma \precsim \pi$ (where $\sigma = \{C_1, ..., C_{m'}\}$) if for every $B_j \in \pi$, there is a $C_{j'} \in \sigma$ such that $B_j \subseteq C_{j'}$, but it is easier to just define it by $\sigma \precsim \pi$ if $\operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$. The join (least upper bound) and meet (greatest lower bound) operations on partitions on U form the partition lattice $\Pi(U)$. The most important operation for our purposes is the join operation where the *join* $\pi \vee \sigma$ is the partition on U whose blocks are the nonempty subsets $B_j \cap C_{j'}$ for $j = 1, ..., m$ and $j' = 1, ..., m'$. It could also be defined using ditsets since: $\operatorname{dit}(\pi \vee \sigma) = \operatorname{dit}(\pi) \cup \operatorname{dit}(\sigma)$. The partition lattice $\Pi(U)$ also has a top and bottom. The top is the *discrete partition* $\mathbf{1}_U = \{\{u_1\}, ..., \{u_n\}\}$ with only singleton blocks which makes all possible distinctions, i.e., $\operatorname{dit}(\mathbf{1}_U) = U \times U - \Delta$ (where Δ is the diagonal of self-pairs (u_i, u_i)). The bottom is the *indiscrete partition* (or "The Blob"[3]) $\mathbf{0}_U = \{U\}$ with only one block U and it makes no distinctions so $\operatorname{dit}(\mathbf{0}_U) = \emptyset$ and $\operatorname{indit}(\mathbf{0}_U) = U \times U$.

There are two notions of 'becoming' illustrated as going from the bottom to top of the Boolean lattice of subsets and the lattice of partitions in terms of the creation of 'its' or dits. The partition notion of becoming is particularly important for our purposes since it prefigures the notion of quantum (projective) measurement.



[2] We stick to the finite case since our purpose is conceptual rather than obtaining mathematical generality.

[3] Since $\mathbf{0}_U$ is below all other partitions π on U, it is called "The Blob" because, as in the Hollywood movie of that name, the Blog absorbs everything it meets, i.e., $\mathbf{0}_U \wedge \pi = \mathbf{0}_U$.

Figure 1: The two dual notions of becoming.

Applied to the 'universe,' this gives two stories of creation.

Subset creation story: In the Beginning was the Void (no substance) and then fully definite elements ("Its") were created until the full universe U was created.

Partition creation story: In the Beginning was the Blob–all substance with no form (i.e., perfect symmetry)–and then, in a Big Bang, distinctions ("Dits") were created (i.e., symmetries were broken) as the substance was increasingly in-formed to reach the universe U where everything is fully distinguished.

3 The logic of partitions

The partition join and meet operations were known in the nineteenth century (e.g., Dedekind and Schröder). Ordinary logic is based on the Boolean logic of subsets of a universe set U; propositional logic is the special case of a one element universe U. And subsets (or generally subobjects) are category-theoretically dual to partitions (or generally quotient objects). Hence one would naturally expect there to be a dual *logic* of partitions, but that would require at least the operation of implication on partitions (corresponding to the Boolean conditional $S \supset T$), but no new operations on partitions were defined in the twentieth century. As acknowledged in a 2001 volume commemorating Gian-Carlo Rota: "the only operations on the family of equivalence relations fully studied, understood and deployed are the binary join $\vee$ and meet $\wedge$ operations." [5, p. 445] Only in the current century was the implication operation $\sigma \Rightarrow \pi$ on partitions (which turns the lattice $\Pi(U)$ into an algebra) defined along with general algorithms to turn subset logical operations into partition logical operations. The resulting logic of partitions cemented the notion of a partition as not just a mathematical concept of combinatorics but a logical concept. [11]

There is a parallel development of subset logic and partition logic based on the dual connection between the elements or "its" of a subset and the distinctions or "dits" of a partition–which is summarized in Table 1.

Its & Dits	Algebra of subsets $\wp(U)$ of U	Algebra of partitions $\Pi(U)$ on U
Its or Dits	Elements of subsets	Distinctions of partitions
Partial order	Inclusion of subsets $S \subseteq T$	Inclusion of ditsets $\operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$
Logical maps	Injection $S \rightarrowtail T$	Surjection $\pi \twoheadrightarrow \sigma$
Join	Union of subsets	Union of ditsets
Meet	Subset of common elements	Ditset of common dits
Top	Subset U with all elements	Partition $\mathbf{1}_U$ with all distinctions
Bottom	Subset $\emptyset$ with no elements	Partition $\mathbf{0}_U$ with no distinctions
Implication	$S \supset T = U$ iff $S \subseteq T$	$\sigma \Rightarrow \pi = \mathbf{1}_U$ iff $\sigma \precsim \pi$

Table 1: Elements and Distinctions (Its & Dits) duality between the two lattices

In subset logic, a formula is valid if for any U ($|U| \geq 1$) and any subsets of U substituted for the atomic variables, the formula evaluates by the logical subset operations to the top U. Similarly in partition logic, a formula is valid if for any U ($|U| \geq 2$) and any partitions on U substituted for the atomic variables, the formula evaluates by the logical partition operations to the top $\mathbf{1}_U$.

4 Logical information theory: Logical entropy

In his writings (and MIT lectures), Gian-Carlo Rota further developed the parallelism between subsets and partitions by considering their quantitative versions: "The lattice of partitions plays for information the role that the Boolean algebra of subsets plays for size or probability." [28, p. 30]

Since the normalized size of a subset $\Pr(S) = \frac{|S|}{|U|}$ gives its Boole-Laplace finite probability, so the "size" of a partition would play a similar role for information:

$$\frac{\text{Information}}{\text{Partitions}} \approx \frac{\text{Probability}}{\text{Subsets}}.$$

Since "Probability is a measure on the Boolean algebra of events" that gives quantitatively the "intuitive idea of the size of a set", we may ask by "analogy" for some measure "which will capture some property that will turn out to be for [partitions] what size is to a set." [33, p. 67] The duality tells us that it is the number of dits in a partition that gives its size (maximum at the top and minimum at the bottom of the partition lattice) that is parallel to the number of 'its' in a subset (maximum at the top and minimum at the bottom of the subset lattice).

That is the reasoning that motivates the definition of the *logical entropy* of a partition as the normalized size of its ditset (equiprobable points):

$$h(\pi) = \frac{|\text{dit}(\pi)|}{|U \times U|} = \frac{|U \times U| - |\text{indit}(\pi)|}{|U \times U|} = 1 - \frac{\cup_j |B_j \times B_j|}{|U \times U|} = 1 - \sum_j \left(\frac{|B_j|}{|U|}\right)^2.$$

In general, if the points of $U = \{u_1, ..., u_n\}$ have general probabilities $p = \{p_1, ..., p_n\}$, then $\Pr(B_j) = \sum_{u_i \in B_j} p_i$, so that:

$$h(\pi) = 1 - \sum_{j=1}^m \Pr(B_j)^2 = \sum_{j=1}^m \Pr(B_j)(1 - \Pr(B_j)).$$

The parallelism carries through to the interpretation: $\Pr(S) = \sum_{u_i \in S} p_i$ is the one-draw probability of getting an 'it' of S and $h(\pi)$ is the two-draw probability of getting a 'dit' of π.

The logical entropy $h(\pi)$ is a (probability) measure in the sense of measure theory, i.e.,, $h(\pi)$ is the product measure $p \times p$ on the ditset $\text{dit}(\pi) \subseteq U \times U$. As a measure, the compound notions of joint, conditional, and mutual logical entropy satisfy the usual Venn diagram relationships. The well-known Shannon entropy $H(\pi) = \sum_{j=1}^m \Pr(B_j) \log_2 \left(\frac{1}{\Pr(B_j)}\right)$ can also be interpreted in terms of partitions; it is the minimum average number of binary partitions (bits) it takes to distinguish the blocks of π. The Shannon entropy is not a measure in the sense of measure theory but the compound notions of joint, conditional, and mutual Shannon entropy were defined so that they satisfy similar Venn-like diagrams. That is possible because there is a non-linear but monotonic dit-to-bit transform, i.e., $1 - \Pr(B_j) \rightsquigarrow \log_2 \left(\frac{1}{\Pr(B_j)}\right)$, that takes $h(\pi) = \sum_j \Pr(B_j)(1 - \Pr(B_j))$ to $H(\pi) = \sum_j \Pr(B_j) \log_2 \left(\frac{1}{\Pr(B_j)}\right)$ and which preserves Venn diagrams. [13]

If a partition π is the inverse-image partition $\pi = \left\{f^{-1}(r)\right\}_{r \in f(U)}$ of a numerical attribute $f : U \to \mathbb{R}$, then $h(\pi)$ is the two-draw probability of getting different f-values. The notion of logical entropy generalizes naturally to the notion of quantum logical entropy ([13]; [39]) where it gives the probability in two independent measurements of the same state by the same observable that the result gives different eigenvalues.

5 Partitions as skeletonized quantum states

There is a very simple way to skeletonize a quantum state to arrive at the corresponding set notion. Consider $U = \{a, b, c, d\}$ as both a set of distinct points and also as a orthonormal basis for a 4-dimensional Hilbert space. Then a superposition state vector of the form (say) $\alpha |a\rangle + \beta |b\rangle$ is *skeletonized* by deleting the complex scalars α and β, the Dirac kets, and the addition operation to yield just the set $\{a, b\}$ as a block in a partition. This sets up the skeletal (many-to-one) dictionary between pure, non-classical mixed states (i.e., mixed states containing at least one superposition), and the classical mixed state–with their skeletonized partition versions as in Figure 2–where we have used the partition shorthand of representing the partition $\{\{a, b\}, \{c, d\}\}$ as $\{ab, cd\}$ and similarly for the other partitions. The interpretation is that a, b, c, d are distinct eigenstates of a particle according to some observable, where "particle" does not mean a classical (or Bohmian) particle but an entity that can have different levels of objective indefiniteness (superpositions of the eigenstates such as a, b, c, or d) or the definiteness of an eigenstate.[4]

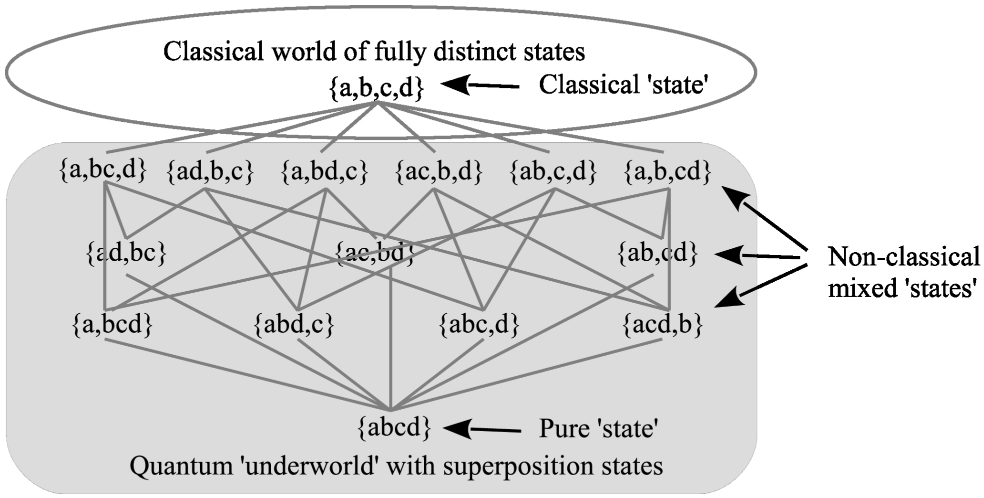


Figure 2: Skeletonized quantum states of a particle as the lattice of partitions.

Figure 2 is the Hasse diagram of the partition lattice on a four element set which means that each line between two partitions represents the partial order of refinement with no intermediate partitions. Then the change from a partition to a more refined one is a "jump." The set level precursor of such a quantum jump is the "choice function" [21, p. 60] which assigns to each nonempty subset (like a block in a partition) an element of the subset. That determination of an element is non-deterministic except in the special case of a singleton set which is the precursor of the quantum measurement when the outcome has probability one, namely when state being measured is a single eigenstate of the observable being measured (i.e., a singleton block in Figure 2).

The top discrete partition $\mathbf{1}_U$ is the skeletal version of a classical mixed state like randomly choosing a leaf in a four-leaf clover or randomly choosing a 'letter' in the four-letter genetic code U, C, A, G. One criterion of classical reality was the idea that it was fully definite or definite all-the-way-down as in Leibniz's Principle of Identity of Indistinguishables (PII) [3, Fourth letter, p. 22] or Kant's Principle of Complete Determination (*omnimoda determinatio*).

> Every thing, however, as to its possibility, further stands under the principle of thoroughgoing determination; according to which, among all possible predicates of things, insofar as they are compared with their opposites, one must apply to it. [25, B600]

Thus two distinct things must have some predicate to distinguish them or if there is no way to distinguish them, then they are the same thing. This principle of classicality characterizes the 'classical' state $\mathbf{1}_U$:

For any $u, u' \in U$, if $(u, u') \in \text{indit}(\mathbf{1}_U)$, then $u = u'$
Partition logical version of Principle of Identity of Indistinguishables.

Any non-classical state in the skeletal representation is a partition π with a non-singleton 'superposition' block, e.g., $\{a, b\}$ so $(a, b) \in \text{indit}(\pi)$ but $a \neq b$. As noted above, any idealized measurement of a classical state (i.e., a singleton) gives the outcome of that state with probability one.

[4] The analysis is of standard von Neumann/Dirac quantum physics, not about quantum field theory where a particle might be analyzed as a "disturbance in a field."

In addition to PII, Leibniz had other metaphysical principles characteristic of the classical notion of reality. His Principle of Continuity was expressed by "*Natura non facit saltus*" (Nature does not make jumps) [29, Bk. IV, chap. xvi] and his Principle of Sufficient Reason was expressed as the statement "that nothing happens without a reason why it should be so rather than otherwise" [3, Second letter, p. 7]. All these classical principles are violated in the quantum world; PII is violated by superpositions of bosons, Continuity is violated by the quantum jumps, and Sufficient Reason is violated by the objective probabilities of QM.

This skeletal representation of quantum states is summarized in Table 2.

Partition concept	Corresponding quantum concept
Non-singleton block, e.g., $\{a, b\}$	Superposition pure state
Indiscrete partition $\mathbf{0}_U = \{\{a, b, c, d\}\}$	Largest pure state
Singleton block, e.g., $\{d\}$	Classical state (no superposition)
Discrete partition $\mathbf{1}_U = \{\{a\}, \{b\}, \{c\}, \{d\}\}$	Classical mixture of states
Partition, e.g., $\{\{a, b, c\}, \{d\}\}$	Mixture of orthogonal states

Table 2: Corresponding partition and quantum concepts

6 Superposition as indefiniteness in the quantum 'underworld'

The biggest 'enemy' to understanding QM is the wave imagery, not to mention the name "wave mechanics." That imagery interprets superposition as the sum of two definite waves to give another *definite* wave as in Figure 3.

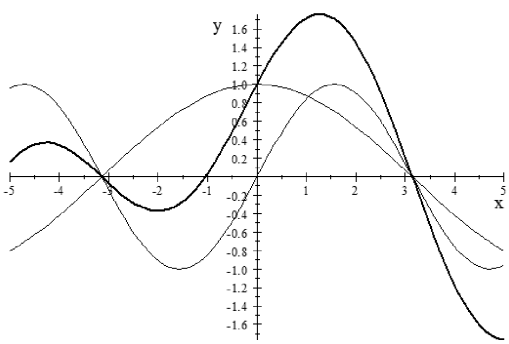


Figure 3: 'Wrong' image of superposition in QM as sum of waves

It is the wave *imagery* that is wrong, not the math of waves since any vector in a space over the complex numbers $\mathbb{C}$ automatically has a wave imagery in the polar representation as having an amplitude and phase. The wave interpretation is a misleading artefact of the use of complex numbers in the math of QM, which is because (among other reasons) they are algebraically complete so that the observable operators will have a complete set of eigenvectors [41, p. 67, fn. 7], not because the 'wave function' describes any physical waves.[5] It is a fact of the mathematics that the addition of vectors in a vector space over $\mathbb{C}$ can *always* be represented as the superposition of waves with interference effects.

> The wave formalism offers a convenient mathematical representation of this latency, for not only can the mathematics of wave effects, like interference and diffraction, be

[5] In view of the century-long difficulties in interpreting QM as "wave mechanics," Einstein's statement: "The Lord is subtle, but not malicious," may be too optimistic.

expressed in terms of the addition of vectors (that is, their linear superposition; see [17, Chap. 29.5]), but the converse, also holds. [24, p. 303]

On the partitional (insstead of wave) approach, a superposition of two definite states $\{a\}$ and $\{b\}$ is a state $\{a, b\}$ that is indefinite between the two definite states. For a pictorial image, Figure 4 gives the superposition of two isosceles triangles with labelled edges and vertices as the triangle that is indefinite on the edges and vertices where they differ and only definite where the two triangles are the same–and certainly not the triangle that is doubly definite (like a double-exposure photograph).

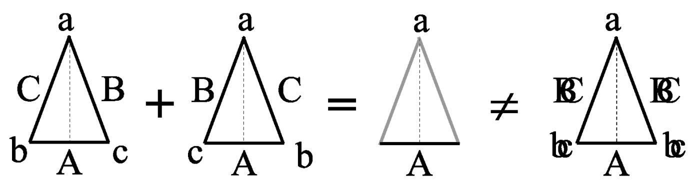


Figure 4: Imagery of superposition as indefiniteness

The fact that where superposed states have the same property is still definite (like vertex label a, the side label A) in the superposition is a little noticed fact about quantum superpositions.

> It follows from the linearity of the operators which represent observables of quantum mechanical systems that any measurable physical property which happens to be shared by all of the individual mathematical terms of some particular superposition (written down in any particular basis) will necessarily also be shared by the full superposition, considered as a single quantum-mechanical state, as well. [2, p. 234]

This means that the notion of superposition in QM and the notion of abstraction (e.g., in mathematics) are 'essentially' the same notion viewed from different angles [12]. In a superposition, the emphasis is on the indefiniteness resulting from where the elements of a set (i.e., a non-singleton equivalence class in an equivalence relation) differ like the vertex labels b and c, and the side labels B and C in Figure 4, while in abstraction, the emphasis is on the properties that are the same for the elements in the set like the vertex label a and side label A. It is like the different viewpoints of seeing a glass as half-empty or as half-full.

It is not a new idea that there is a quantum 'underworld' of indefinite superposition states 'beneath' the classical space-time world of definite states. That view was previously expressed in the language of the quantum world of potentialities (or latencies) versus the classical world of actualities ([23]; [31]; [24, Sec. 10.2]; [36]; [20]; [27], [26]; [8]).

> Heisenberg [23, p. 53]... used the term "potentiality" to characterize a property which is objectively indefinite, whose value when actualized is a matter of objective chance, and which is assigned a definite probability by an algorithm presupposing a definite mathematical structure of states and properties. Potentiality is a modality that is somehow intermediate between actuality and mere logical possibility. That properties can have this modality, and that states of physical systems are characterized partially by the potentialities they determine and not just by the catalogue of properties to which they assign definite values, are profound discoveries about the world, rather than about human knowledge. [36, p. 6]

Ruth Kastner even uses the imagery of an iceberg [26, p. 3] with the classical world above water and the quantum world beneath the water–an imagery that is filled out by the Figure 2 imagery of the partition lattice as the skeletonized classical and quantum states. The language of "potentialities" or the Aristotelean notion of "potentia" is not very felicitous since they are taken to be realities, not mere possibilities. Hence some prefer the language of "latencies" (e.g., Henry Margenau and R. I. G. Hughes), but in both the cases of "potentialities" and "latencies," the key idea is objective indefiniteness.

> The historical reference should perhaps be dismissed, since quantum mechanical potentiality is completely devoid of teleological significance, which is central to Aristotle's conception. What it has in common with Aristotle's conception is the indefinite character of certain properties of the system. [35, pp. 313-4]

And Margenau notes that the measurement of observables "forces them out of indiscriminacy or latency" [31, p. 10]–which indicates that Margenau also interprets "latency" in terms of indeterminacy or indefiniteness. Kastner also considers the indeterminacy of values as a key characteristic of the real potentia [26, p. 3].

7 Quantum states

To demonstrate our thesis that the math of QM is the Hilbert space version of the math of partitions, we need to first focus on the three main concepts in the math of QM: 1) the quantum state, 2) the quantum observable, and 3) the quantum (always projective) measurement.

A quantum state can be presented either as a state vector or as a density matrix [40]. The density matrix approach best displays the relevant information for the partitional interpretation. Hence we start by transferring the structure of a partition π into its density matrix form $\rho(\pi)$. The initial data is a partition $\pi = \{B_1, ..., B_m\}$ on U with (positive) point probabilities $p = (p_1, ..., p_n)$. For each $B_j \in \pi$, define $|b_j\rangle$ as the column vector with the i^{th} entry being $\sqrt{p_i / \Pr(B_j)}$ if $u_i \in B_j$, else 0 so that $\langle b_{j'}|b_j\rangle = \delta_{jj'}$. We form the projection matrix $\rho(B_j) = |b_j\rangle\langle b_j|$ with the i,k-entry being $\rho(B_j)_{ik} = \frac{\sqrt{p_i p_k}}{\Pr(B_j)}$ if $u_i, u_k \in B_j$, else 0. Then the density matrix $\rho(\pi)$ is the probability sum of these projectors:

$$\rho(\pi) = \sum_{j=1}^{m} \Pr(B_j) |b_j\rangle \langle b_j|.$$

Then it is easily checked that $\rho(\pi)_{ik} = \sqrt{p_i p_k}$ if $(u_i, u_k) \in \operatorname{indit}(\pi)$, else 0. Thus the non-zero entries in $\rho(\pi)$ represent the indits of π and the zero entries represent the distinctions of π. A density matrix is not only Hermitian but positive so its eigenvalues are non-negative real numbers λ_i which sum to 1, i.e., $\sum_{i=1}^{n} \lambda_i = 1$. In the case of $\rho(\pi)$, there are m non-zero eigenvalues $\Pr(B_j)$ with the remaining $n - m$ eigenvalues of 0.

For the classical state $\mathbf{1}_U$, its density matrix $\rho(\mathbf{1}_U)$ is a diagonal matrix with the point probabilities along the diagonal, e.g. "the statistical mixture describing the state of a classical dice before the outcome of the throw" [4, p. 176]. Thus the non-classical states in the skeletal representation are the ones where $\rho(\pi)$ has non-zero off-diagonal elements indicating the 'amplitudes' $\sqrt{p_i p_k}$ of the corresponding diagonal states (u_i and u_k) blobbing or cohering together in a superposition. Since superposition states are the key non-classical states, it is these non-zero off-diagonal "coherences" [7, p. 303] that account for the non-classical interference effects in the Hilbert space version.

> [T]he off-diagonal terms of a density matrix ... are often called quantum coherences because they are responsible for the interference effects typical of quantum mechanics that are absent in classical dynamics. [4, p. 177].

In the full non-skeletal Hilbert space case of a density matrix ρ, it has a spectral decomposition $\rho = \sum_{i=1}^{n} \lambda_i |u_i\rangle\langle u_i|$ with an orthonormal basis $\{|u_i\rangle\}_{i=1}^{n}$so $\langle u_{i'}|u_i\rangle = \delta_{ii'}$ and where the non-negative eigenvalues λ_i sum to 1. Thus for the concept of a quantum state, we have the skeletal set level presentation of a partition and the corresponding Hilbert space version of that partition math as summarized in Table 3.

Quantum States	Partition math	Hilbert space math
Density matrix	$\rho(\pi)$	ρ
ON vectors	$\langle b_{j'} \vert b_j\rangle = \delta_{jj'}$	$\langle u_{i'} \vert u_i\rangle = \delta_{ii'}$
Non-negative eigenvalues	$\Pr(B_1), ..., \Pr(B_m), 0, ..., 0$	$\lambda_1, ..., \lambda_n$
Spectral decomposition	$\rho(\pi) = \sum_{j=1}^{m} \Pr(B_j) \vert b_j\rangle \langle b_j \vert$	$\rho = \sum_{i=1}^{n} \lambda_i \vert u_i\rangle \langle u_i \vert$
Non-zero off-diagonal entries	Cohering of diag. elements	Coherence of diag. elements

Table 3: Quantum state ρ as Hilbert space version of partition $\rho(\pi)$

8 Quantum observables

We have seen how the notion of a quantum state was prefigured at the set level by (i.e., has the set level precursor of) a partition on a set with point probabilities. In a similar manner, the notion of a quantum observable is prefigured at the set level by the inverse-image partition $\left\{f^{-1}(r)\right\}_{r\in f(U)}$ of a real-value numerical attribute $f : U \to \mathbb{R}$.

In the folklore of mathematics, there is a semi-algorithmic procedure to connect set concepts with the corresponding vector (Hilbert) space concepts. We will call it the "Yoga of Linearization":

For any given set-concept, apply it to a basis set of a vector space
and whatever is linearly generated is the corresponding vector space concept.
The Yoga of Linearization.

To apply the Yoga, we first take U as just a universal set and consider some set-based concept, and then we consider U as a basis set (e.g., ON basis of a Hilbert space) of a vector space V and see what is linearly generated. For instance, a subset S of a basis set U generates a subspace $[S]$ of the space V. The cardinality of the subset gives the dimension of the subspace. A real-valued numerical attribute $f : U \to \mathbb{R}$ defines a Hermitian operator $F : V \to V$ (where $V = [U]$) by defining F on the basis set U as $Fu = f(u)u$ or, using the fancier notation, $F|u\rangle = f(u)|u\rangle$. To better analyze the numerical attribute, let $f \upharpoonright S = rS$ stand for "the value of f on the subset S is r". That is the set level version of the eigenvalue/eigenvector equation $F|u\rangle = r|u\rangle$. Hence we see that the set version of an eigenvector is a constant set S of f and the set version of an eigenvalue of an eigenvector is the constant value r on a constant set S. A characteristic function $\chi_S : U \to \{0,1\} \subseteq \mathbb{R}$ has only two constant sets $S = \chi_S^{-1}(1)$ and $S^c = U - S = \chi_S^{-1}(0)$. The Yoga yields the corresponding vector space notion which is a projection operator $P_{[S]}$ which is defined by $P_{[S]}|u\rangle = |u\rangle$ if $u \in S$, else $\mathbf{0}$ (zero vector) which has the eigenvalues of 0 and 1. A Hermitian (or self-adjoint) operator F in QM has spectral decomposition $F = \sum_{\lambda_i} \lambda_i P_{V_i}$ where the sum is over the real eigenvalues λ_i and the projections P_{V_i} to their eigenspaces. Bearing in mind the correlation given by the Yoga, we can define the 'spectral decomposition' of the numerical attribute $f : U \to \mathbb{R}$ as $f = \sum_{r\in f(U)} r\chi_{f^{-1}(r)}$. Starting at the quantum level with a Hermitian operator $F : V \to V$ and a basis set U of eigenvectors of F, then f is obtained as the eigenvalue function $f : U \to \mathbb{R}$.

An important application of the Yoga is to the notion of a set partition $\pi = \left\{f^{-1}(r)\right\}_{r\in f(U)}$ as the inverse-image of a numerical attribute. Applied to the basis set $\{|u_i\rangle\}_{i=1}^{n}$ used to define F by $F|u_i\rangle = f(u_i)|u_i\rangle$, each block $f^{-1}(r)$ of π generates the eigenspace of eigenvectors for the eigenvalue r. This eigenspaces $\{V_r\}_{r\in f(U)}$ form a *direct-sum decomposition* (DSD) of V, i.e., $V = \oplus_{r\in f(U)} V_r$, where a DSD is defined as a set of non-zero subspaces $\{V_r\}_{r\in f(U)}$ such that every non-zero vector $v \in V$ has a unique representation as a sum $v = \sum_{r\in f(U)} v_r$ of vectors $v_r \in V_r$. It was noted previously that a set partition $\pi = \{B_1, ..., B_m\}$ has a similar definition since every non-empty subset S has a unique representation as the union of subsets of the $\{B_j\}_{j=1}^{m}$. If the union of the B_j's was not all of U, then $U - \cup_{j=1}^{m} B_j$ would have no representation, and if $S = B_j \cap B_{j'} \neq \emptyset$, then S has two representations as subsets of the B_j's. Moreover, $S \cap () : \wp(U) \to \wp(U)$ is a projection operator that takes any subset $T \in \wp(U)$ to $S \cap T \in \wp(U)$. Thus we have $\cup_{r\in f(U)} \left(f^{-1}(r) \cap ()\right) =$

$I:\wp(U)\rightarrow\wp(U)$ whose quantum version is resolution of unity $\sum_{r\in f(U)}P_{V_r}=I:V\rightarrow V$. And lastly, we might apply the Yoga to the Cartesian product $U\times U'$ where U and U' are basis sets for V and V'. Then the ordered pairs $(u,u')\in U\times U'$ (bi)linearly generate the tensor product $V\otimes V'$ where the ordered pair (u,u') is customarily written as $u\otimes u'$or $|u\rangle\otimes|u'\rangle$.

These results of the Yoga of Linearization are summarized in Table 4.

Set concept (skeletons)	Vector-space concept
Partition $\left\{f^{-1}(r)\right\}_{r\in f(U)}$	DSD $\{V_r\}_{r\in f(U)}$
$U=\uplus_{r\in f(U)}f^{-1}(r)$	$V=\oplus_{r\in f(U)}V_r$
Numerical attribute $f:U\rightarrow\mathbb{R}$	Observable $Fu_i=f(u_i)\,u_i$
$f\upharpoonright S=rS$	$Fu_i=ru_i$
Constant set S of f	Eigenvector u_i of F
Value r on constant set S	Eigenvalue r of eigenvector u_i
Characteristic fcn. $\chi_S:U\rightarrow\{0,1\}$	Projection operator $P_{[S]}u_i=\chi_S(u_i)u_i$
$\cup_{r\in f(U)}\left(f^{-1}(r)\cap()\right)=I:\wp(U)\rightarrow\wp(U)$	$\sum_{r\in f(U)}P_{V_r}=I:V\rightarrow V$
Spectral Decomp. $f=\sum_{r\in f(U)}r\chi_{f^{-1}(r)}$	Spectral Decomp. $F=\sum_{r\in f(U)}rP_{V_r}$
Set of r-constant sets $\wp\left(f^{-1}(r)\right)$	Eigenspace V_r of r-eigenvectors
Cartesian product $U\times U'$	Tensor product $V\otimes V'$

Table 4: Skeletal set-level concepts and the corresponding vector (Hilbert) space concepts

9 Quantum Measurement

The third basic concept to be analyzed is quantum measurement (always projective). The connection between the set level notion of measurement and the quantum level is the Lüders mixture operation ([30]; [4, p. 279]) that can be applied at both levels. At the set level, we have the skeletal state represented by a density matrix $\rho(\pi)$ and we have an 'observable' or real-value numerical attribute, say, $g:U\rightarrow\mathbb{R}$ whose inverse-image is the partition $\sigma=\left\{g^{-1}(r)\right\}_{r\in g(U)}$. The Lüders mixture operation applies the 'observable' to the density matrix $\rho(\pi)$ to arrive at the post-measurement density matrix $\hat{\rho}(\pi)$. The operation uses the $n\times n$ projection matrices for the blocks $g^{-1}(r)$ of σ which are diagonal matrices $P_{g^{-1}(r)}$ whose diagonal elements are the values of the characteristic function $\chi_{g^{-1}(r)}$. Then the post-measurement density matrix is:

$$\hat{\rho}(\pi)=\sum_{r\in g(U)}P_{g^{-1}(r)}\rho(\pi)\,P_{g^{-1}(r)}$$
Set version of Lüders mixture operation.

Then it is an easy result:

Theorem: $\hat{\rho}(\pi)=\rho(\pi\vee\sigma)$.

Thus the set version of quantum level projective measurement in the math of QM is the partition join operation where $\operatorname{dit}(\pi\vee\sigma)=\operatorname{dit}(\pi)\cup\operatorname{dit}(\sigma)$ and, by DeMorgan's Law, $\operatorname{indit}(\pi\vee\sigma)=\operatorname{indit}(\pi)\cap\operatorname{indit}(\sigma)$.

Example: Let $\pi=\{\{a\},\{b,c\}\}$ with probabilities $\Pr(\{a\})=\frac{1}{3}$, $\Pr(\{b\})=\frac{1}{4}$, and $\Pr(\{c\})=\frac{5}{12}$ in $U=\{a,b,c\}$ and $\sigma=\{\{a,b\},\{c\}\}$ so $\sigma=g^{-1}$ for any $g:U\rightarrow\mathbb{R}$ that assigns the same g-value to a and b with a different value for c. Then the density matrix for π and the projections matrices for the blocks of σ are:

$$\rho(\pi)=\begin{bmatrix}\frac{1}{3}&0&0\\0&\frac{1}{4}&\frac{\sqrt{5}}{4\sqrt{3}}\\0&\frac{\sqrt{5}}{4\sqrt{3}}&\frac{5}{12}\end{bmatrix},\ P_{\{a,b\}}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\text{ and }P_{\{c\}}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}.$$

The Lüders mixture operation is:

$$
\begin{aligned}
\hat{\rho}(\pi) &= P_{\{a,b\}}\rho(\pi)P_{\{a,b\}} + P_{\{c\}}\rho(\pi)P_{\{c\}} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{\sqrt{5}}{4\sqrt{3}} \\ 0 & \frac{\sqrt{5}}{4\sqrt{3}} & \frac{5}{12} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{\sqrt{5}}{4\sqrt{3}} \\ 0 & \frac{\sqrt{5}}{4\sqrt{3}} & \frac{5}{12} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{5}{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{5}{12} \end{bmatrix}.
\end{aligned}
$$

Since $\pi \vee \sigma = \{\{a\},\{b\},\{c\}\} = \mathbf{1}_U$, we see that the post-measurement density matrix is $\hat{\rho}(\pi) = \rho(\pi \vee \sigma) = \rho(\mathbf{1}_U)$. Thus the superposition of $\{b\}$ and $\{c\}$ in π got distinguished since b and c had different g-values. The new distinctions in $\operatorname{dit}(\sigma) - \operatorname{dit}(\pi)$ are (b,c) (along with (c,b)) and those were the non-zero off-diagonal elements (coherences) of $\rho(\pi)$ that got zeroed (distinguished or decohered) in $\rho(\pi \vee \sigma)$. Figure 5 shows the join of $\{\{a\},\{b,c\}\}$ and $\{\{a,b\},\{c\}\}$ is their least upper bound $\mathbf{1}_U = \{\{a\},\{b\},\{c\}\}$.

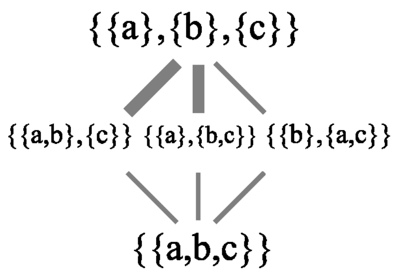


Figure 5: Partition lattice with join $\{\{a\},\{b,c\}\} \vee \{\{a,b\},\{c\}\} = \{\{a\},\{b\},\{c\}\}$

The Lüders mixture is not the end of the measurement process. The measurement returns one of the g-values, say the (degenerate) one for $\{a,b\}$. Then the Lüders Rule [24, Appendix B] gives the final density matrix which is the corresponding term in the Lüders mixture sum, e.g., $P_{\{a,b\}}\rho(\pi)P_{\{a,b\}}$, normalized so the final density matrix is the (in this case, classical) mixed state:

$$
\frac{P_{\{a,b\}}\rho(\pi)P_{\{a,b\}}}{\operatorname{tr}\left[P_{\{a,b\}}\rho(\pi)P_{\{a,b\}}\right]} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{1}{7/12} = \begin{bmatrix} \frac{4}{7} & 0 & 0 \\ 0 & \frac{3}{7} & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

In the general Hilbert space case, the Hermitian operator G is given by its DSD of eigenspaces $\{V_r\}_{r \in g(U)}$ (where $g : U \to \mathbb{R}$ is the eigenvalue function assigning the appropriate real eigenvalue to each vector in an ON basis U of eigenvectors of G). The state being measured is given by density matrix ρ expressed in the ON basis U, and the Lüders mixture operation uses the projection matrices P_{V_r} to the eigenspaces of G to determine the post-measurement density matrix $\hat{\rho}$. The Hilbert space version of the set operation is:

$$
\hat{\rho} = \sum_{r \in g(U)} P_{V_r} \rho P_{V_r}
$$

Hilbert space version of Lüders mixture operation.

We saw previously how the notion of logical entropy (and its quantum counterpart) was based on the notion of a quantitative measure of distinctions of a partition. Hence logical entropy is the *natural* notion to measure the changes in a density matrix under a measurement. For instance, in the above example where $\pi = \{\{a\}, \{b,c\}\}$, the block probabilities are $\Pr(\{a\}) = \frac{1}{3}$ and $\Pr(\{b,c\}) = \frac{2}{3}$, and the logical entropy is: $h(\pi) = 1 - \Pr(\{a\})^2 - \Pr(\{b,c\})^2 = 1 - \frac{1}{9} - \frac{4}{4} = \frac{4}{9}$. When the partition π is represented as a density matrix $\rho(\pi)$, then the logical entropy could also be computed as:

$$h(\pi) = h(\rho(\pi)) = 1 - \mathrm{tr}\left[\rho(\pi)^2\right] = 1 - \mathrm{tr}\begin{bmatrix} \frac{1}{9} & 0 & 0 \\ 0 & \frac{1}{6} & \frac{1}{18}\sqrt{3}\sqrt{5} \\ 0 & \frac{1}{18}\sqrt{3}\sqrt{5} & \frac{5}{18} \end{bmatrix} = 1 - \frac{10}{18} = \frac{4}{9}.$$

The logical entropy is the two-draw probability of drawing a distinction of π so it could also be computed as the sum of all the distinction probabilities (remembering that a distinction is an ordered pair of elements in different blocks so the probability of an unordered pair is doubled). Hence in general we have: $h(\pi) = \sum_{(u_i,u_k)\in \mathrm{dit}(\pi)} p_i p_k$, or in the case at hand:

$$h(\pi) = 2p_a p_b + 2p_a p_c = 2\frac{1}{3}\frac{1}{4} + 2\frac{1}{3}\frac{5}{12} = \frac{1}{6} + \frac{5}{18} = \frac{4}{9}.$$

There is then a general theorem [13] showing how logical entropy measures measurement.

Theorem (set case of measuring measurement): In the Lüders mixture operation $\hat{\rho}(\pi) = \sum_{r\in g(U)} P_{g^{-1}(r)}\rho(\pi) P_{g^{-1}(r)}$, the increase in logical entropy from $h(\rho(\pi))$ to $h(\hat{\rho}(\pi))$ is the sum of the squares of the off-diagonal non-zero entries in $\rho(\pi)$ that were zeroed in the measurement operation $\rho(\pi) \rightsquigarrow \hat{\rho}(\pi)$.

In the example, the logical entropy of the post-measurement state is:

$$h(\hat{\rho}(\pi)) = 1 - \mathrm{tr}\left[\hat{\rho}(\pi)^2\right] = 1 - \mathrm{tr}\begin{bmatrix} \frac{1}{9} & 0 & 0 \\ 0 & \frac{1}{16} & 0 \\ 0 & 0 & \frac{25}{144} \end{bmatrix} = 1 - \left(\frac{1}{9} + \frac{1}{16} + \frac{25}{144}\right) = 1 - \frac{16+9+25}{144} = \frac{94}{144}.$$

The sum of the squares of the non-zero off-diagonal terms (representing the coherences) of $\rho(\pi)$ that were zeroed (decohered) in $\hat{\rho}(\pi)$ is:

$$2\left(\frac{\sqrt{5}}{4\sqrt{3}}\right)^2 = 2\frac{5}{48} = \frac{5}{24}$$

and the increase in logical entropy due to the making of distinctions is:

$$h(\hat{\rho}(\pi)) - h(\rho(\pi)) = \frac{94}{144} - \frac{4}{9} = \frac{94}{144} - \frac{64}{144} = \frac{30}{144} = \frac{5}{24}.\checkmark$$

And the quantum case is *mutatis mutandis*.

Theorem (quantum case of measuring measurement): In the Lüders mixture operation $\hat{\rho} = \sum_{r\in g(U)} P_{V_r}\rho P_{V_r}$, the increase in quantum logical entropy from $h(\rho)$ to $h(\hat{\rho})$ is the sum of the absolute squares of the off-diagonal entries in ρ that were zeroed in the measurement operation $\rho \rightsquigarrow \hat{\rho}$.

The dictionary relating the three basic concepts in the math of QM to their set partitional precursors is given in the following Table 5.

Dictionary	Partition math	Hilbert space math
Notion of state	$\rho(\pi) = \sum_{j=1}^{m} \Pr(B_j)\,\lvert b_j\rangle\langle b_j\rvert$	$\rho = \sum_{i=1}^{n} \lambda_i \lvert u_i\rangle\langle u_i\rvert$
Notion of observable	$g = \sum_{r\in g(U)} r\chi_{g^{-1}(r)} : U \to \mathbb{R}$	$G = \sum_{r\in g(U)} rP_{V_r}$
Notion of measurement	$\hat{\rho}(\pi) = \sum_{r\in g(U)} P_{g^{-1}(r)}\rho(\pi) P_{g^{-1}(r)}$	$\hat{\rho} = \sum_{r\in g(U)} P_{V_r}\rho P_{V_r}$

Table 5: Three basic notions: set version and corresponding Hilbert space version

10 Other aspects of QM mathematics

10.1 Commuting, non-commuting, and conjugate operators

We have seen that a Hermitian operator is the QM math version of a real-valued numerical attribute and that the direct-sum decomposition of the operator's eigenspaces is the QM math version of the inverse-image partition of the numerical attribute. Let $F, G : V \to V$ be two Hermitian operators with the corresponding DSDs of $\{V_j\}_{j \in J}$ and $\{W_{j'}\}_{j' \in J'}$. Since the two DSDs are the vector space version of partitions, consider the join-like operation giving the set of non-zero subspaces formed by the intersections $V_j \cap W_{j'}$. The additional generality gained over the join of set partitions is that these subspaces may not span the whole space V. Since the vectors in those intersections are simultaneous eigenvectors of F and G, let $\mathcal{SE}$ be the subspace spanned by the simultaneous eigenvectors of F and G. The commutator $[F, G] = FG - GF : V \to V$ is a linear operator on V so it has a kernel $\ker[F, G]$ consisting of the vectors v such that $[F, G]v = \mathbf{0}$. Then there is a:

Theorem: $\mathcal{SE} = \ker[F, G]$. [14, Proposition 1]

Since commutativity is defined as $\ker[F, G] = V$, we have the following definitions in terms of the vector space partitions or DSDs:

- F and G are *commuting* if $SE = V$;
- F and G are *incompatible* if $SE \neq V$;
- F and G are *conjugate* if $SE = 0$ (zero space).

Since the join-like operation on DSDs yields a set of subspaces that do not necessarily span the whole space, that operation is only the *join* of DSDs in the commuting case–or as Hermann Weyl put it: "Thus combination of two gratings presupposes commutability...". [42, p. 257]

The set version of *compatible* partitions for the join operation is simply being defined on the same set. Hence our thesis gives a complete parallelism between compatible partitions and commuting operators.

Set math: A set of compatible partitions $\pi, \sigma, ...\gamma$ defined by $f, g, ...h : U \to \mathbb{R}$ is said to be *complete*, i.e., a Complete Set of Compatible Attributes or CSCA, if their join is the partition whose blocks are of cardinality one (i.e., $\mathbf{1}_U$). Then the elements $u \in U$ are uniquely characterized by the ordered set of values $(f(u), g(u), ..., h(u)$.

QM math: A set of commuting observables $F, G, ..., H$ is said to be *complete*, i.e., a Complete Set of Commuting Observables or CSCO [9], if the join of their eigenspace DSDs is the DSD whose subspaces are of dimension one. Then the simultaneous eigenvectors of the operators are unique characterized by the ordered set of their eigenvalues.

10.2 Feynman's treatment of measurement

The partitional approach to understanding the math of QM shows that the key organizing concepts are indistinction versus distinction, indefiniteness versus definiteness, or indistinguishability versus distinguishability. When a particle in a superposition state undergoes an interaction, what characterizes whether it is a measurement or not? As early as 1951 [16], Richard Feynman gave the analysis of "measurement or not" in terms of distinguishability.

> If you could, *in principle*, distinguish the alternative *final* states (even though you do not bother to do so), the total, final probability is obtained by calculating the *probability* for each state (not the amplitude) and then adding them together. If you *cannot* distinguish the final states *even in principle*, then the probability amplitudes must be summed before taking the absolute square to find the actual probability.[18, p. 3-9]

This analysis has been further explained by another physicist.

> Feynman's approach is based on the contrast between processes that are *distinguishable* within a given physical context and those that are *indistinguishable* within that context. A process is distinguishable if some record of whether or not it has been realized results from the process in question; if no record results, the process is indistinguishable from alternative processes leading to the same end result. [37, p. 314]

Feynman gives a number of examples ([18, § 3-3]; [19, pp. 17-8]) such as a particle scattering off the atoms in a crystal. If there is no physical record of which atom the particle scattered off of (i.e., the indistinguishable case), then no measurement took place so the amplitudes for the superposition state of scattering off the different atoms are added to compute the amplitude of the particle reaching a certain final state. But if all the atoms had, say spin up, and scattering off an atom flipped the spin, then a physical record exists (i.e., the distinguishable case) so a measurement took place and the probabilities of scattering off the different atoms are added to compute the probability of reaching a certain final state.

The same analysis applies to the well-known double-slit experiment where the distinguishable case is where there are detectors at the slits and the indistinguishable case is having no detectors at the slits. But the important thing to notice about Feynman's example is that the measurement is entirely at the quantum level; it involves no macroscopic apparatus. Hence the Feynman analysis bypasses the whole tortured literature trying to analyze measurement in term of the "decoherence" induced by a macroscopic measuring devices (e.g., [43]). Of course, the quantum level physical record in the distinguishable case has to be amplified for humans to record the result but such macroscopic considerations have no role in quantum *theory*.

The implicit principle in Feynman's analysis of measurement is:

If the interaction distinguishes between superposed eigenstates,
then a distinction (state reduction) is made.
The State Reduction Principle

The mathematics of the State Reduction Principle can be stated in both the set case and the QM case.

Theorem (State Reduction Principle–set case). Measurement is described in the set case by the Lüders mixture operation $\hat{\rho}(\pi) = \sum_{r \in g(U)} P_{g^{-1}(r)} \rho(\pi) P_{g^{-1}(r)}$. The State Reduction Principle then states: if an off-diagonal entry $\rho(\pi)_{ik} \neq 0$ (i.e., u_i and u_k are in a same-block superposition), then: if $g(u_i) \neq g(u_k)$ (i.e., the interaction distinguishes u_i and u_k), then $\hat{\rho}(\pi)_{ik} = 0$ (i.e., the 'coherence' between u_i and u_k is decohered and a distinction is made).

Theorem (State Reduction Principle–QM case). Measurement is described in QM by the Lüders mixture operation $\hat{\rho} = \sum_{r \in g(U)} P_{V_r} \rho P_{V_r}$ (measuring ρ by G). The State Reduction Principle then states: if an off-diagonal entry $\rho_{ik} \neq 0$ (i.e., the G-eigenvectors $|u_i\rangle$ and $|u_k\rangle$ are in a superposition in ρ), then: if $|u_i\rangle$ and $|u_k\rangle$ have different G-eigenvalues (i.e., the vectors are distinguished by G), then $\hat{\rho}_{ik} = 0$ (i.e., the vectors are decohered[6] and a distinction is made).

If no distinctions were made by the interaction, then no measurement took place.

10.3 Von Neumann's type I and type II processes

John von Neumann made his famous distinction between the processes:

1. Type I process of measurement and state reduction, and
2. Type II process obeying the Schrödinger equation.

[6] This is not the Zeh/Zurek "decoherence" [43] but the old-fashioned change from the coherence of a superposition pure state into a decohered mixture of states.

We have seen that the Type I processes of measurement involves distinguishability, i.e., the making of distinctions (like which atom the particle scattered off of), so a natural way to designate the Type II processes would be ones that do not make distinctions by preserving distinguishability or indistinguishability. The measure of indistinctness of two quantum states is their overlap or inner product. For instance, two states have zero indistinctness (zero inner product) then they are fully distinct (orthogonal). Hence the natural characterization of a Type II process is one that preserves inner products, i.e., a unitary transformation.[7]

The partitional approach highlights the key analytical concepts of indistinctions versus distinctions and the cognate notions of indefiniteness versus definiteness or indistinguishability versus distinguishability. Many people working on quantum foundations seem to ignore those key concepts, and then the division between the measurement and unitary evolution seems unfounded, if not "unbelievable."

> [I]t seems unbelievable that there is a fundamental distinction between "measurement" and "non-measurement" processes. Somehow, the true fundamental theory should treat all processes in a consistent, uniform fashion. [32, p. 245]

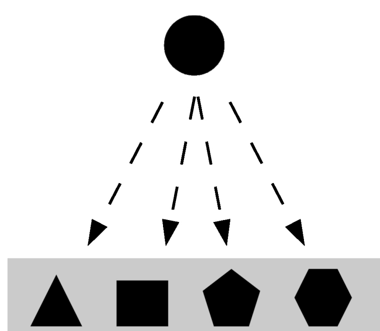
10.4 Hermann Weyl's imagery for measurement

An industrial sieve is used to distinguish particles of matter of different sizes so it might serve as a helpful metaphor for the quantum process of making distinctions, namely measurement.

> In Einstein's theory of relativity the observer is a man who sets out in quest of truth armed with a measuring-rod. In quantum theory he sets out armed with a sieve.[10, p. 267]

Hermann Weyl quotes Eddington's passage [42, p. 255] but uses his own expository notion of a "grating." Weyl in effect uses the Yoga from the mathematical folklore to develop both the set notion of a grating as an "aggregate [which] is used in the sense of 'set of elements with equivalence relation.'" [42, p. 239] and the vector space notion of a direct-sum decomposition. In the set to vector space move of the Yoga, the "aggregate of n states has to be replaced by an n-dimensional Euclidean vector space" [42, p. 256] ("Euclidean" is an old name for an inner product space). The notion of a vector space partition or "grating" in QM is a "splitting of the total vector space into mutually orthogonal subspaces" so that "each vector $\overrightarrow{x}$ splits into r component vectors lying in the several subspaces" [42, p. 256], i.e., a DSD. After thus referring to a partition and a DSD as a "grating" or "sieve," Weyl notes that "Measurement means application of a sieve or grating" [42, p. 259], i.e., the making of distinctions by the join-like process described by the Lüders mixture operation.

This imagery of measurement as passing through a sieve or grating is illustrated in Figure 6.



[7] The connection to solutions to the Schrödinger equation in Hilbert space math is provided by Stone's Theorem ([38]; [24, p. 114]).

Figure 6: Measurement imaged as an indefinite blob of dough passing through a grating to get a definite shape

One should imagine the roundish blob of dough as the superposition of the definite shapes in the grating or sieve. The interaction between the superposed blob and the sieve/grating forces a distinction, so a distinction is made as the blob must pass through one of the definite-shaped holes. In general, a state reduction ('measurement') from an indefinite superposition to a more definite state takes place when the particle in the superposition state undergoes an interaction that distinguishes the superposed states.

10.5 A skeletal analysis of the double-slit experiment

Consider the skeletal case of a particle have three possible states $U = \{a, b, c\}$ which are interpreted as vertical positions in the setup for the double-slit experiment in Figure 7.

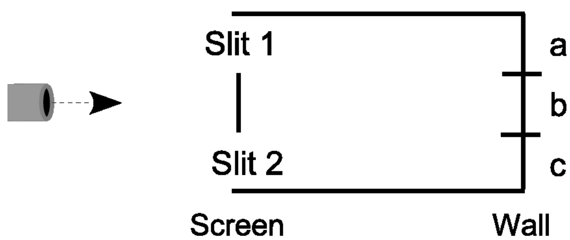


Figure 7: Skeletal setup for the double-slit experiment

In the set level skeletal analysis, we have discarded the scalars from $\mathbb{C}$ but we are nevertheless left with the scalars 0 and 1 which are the elements of the field $\mathbb{Z}_2$. There is the natural correspondence between the zero-one vectors in the three-dimensional vector space $\mathbb{Z}_2^3$ (i.e., the column vectors $[1,0,0]^t$ is associated with $\{a\}$, and so forth) which establishes an isomorphism: $\mathbb{Z}_2^3 \cong \wp(U)$, where the set addition is the symmetric difference, i.e., for $S, T \in \wp(U)$, $S + T = (S - T) \cup (T - S)$. That mimics the addition mod 2 in $\mathbb{Z}_2^3$ since, for instance, $\{a, b\} + \{b, c\} = \{a, c\}$. For our dynamics, we assume a non-singular linear transformation $\{a\} \rightsquigarrow \{a'\} = \{a, b\}$, $\{b\} \rightsquigarrow \{b'\} = \{a, b, c\}$, and $\{c\} \rightsquigarrow \{c'\} = \{b, c\}$ which is non-singular since $\{a'\} = \{a, b\}$, $\{b'\} = \{a, b, c\}$, and $\{c'\} = \{b, c\}$ also form a basis set for $\wp(U)$–so we also have a partition lattice $\Pi(U')$ on the basis set $U' = \{a', b', c'\}$.

We are interested in the analysis when the particle arrives at the screen in the superposition of $|\text{slit } 1\rangle + |\text{slit } 2\rangle$, or in skeletal terms $\{a, c\}$.

Case 1: There are detectors at the slits to distinguish between the two superposed states so the state reduces to the half-half mixture of $\{a\}$ and $\{c\}$. Then $\{a\}$ evolves by the non-singular dynamics to $\{a, b\}$ which hits the wall and reduces to $\{a\}$ or $\{b\}$ with half-half probability. Similar $\{c\}$ evolves to $\{b, c\}$ which hits the wall and reduces to $\{b\}$ or $\{c\}$ with half-half probability. Since this is the case of distinctions between the alternative paths to $\{a\}$, $\{b\}$, or $\{c\}$ we add the probabilities to obtain:

$$\begin{gathered}
\Pr(\{a\} \text{ at wall} | \{a,c\} \text{ at screen}) \\
= \Pr(\{a\} \text{ at wall} | \{a\} \text{ at screen}) \Pr(\{a\} \text{ at screen} | \{a,c\} \text{ at screen}) = \tfrac{1}{2}\tfrac{1}{2} = \tfrac{1}{4}. \\
\Pr(\{b\} \text{ at wall} | \{a,c\} \text{ at screen}) \\
= \Pr(\{b\} \text{ at wall} | \{a\} \text{ at screen}) \Pr(\{a\} \text{ at screen} | \{a,c\} \text{ at screen}) \\
+ \Pr(\{b\} \text{ at wall} | \{c\} \text{ at screen}) \Pr(\{c\} \text{ at screen} | \{a,c\} \text{ at screen}) = \tfrac{1}{2}\tfrac{1}{2} + \tfrac{1}{2}\tfrac{1}{2} = \tfrac{1}{2}. \\
\Pr(\{c\} \text{ at wall} | \{a,c\} \text{ at screen}) \\
= \Pr(\{c\} \text{ at wall} | \{c\} \text{ at screen}) \Pr(\{c\} \text{ at screen} | \{a,c\} \text{ at screen}) = \tfrac{1}{2}\tfrac{1}{2} = \tfrac{1}{4}.
\end{gathered}$$

Hence the probability distribution in the Case 1 of measurement at the screen is given in Figure 8.

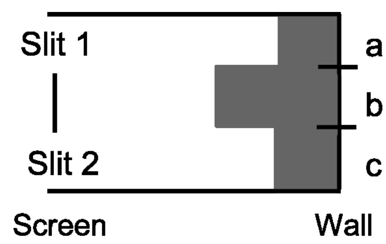


Figure 8: Probabilities at the wall with distinctions at the screen

Case 2: There are no detectors to distinguish between the slits in the superposition $\{a,c\}$ so it linearly evolves by the dynamics: $\{a,c\} = \{a\} + \{c\} \rightsquigarrow \{a,b\} + \{b,c\} = \{a,c\}$. Hence the probabilities at the wall are:

$$\begin{gathered}
\Pr\left(\{a\} \text{ at wall } | \{a,c\} \text{ at screen}\right) \\
= \Pr\left(\{a\} \text{ at wall} | \{a,c\} \text{ at wall}\right) \Pr\left(\{a,c\} \text{ at wall} | \{a,c\} \text{ at screen}\right) = \tfrac{1}{2} \times 1 = \tfrac{1}{2}. \\
\Pr\left(\{b\} \text{ at wall} | \{a,c\} \text{ at screen}\right) \\
= \Pr\left(\{b\} \text{ at wall} | \{a,c\} \text{ at wall}\right) \Pr\left(\{a,c\} \text{ at wall} | \{a,c\} \text{ at screen}\right) = 0 \times 1 = 0. \\
\Pr\left(\{c\} \text{ at wall } | \{a,c\} \text{ at screen}\right) \\
= \Pr\left(\{c\} \text{ at wall} | \{a,c\} \text{ at wall}\right) \Pr\left(\{a,c\} \text{ at wall} | \{a,c\} \text{ at screen}\right) = \tfrac{1}{2} \times 1 = \tfrac{1}{2}.
\end{gathered}$$

Hence the probability distribution in the Case 2 of no distinctions at the screen is given in Figure 9.

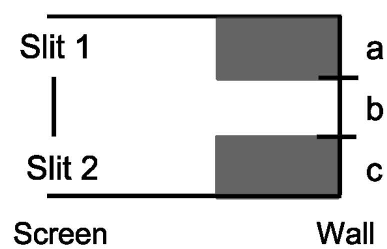


Figure 9: Probabilities at the wall with no distinctions at the screen

The Case 2 distribution shows the usual probability stripes due to the interference in the linear evolution of the superposition state $\{a,c\}$, i.e., the destructive interference in the evolved superposition $\{a,b\} + \{b,c\} = \{a,c\}$.

Our classical intuitions insist on asking: "Which slit did the particle go through in Case 2?". That question assumes that the evolution of the state $\{a,c\}$ was at the classical level where the slits were distinguished. But in Case 2, the slits were not distinguished so the evolution took place at the lower level in the skeletal lattice of partitions. In Figure 10, the Case 2 evolution is illustrated as going from the superposition state $\{a,c\}$ in the partition lattice of states on $U = \{a,b,c\}$ to the superposition state $\{a',c'\}$ lattice of states on $U' = \{a',b',c'\}$.

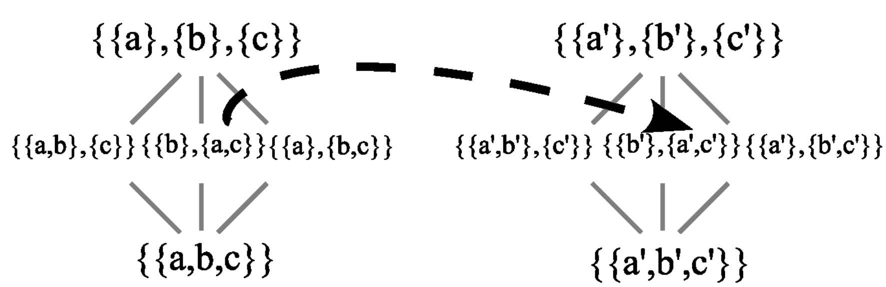


Figure 10: Evolution taking place at a non-classical level of indefiniteness

The important 'take-away' is that there are different levels of indefiniteness (as illustrated in the partition lattice) and evolution can take place at a non-classical level of indefiniteness so, in that case, there is no matter of fact of the particle going through one slit or the other at the classical level.

Sometimes metaphors can serve as an aid or crutch to our biologically evolved intuitions. Consider the "hawks and hounds" of Shakespeare's Sonnet 91. There is a high fence across a field with two slits or gates. To get from A on one side of the fence to B on the other side, the hound (like a classical particle) is limited to horizontal 'classical' trajectory on the "flatland" [1] so it has to go through one gate of another. But the hawk's "flights and perches" [22, p. 198] can go from ground perch A to ground perch B without going through one gate or the other. We make the unrealistic assumption that a light source above the hawk (like the sun) will 'project' the hawk down to the 'classical' definite ground (like Icarus!). Then the grounded hawk, like the hound, must go through one gate or the other to get from A to B. But with no light source, then the hawk (like Hegel's owl of minerva who only flies at night) has an indefinite flight trajectory and can go from A to B without going through a gate. Our classical ("flatlander") intuitions see only the definite ground-level paths or trajectories and, in the absence of either projecting light source as in Case 2 above, will insist on asking: "Which gate did the hawk go through?". But with no detections at the slits in the double-slit experiment, there is no matter of fact of the particle going through a slit at the classical level since the evolution is at the non-classical quantum level (illustrated by the third dimension in our flatlander metaphor) as in Figure 10.

11 Final remarks

Our thesis is that the math of QM is the Hilbert space version of the math of partitions, or, put the other way around, the math of partitions is the skeletonized version of QM math. There are many other aspects of QM math that could be investigated such as group representations on sets or on vector spaces over $\mathbb{C}$ since a group is essentially a 'dynamic' algebraic way to define an equivalence relation or DSD [6] (e.g., the orbit partition in a set representation or the DSD of irreducible subspaces in the vector space over $\mathbb{C}$ representation). [14] But in this introductory treatment, we have hopefully analyzed enough aspects of QM math to illustrate our thesis.

Since partitions are the mathematical tool to analyze indistinctions and distinctions or indefiniteness and definiteness, the thesis shows that the key QM notion of superposition should be interpreted in terms of (objective) indefiniteness, and that measurement should be interpreted as an interaction that makes distinctions so it turns an indefinite state into a state with more definiteness. This approach to better understanding or interpreting QM works with the standard von Neumann/Dirac quantum theory. It does not involve any new physics, unlike the pilot-wave or spontaneous localization theories, or any many-worldly interpretations of measurement. In that sense, the partitional approach shows how to develop Shimony's idea of the Literal Interpretation of the math or "formalism of quantum mechanics" [36, pp. 6-7]. Furthermore, the partitional analysis substantiates the analysis of Heisenberg, Shimony, and others which describes the quantum world in terms of potentialities or latencies, where, in both cases, the key attribute was the *reality* of objective indefiniteness. Hence this way of understanding or interpreting quantum mechanics might be called the *Objective Indefiniteness* or *Literal Interpretation* ([14], [15]).

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