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How many postulates are needed to derive the Lorentz transformation?

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Abstract

It is generally believed that Einstein derived special relativity from two postulates, the principle of relativity and the constancy of the speed of light, without paying much attention to the results of the Michelson-Morley experiment and Lorentz's interpretation. Some researchers even assert that one postulate, i.e. the principle of relativity, is enough to derive a Lorentz-type transformation. The present study investigates how many postulates are needed to derive the Lorentz transformation without covertly adding more postulates. We found that at least four postulates are necessary to derive the Lorentz transformation when applying only mathematical and logical rules in derivation. They are 1) the principle of relativity; 2) the constancy of the speed of light; 3) motion has no impact on the length or spatial distance in the directions perpendicular to its velocity; 4) the coefficient of the spatial coordinate term such as x (or x') term equals that of the velocity-time term vt (or $v't'$) on the right-hand side of the Lorentz spatial transformation equations. Without the third and the fourth postulates as constraints, an infinite number of space and time transformations satisfy the constancy of the speed of light and the principle of relativity. The well-known Voigt transformation is one of them. The Lorentz transformation that satisfies the principle of relativity is a sufficient condition for the constancy of the speed of light, but not a necessary condition. Therefore, it cannot be logically derived from the constancy of the speed of light and the principle of relativity. All derivations using Einstein's two postulates or the principle of relativity alone involve covertly adding more postulates or/and making mathematical and logical mistakes. Since the Lorentz transformation is a sufficient condition for the constancy of the speed of light while the latter and the principle of relativity are not sufficient conditions for the former, it is more logically appropriate to postulate the Lorentz transformation rather than the constancy of the speed of light as the primary principle of the theory of relativity.

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1. Introduction

At the core of special relativity is the Lorentz transformation, proposed by Lorentz in 1904 following a series of attempts (Fitzgerald 1889; Larmor 1893, 1900; Lorentz 1892, 1898, 1937; Poincaré 1905, 1906) to explain the result of the Michelson-Morley experiment (Michelson 1881; Michelson and Morley 1887). However, most physicists and the general public believe that Einstein derived special relativity from two postulates, the principle of relativity and the constancy of the speed of light (Einstein 1905). Generations of scientists and people from other walks of life have admired this feat of Einstein. According to Einstein's account, the idea of special relativity occurred to him when he reflected on the propagation of light and Maxwell's electromagnetic theory; the result of the Michelson-Morley experiment had little influence on this achievement (Pais 1987). Later many researchers argued that the principle of relativity alone is sufficient for deriving a Lorentz-type transformation with an unspecified cosmic speed limit, which could be determined by experiments (Berzi and Gorini 1969; Ignatowsky 1910; Coleman 2003b; Shen 2008; Field 1997). However, some researchers think that the constancy of the speed of light and the principle of relativity are insufficient for deriving the Lorentz transformation, so additional postulates or assumptions are needed (Ma 2004, 2013).

Whether the derivation of Lorentz-type transformations needs the constancy of the speed of light or other speed limits has significant theoretical importance for the foundation of physics. The present study aims to investigate how many postulates are needed to derive the Lorentz transformation and what additional implicit postulates have been inserted in those claims that one principle is sufficient to give rise to a Lorentz-type transformation. Since the meaning of Einstein's constancy of the speed of light is uncontroversial, we will first examine what the principle of relativity means and what conditions can be used without being considered as adding more postulates. Then, we will derive space and time transformation equations using Einstein's two postulates and examine how many postulates are necessary to derive the Lorentz transformation logically.

The rest of the paper is organized as follows: Section 2 examines what the principle of relativity implies and what rules should be followed in the derivation; Section 3 derives the space and time transformation equations using Einstein's two postulates; Section 4 derives the Lorentz transformation based on these general requirements; Section 5 examines derivations using only the principle of relativity; Section 6 derives the Poincaré transformation and analyzes the nature of the Lorentz transformation; Section 7 discusses the findings in the present study; Section 8 concludes.

2. The meaning of the principle of relativity and the rules of derivation

Whether we can derive the Lorentz transformation from the constancy of the speed of light and the principle of relativity depends on how we interpret them. The generally accepted definition of the principle of relativity is that physical laws have the same function forms in all inertial reference frames. This definition says nothing about the values of the variables in the functions describing those physical laws. Therefore, for two inertial reference frames the principle of relativity does not require a variable to have the same value in them. In all derivations of the Lorentz transformation, v and v', the relative velocity measured by the two frames K and K' respectively, are assumed to be equal. However, this is not a consequence of the principle of relativity, unless we can also assume that d and d', the same distance between them measured by the two frames K and K' respectively, are also equal. The same goes for t and t', the same time interval for crossing the same distance between them measured by the two frames K and K' respectively. If we use $v=v'$ in our derivation, it is a postulate on the value of a variable in addition to the requirement on the function forms by the principle of relativity. As illustrated in Fig.1, any reasoning to justify $v=v'$ with the principle of relativity should also justify $d=d'$ and $t=t'$.

If two inertial frames or observers A and B moving toward each other with identical measuring sticks and clocks cannot obtain an equal value for the distance between them, the time needed for crossing that distance, and (as this paper argued) the velocity between them, coefficients of the physical law functions in two reference frames would likely have different values even though they have the same function forms. Considering these, we might find that people have implicitly assumed too many conditions unwittingly in deriving the Lorentz transformation. In the present study, we try to expose the implicit assumptions (postulates) and take only mathematical and logical operations for granted.

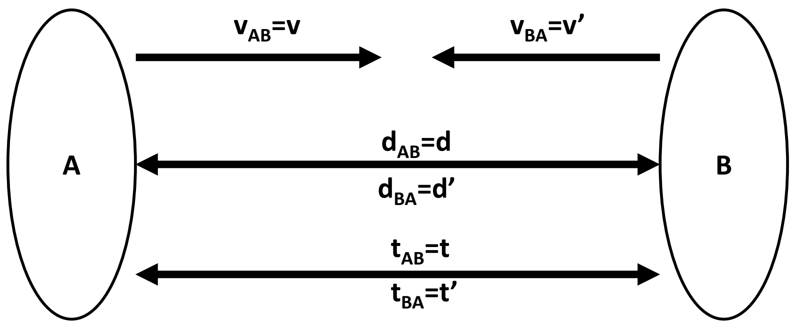

Fig.1 The velocity *v* and the distance *d* between two inertial observers A and B and the time interval *t* used for crossing the distance *d* at the velocity *v*. Using the principle of relativity to justify $v_{AB}=v_{BA}$ will also justify $d_{AB}=d_{BA}$ and $t_{AB}=t_{BA}$.

A derivation of a theorem is a logical inference from premises to conclusions. In a Hilbert-style deduction system, axioms generally define the syntax of the logical operators, and the only inference rule used, *modus ponens*, has the form

$$((P \rightarrow Q) \land P) \rightarrow Q. \tag{1}$$

In Equation (1), $P \rightarrow Q$, i.e. "If *P* then *Q*" means P is a sufficient condition of *Q*, and "*P*" are two premises, and "*Q*" is the conclusion. For the derivation of the Lorentz transformation, the two postulates are premises and the transformation equations are conclusions. A true derivation should use only the premises and mathematic or logical operation rules, and no other conditions should be introduced during the derivation process. Any conditions introduced during the derivation will constrain the conclusions to the situations that meet the introduced conditions, so they are implicit premises that should be introduced explicitly as postulates or premises at the beginning of the derivation.

Science historians often incorrectly assert that Voigt first proposed the Lorentz transformation in 1887 (Voigt 1887). Although Voigt's transformation satisfies both the constancy of the speed of light and the principle of relativity, it has different implications from those of the Lorentz transformation. Since both transformations satisfy the constancy of the speed of light and the principle of relativity but give opposite conclusions regarding the effect of velocity on lengths in different directions, we can confidently conclude that Einstein's two postulates are not sufficient conditions for deriving the Lorentz transformation. Otherwise, other transformations such as the Voigt's would not satisfy them. Therefore, while Voigt's and Lorentz's transformations are sufficient conditions for the constancy of the speed of light, the two postulates are only necessary conditions for the two transformations.

It is easy to verify that the following general form of the Voigt transformation satisfies the constancy of the speed of light and the principle of relativity.

$$x' = \gamma^n (x - vt), \tag{2}$$

$$y' = \gamma^{n-1} y, \tag{3}$$

$$z' = \gamma^{n-1} z, \tag{4}$$

$$t' = \gamma^n (t - vx/c^2). \tag{5}$$

In the above equations, *n* is any integer including 0, and

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \quad . \tag{6}$$

When *n*=0, the above equations reduce to the Voigt transformation. The Lorentz transformation arises when *n*=1. The existence of the Voigt transformation and others implying effects on lengths perpendicular to the velocity suggests that we need a third postulate to exclude them.

3. Derivation using Einstein's two postulates plus one

Now we start to derive the space and time transformations from three postulates. The first postulate is the principle of relativity, represented by the following equations in general forms.

$$x' = \alpha_x x + bvt \ , \tag{7}$$

$$y' = \alpha_y y, \tag{8}$$

$$z' = \alpha_z z \ , \tag{9}$$

$$t' = nt + \delta x. \tag{10}$$

$$x = \alpha'_x x' + b'v't' \ , \tag{11}$$

$$y = \alpha'_y y', \tag{12}$$

$$z = \alpha'_z z' \ , \tag{13}$$

$$t = n't' + \delta' x'. \tag{14}$$

In the above equations, *x'* (or *x*) is not just a variable, it is a function of time *t* and possibly other variables and should be written as *x'*(*t*) (or *x*(*t*)) because the main purpose of the Lorentz transformation is to relate motions in two reference frames such as the propagation of light. The full notation of (7) should be

$$x'(x(t), t) = \alpha_x(v)x(t) + b(v)vt \ . \tag{7a}$$

In (7a), *x'*(*x*(*t*),*t*) on the left-hand side of the equality is a function of *x*(*t*) and *t*, while the coefficients $\alpha_x(v)$ and $b(v)$ are functions of *v*. Since *x*(*t*) on the right-hand side could take any function form to describe physical motions, solutions for $\alpha_x(v)$ and $b(v)$ obtained by imposing a specific function such as $x = vt$ will apply only to transformation for $x = vt$ between two frames. Applying the results from $x' = 0$ and $x = vt$ to other function forms of *x*(*t*) is to add another postulate. Similarly, the full notation of (11) should be

$$x(x'(t'), t') = \alpha'_x(v')x'(t') + b'(v')v't' \ . \tag{11a}$$

Imposing $x' = v't'$ when solving for $\alpha'_x(v)$ and $b'(v)$ restricts the results applicable only to scenarios where $x' = v't'$. Extending the applicability of results from $x = 0$ and $x' = v't'$ to other function forms of *x'*(*t*) adds another postulate. Therefore, it is logically sounder to specify the relationship between $\alpha_x(v)$ and $b(v)$ as a postulate than pretending they are derived for arbitrary *x*(*t*) (or *x'*(*t*)) functions without implicitly adding a postulate.

The second postulate is the constancy of the speed of light, defined by the following propagation equations.

$$x^2 + y^2 + z^2 = c^2t^2 \ , \tag{15}$$

$$x'^2 + y'^2 + z'^2 = c^2t'^2. \tag{16}$$

The third postulate is "no length or space effect in directions perpendicular to the velocity", represented by the following equations.

$$\alpha_y = \alpha'_y = 1 \ , \qquad \alpha_z = \alpha'_z = 1. \tag{17}$$

Equivalently,

$$y' = y, z' = z. \tag{18}$$

Substituting equations (7), (10), and (18) into (16), and letting $\alpha_x = \alpha$,

$$(\alpha x + bvt)^2 + y^2 + z^2 = c^2(\delta x + nt)^2. \tag{19}$$

Expanding the above equation,

$$\alpha^2x^2 - c^2\delta^2x^2 + 2\alpha bxvt - 2c^2\delta nxt + y^2 + z^2 = c^2n^2t^2 - b^2v^2t^2, \tag{20}$$

Eq. (20) can be simplified to

$$(\alpha^2 - c^2\delta^2)x^2 + 2(\alpha bv - c^2\delta n)xt + y^2 + z^2 = (c^2n^2 - b^2v^2)t^2. \tag{21}$$

Comparing (21) with (15), we can see that if

$$\alpha^2 - c^2\delta^2 = 1, \tag{22}$$

$$\alpha bv - c^2\delta n = 0, \tag{23}$$

$$c^2n^2 - b^2v^2 = c^2. \tag{24}$$

the transformation equations will ensure that the speed of light will be the same *c* in both reference systems. From Eq. (23),

$$b = \frac{c^2 \delta n}{a v}. \tag{25}$$

Substituting (25) into (24),

$$c^2 n^2 - \frac{c^4 \delta^2 n^2}{\alpha^2} = c^2. \tag{26}$$

which can be simplified to

$$\alpha^2 n^2 - c^2 \delta^2 n^2 = \alpha^2. \tag{27}$$

From (22),

$$\alpha^2 = 1 + c^2 \delta^2. \tag{28}$$

Substituting (28) into (27),

$$(1 + c^2 \delta^2) n^2 - c^2 \delta^2 n^2 = 1 + c^2 \delta^2, \tag{29}$$

which can be simplified to

$$n^2 = 1 + c^2 \delta^2. \tag{30}$$

From (28) and (30), it is obvious that

$$n^2 = \alpha^2, \ n = \alpha. \tag{31}$$

The original Eqs. (23) and (24) can be simplified to

$$bv - c^2 \delta = 0, \tag{32}$$

$$c^2 \alpha^2 - b^2 v^2 = c^2. \tag{33}$$

Since given Eq. (32), Eqs. (22) and (33) are the same equation, there are effectively only two equations for three unknowns, α , b and δ . So, there are infinite groups of solutions for parameters α , b and δ . From (22) and (33), we obtain

$$\alpha = \sqrt{1 + c^2 \delta^2}, \tag{34}$$

$$b = \frac{c^2\delta}{v}. \tag{35}$$

Any real valued δ with α and b are chosen according to Eqs. (35) and (36) will ensure Eqs. (7) - (14) satisfy the principle of relativity and the constancy of the speed of light. This shows there are an infinite number of such transformations, hence three postulates are not sufficient for deriving the Lorentz transformation.

4. Derivation of the Lorentz transformation

We have shown that infinite transformations satisfy Einstein's two postulates plus one prohibiting spatial effect in directions perpendicular to the velocity. To derive the Lorentz transformation, we need another postulate to restrict the function form of the transformation equation, which can be achieved by imposing $\alpha_x = -b$ in Eq. (7) and (11). Now we add the fourth postulate, i.e. the space transformation equations in the direction of velocity along the x-axis have the function forms

$$x' = \alpha x - avt \tag{36}$$

$$x = \alpha' x' - a' v' t' \ . \tag{37}$$

Substituting $b = -\alpha$ into Eqs. (34) and (35), we have

$$-\frac{c^2\delta}{v} = \sqrt{1 + c^2\delta^2}$$

which implies

$$\delta^2 = \frac{1}{\frac{c^4}{v^2} - c^2} = \frac{v^2/c^4}{1 - v^2/c^2}$$

$$\alpha = \sqrt{1 + c^2\delta^2} = \sqrt{1 + \frac{v^2/c^2}{1 - v^2/c^2}} = \sqrt{\frac{1}{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{38}$$

$$b = -\alpha = -\frac{1}{\sqrt{1 - v^2/c^2}} \tag{39}$$

$$\delta = \frac{bv}{c^2} = -\frac{v/c^2}{\sqrt{1 - v^2/c^2}} \tag{40}$$

Substituting these coefficients, $n=\alpha$ and equations (17) in equations (7)-(10), we obtain the Lorentz transformation

$$x'=\frac{x-vt}{\sqrt{1-v^2/c^2}},\qquad y'=y,\qquad z'=z,$$

$$t'=\frac{t-vx/c^2}{\sqrt{1-v^2/c^2}} \tag{41}$$

Because of the symmetry in the specifications of the two reference frames, we can use the same approach to derive the transformation equations from frame *K'* to frame *K*. The transformation equations in the opposite direction are

$$x=\frac{x'+v't'}{\sqrt{1-v'^2/c^2}},\qquad y=y',\qquad z=z',$$

$$t=\frac{t'+v'x'/c^2}{\sqrt{1-v'^2/c^2}} \tag{42}$$

Therefore, deriving the Lorentz transformation needs at least four postulates.

In the above derivation, we use the results from the preceding section. Directly substituting Eq. (36) in Eq. (16), we obtain relationships between coefficients

$$\alpha^2-c^2\delta^2=1,$$

$$\alpha^2v+c^2\delta n=0, \tag{43}$$

$$c^2n^2-\alpha^2v^2=c^2. \tag{44}$$

From (43),

$$\delta=\frac{-\alpha^2v}{c^2n}. \tag{45}$$

Substituting (45) in (22)

$$\alpha^2-\frac{\alpha^4v^2}{c^2n^2}=1,$$

$$\alpha^2c^2n^2-\alpha^4v^2=c^2n^2. \tag{46}$$

From (44),

$$n^2 = \frac{c^2+\alpha^2 v^2}{c^2}. \quad (47)$$

Substituting (47) in (46), we obtain

$$\alpha^2(c^2 + \alpha^2 v^2) - \alpha^4 v^2 = c^2 + \alpha^2 v^2,$$

$$\alpha^2 = \frac{c^2}{c^2 - v^2} = \frac{1}{1-\frac{v^2}{c^2}}.$$

$$n^2 = \frac{c^2 + \frac{c^2}{c^2 - v^2}v^2}{c^2} = \frac{c^2}{c^2 - v^2} = \alpha^2.$$

$$\alpha = n = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

$$\delta = \frac{-\alpha^2 v}{c^2 n} = \frac{-v/c^2}{\sqrt{1-\frac{v^2}{c^2}}}.$$

By specifying the function forms as the fourth postulate, we can derive a unique set of space and time transformation equations, the Lorentz transformation.

Several popular derivations of the Lorentz transformation have "derived" the fourth postulate from Eqs. (7) and (11). The approach is as follows:

Let $x' = 0$, from (7) we have

$$0 = \alpha x + bvt.$$

Since $x = vt$, we obtain

$$bvt = -avt.$$

Hence, $b = -a$.

The reason the above derivation of $b = -a$ cannot be accepted is that *x* in Eq. (7) can have any relationship with *t*, for example, the light wavefront has the relationship $x = ct$. The conditions $x' = 0$ and $x = vt$ apply only to the origin of frame *K'*, not to light wavefront or other objects and places. Imposing a result from one special condition on all scenarios described by Eq. (7) is the insertion of a postulate by stealth. If we want to derive a transformation equation for the origins of two reference frames, the above derivation of $b = -a$ is acceptable, but not for a transformation applying to all objects moving relative to both frames or stationary in one frame. Extending the result from one special condition to all scenarios is an additional postulate.

How should we analyze the implications of $x' = 0$? If $b = -a$ is a postulate, the implications depend on what object or phenomenon we describe. For the light wavefront, $x' = 0$ implies $t = 0$ in the settings of derivation, and therefore $x = 0$. For an object started moving from $x' = -100$ at $u' = 100$ along the positive direction of the *x'*-axis, $x' = 0$ implies $\Delta t' = 1$, $\Delta t = \alpha$, and $\Delta x = au$. For the origin of frame *K'* measured in frame *K*, $x' = 0$ implies $x = vt$. Therefore, using $x = vt$ and $x' = 0$ to derive $b = -a$ but not counting it as a postulate is acceptable only for equations describing the coordinates of a frame's origin; it is not acceptable for transformations describing all moving and stationary objects.

Logical inference rules also suggest that using $x = vt$ and $x' = 0$ to derive $b = -a$ amounts to an additional postulate. If Eqs. (7) and (11) imply $b = -a$, derivations using them should produce no other space-time transformations that satisfy the constancy of the speed of light and the principle of relativity than the Lorentz transformation whose derivation requires $b = -a$ explicitly. As we have shown in Section 3, using Einstein's two postulates plus one postulate of no spatial effects in directions perpendicular to the velocity, we can have an infinite number of space-time transformations that satisfy the constancy of the speed of light and the principle of relativity. This proves that Eqs. (7) and (11) do not imply $b = -a$. Even though we can obtain $b = -a$ from $x = vt$ and $x' = 0$, $x = vt$ does not describe objects moving in the two frames and other objects stationary in one frame but not at its origin. Therefore, using $b = -a$ derived from applying $x = vt$ and $x' = 0$ to Eq. (7) in deriving the Lorentz transformation is adding the fourth postulate.

5. Implicit postulates in derivations using only the principle of relativity

Many researchers have asserted that a Lorentz-type transformation can be derived from the principle of relativity (and the isotropy of space) alone (Coleman 2003b; Field 1997; Berzi and Gorini 1969; Ignatowsky 1910). In the present study, we will use Coleman's derivation (Coleman 2003b) as an example to analyze the implicit assumptions used. In this analysis, we will use notations in this paper to describe his derivation rather than Coleman's notation. Coleman used a scenario where three frames or observers move relative to each other and investigated the spatial and temporal relationships between the three frames.

Coleman began with two frames *I* and *I'* moving along the *x*-axis with relative velocities at *v* and -*v* respectively and assumed the general space transformation equations as

$$x' = a_v x + G_v t \qquad \text{and} \qquad x = a_v' x' + G_v' t'. \tag{48}$$

"Likewise familiar scenarios involving mutual perception of origin trajectories and unit-length rods then easily resolve three of these unknown parameters:"

$$a_v v = -G_v, \quad a_v' v = G_v', \quad \text{and} \quad a_v = a_v'. \tag{49}$$

These reduce (48) to

$$x' = a_v(x - vt) \qquad \text{and} \qquad x = a_v(x' + vt'). \tag{50}$$

Hence,

$$t' = a_v\left(t - \frac{x\left(1 - \frac{1}{a_v{}^2}\right)}{v}\right) . \tag{51}$$

Coleman defined a new term chronicity

$$K_v = \frac{1 - \frac{1}{a_v{}^2}}{v} . \tag{52}$$

From (50)-(52), we can relate *x'* and *t'* respectively in terms of *x* and *t* symmetrically as

$$x' = a_v(x - vt) \qquad \text{and} \qquad t' = a_v(t - xK_v). \tag{53}$$

Although Coleman did not explain the procedures in the above quotation, from our analysis in the preceding section we know that he has assumed more than what the principle of relativity implies. He has added two more postulates: $v' = -v$ and $a_v v = -G_v$ (because he certainly obtained it by applying $x = vt$ and $x' = 0$ to Eq. (48)) to begin with.

Then, a third frame *I''* traveling relative to *I'* in the positive *x'*-direction at velocity *u collinear* with v is introduced. The velocity of *I* as perceived by *I''* is denoted by w whose orientation is chosen for *cyclic symmetry (Fig.2)*. So, the equations for *x''* and *t''* in terms of *x'* and *t'* depend only on *u*,

$$x'' = a_u(x' + ut') \quad \text{and} \quad t'' = a_u(t' - x'K_u). \tag{54}$$

Substituting (53) in (54), we can relate *x''* and *t''* respectively in terms of *x* and *t*,

$$x'' = a_v a_u(x(1 + uK_v) - t(u + v)) \quad \text{and}$$

$$t'' = a_v a_u(t(1 + vK_u) - x(u + v)). \tag{55}$$

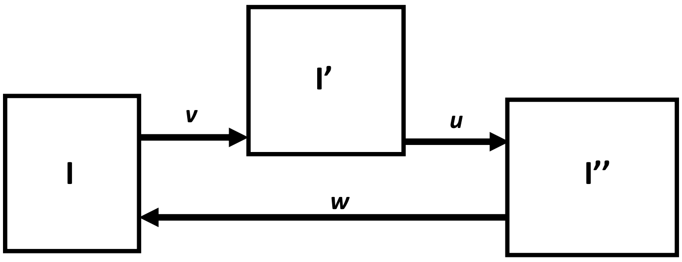

Fig.2 Three reference frames in relative motion. The velocity between *I* and *I'* is *v*; that between *I'* and *I''* is *u*; and that between *I* and *I''* is *w*.

Coleman commented that Terletskii (1968) and Rindler (1977) used equations similar to (55) but required the context of a two-postulate derivation instead of using the direct *physical* arguments as Coleman. He thought many others (Ignatowsky 1910; Frank and Rothe 1911; Lee and Kalotas 1975; Lévy-Leblond 1976; Srivastava 1981; Stachel 1983; Mermin 1984; Schwartz 1984; Sen 1994; Field 1997) also failed to use his following direct physical arguments. However, as we pointed out earlier, a derivation should be an inference from premises to conclusions where premises are the sufficient condition for the conclusions. Once the premises are given, the derivation should use only mathematical and logical operations. Coleman's following direct physical arguments are additional postulates as we analyzed earlier.

Since the velocity of origin I_o (x = 0), as perceived by I'', is $w = x''/t''$, we obtain from (55)

$$x'' = a_v a_u\big(0 \times (1 + uK_v) - t(u + v)\big) = -a_v a_u t(u + v) \qquad \text{and}$$

$$t'' = a_v a_u\big(t(1 + vK_u) - 0 \times (u + v)\big) = a_v a_u t(1 + vK_u).$$

Therefore,

$$w = \frac{x''}{t''} = \frac{-(u+v)}{1+vK_u}. \tag{56}$$

The velocity of I_o'' (x'' = 0), as perceived by I, is $-w = x/t$, we obtain from (55)

$$0 = a_v a_u(x(1 + uK_v) - t(u + v)),$$

$$-w = \frac{x}{t} = \frac{u+v}{1+uK_v}. \tag{57}$$

From (56) and (57), we have

$$\frac{u+v}{1+vK_u} = \frac{u+v}{1+uK_v}, \text{ which implies } vK_u = uK_v.$$

Therefore,

$$\frac{K_v}{v} = \frac{K_u}{u} = \Omega. \tag{58}$$

From (57) and (58), we have

$$-w = \frac{x}{t} = \frac{u+v}{1+uv\Omega}. \tag{59}$$

The above equation is similar to Einstein's equation for velocity addition, which implies a universal constant

$$\Omega = \frac{u+v+w}{-uvw}. \tag{60}$$

As we commented earlier, Coleman has used two additional postulates, $v' = -v$ and $a_v v = -G_v$ (by applying $x = vt$ and $x' = 0$ to Eq. (48)), in deriving (56) and (57) he further assumed $x = 0$ and $x'' = 0$. Eqs. (59) and (60) require the validity of (56) and (57), i.e. $x = 0$ and $x'' = 0$ simultaneously. Since the only space-time location in Coleman's settings that satisfies these conditions is when $x = 0$, $t = 0$, $x'' = 0$, and $t'' =$

0, Eq. (59) is an outcome of the undefined expression *x/t* when $x = 0$, $t = 0$. Therefore, Eqs. (59) and (60) are not the true relationship of the three velocities of the three frames' origins, rather they are abnormal results of using the undefined expression 0/0. Other derivations (Einstein 1905, 1952; Kittle, Knight, and Ruderman 1973; Schutz 2009) employing both $x = 0$ and $x'' = 0$ are all likely to be outcomes of undefined expressions such as *x/t* when $x = 0$, $t = 0$ as well (Ma 2004, 2013). When an expression is undefined, it may give absurd results. It is acceptable to impose (59) as a postulate in deriving certain equations, but claiming that the principle of relativity implies (59), i.e. the principle of relativity is a sufficient condition for (59), is logically unacceptable.

Coleman then introduced a fourth frame I''' with velocities y and $-z$ relative to frames I'' and I respectively. Applying (60) to frames I, I'', and I''', we obtain

$$\Omega = \frac{-w+y+z}{wyz}, \quad \text{or} \qquad w = \frac{y+z}{1+yz\Omega}.$$

We obtain by substituting for w in (60)

$$\Omega = \frac{u+v+y+z}{-(uvy+uvz+uyz+vyz)}. \tag{61}$$

Coleman let u and y both be equal to v for the four-frame case and obtain

$$\Omega = \frac{3v+z}{-v^3-3v^2z} \qquad \text{or} \qquad -z = \frac{3v(1+\frac{\Omega v^2}{3})}{1+3\Omega v^2} \tag{62}$$

Since $v = 1/\sqrt{-3\Omega}$ will lead to infinite velocity for $-z$ if Ω is negative, Coleman thought this velocity addition singularity excludes any finite negative value for Ω (Coleman 2003a). He let u be equal to v for the three-frame case and obtain

$$-w = \frac{2v}{1+\Omega v^2}. \tag{63}$$

Since $-w$ in the above function peaks at $v = 1/\sqrt{\Omega}$, Coleman argued that "Velocity addition uniqueness implies that $1/\sqrt{\Omega}$ corresponds to a cosmic upper speed limit."

We have commented earlier that Eqs. (59) and (60) are consequences of the undefined expression *x/t* due to imposing conditions $x = 0$ and $x'' = 0$, which means $t = 0$ and $t'' = 0$ in the settings specified. Moreover, Coleman's reason to exclude negative values of Ω is not well founded, because v^2 in (62) and (63) comes from the product of u and v which could be positive or negative as vectors. Although Coleman assumes that frame I'' travels relative to I' in the positive x'-direction, as he tries to derive a general transformation function for arbitrary velocities, u in (59) and (60) should be able to take positive and negative values (i.e. u can have the same direction as v or the opposite direction of v). Hence no matter whether Ω has a positive or negative value, there will be a combination of u and v to make $1 + uv\Omega = 0$. For example, for a positive Ω when u and v have opposite directions, $vu = -1/\Omega$ produce zero-valued denominators in (57) and (58), just like for a negative Ω when u and v have the same direction, $vu = 1/\Omega$ produce zero-valued denominators in (57) and (58). Therefore, there is no mathematical or logical reason to exclude negative Ω only; excluding only it ignores that u and v can have opposite directions. If negative Ω should be excluded because some values of u and v can make $1 + uv\Omega = 0$, positive Ω should also be excluded because some values of u and v can make $1 + uv\Omega = 0$ too. Then, Ω can only have a value of 0 by Coleman's criterion of velocity addition non-singularity, leading to the Galilean transformation. Thus, the non-singularity of velocity addition (which per se might be considered a postulate) should exclude negative and positive non-zero Ω for Eq. (59). Only the Galilean transformation can exclude all velocity addition singularity. Coleman's derivation used $v' = -v$, $a_v v = -G_v$, $x = 0$, $x'' = 0$, and no negative Ω, which are additional postulates besides the principle of relativity.

6. Derivation of the Poincaré transformation and its nature

The Lorentz transformation assumes that the origins of the two (or more) frames coincide at time zero. What are the relationships between the coordinates of two frames when an event like a light flash does not occur at the two frames' origins? The time measured by the two frames can also have different values. It is difficult to derive the transformation equations between the coordinates of two reference frames. We can expand the Lorentz transformation to establish the transformation between the coordinate differences of two reference frames.

When an event's initial coordinates are $x = x_0$, $x' = x'_0$, $y = y_0$, $y' = y'_0$, $z = z_0$, $z' = z'_0$, $t = t_0$, and $t' = t'_0$, the propagation of light waves generated at that point can be described as

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = c^2(t - t_0)^2 \qquad (64)$$

$$(x' - x'_0)^2 + (y' - y'_0)^2 + (z' - z'_0)^2 = c^2(t' - t'_0)^2 \qquad (65)$$

Using the results obtained earlier, we write the space and time transformation equation as

$$x' - x'_0 = \alpha(x - x_0) - \alpha v(t - t_0) \tag{66}$$

$$t' - t'_0 = \alpha(t - t_0) - \delta(x - x_0). \tag{67}$$

Substituting (66) and (67) into (65), with similar operations we obtain Eqs. (22), (43), and (44). Solving these equations and substituting the coefficients in (66) and (67), we obtain the following Poincaré transformation

$$x' - x'_0 = \frac{(x - x_0) - v(t - t_0)}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{68}$$

$$y' = y, \qquad z' = z,$$

$$t' - t'_0 = \frac{(t - t_0) - v(x - x_0)/c^2}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{69}$$

$$x - x_0 = \frac{(x' - x'_0) - v'(t' - t'_0)}{\sqrt{1 - \frac{v'^2}{c^2}}}, \tag{70}$$

$$y = y', \qquad z = z',$$

$$t - t_0 = \frac{(t' - t'_0) - v'(x' - x'_0)/c^2}{\sqrt{1 - \frac{v'^2}{c^2}}}. \tag{71}$$

The Poincaré transformation clearly shows that it transforms coordinate intervals or differences rather than coordinates per se. The Lorentz transformation can therefore be viewed as a transformation between the difference of (x, y, z, t) and (x_0, y_0, z_0, t_0) and the difference of (x', y', z', t') and (x'_0, y'_0, z'_0, t'_0) when $x_0 = 0$, $x'_0 = 0$, $t_0 = 0$, and $t'_0 = 0$ (Ma 2004, 2013). The nature of the Lorentz transformation and the Poincaré transformation is

$$dx' = \frac{dx - vdt}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{72}$$

$$dy' = dy, \qquad dz' = dz, \tag{73}$$

$$dt' = \frac{dt - vdx/c^2}{\sqrt{1-\frac{v^2}{c^2}}}. \quad (74)$$

$$dx = \frac{dx' - v'dt'}{\sqrt{1-\frac{v'^2}{c^2}}}, \quad (75)$$

$$dy = dy', \qquad dz = dz',$$

$$dt = \frac{dt' - v'dx'/c^2}{\sqrt{1-\frac{v'^2}{c^2}}}. \quad (76)$$

There is no transformation between x'_0 and x_0 or between t'_0 and t_0 yet. The Poincaré transformation needs the same four postulates as the Lorentz transformation.

7. Discussion

In the present study, we have shown that derivation of the Lorentz transformation needs at least four postulates or premises. A derivation should be a logical deduction or inference from premises to conclusions, where the premises should be a set of sufficient conditions for the conclusions. Since non-Lorentz transformations satisfying the constancy of the speed of light and the principle of relativity such as the Voigt transformation exist, the Lorentz transformation is not a necessary condition for the constancy of the speed of light or the principle of relativity. As basic logic rules tell us, if A is not a necessary condition of B, then we cannot derive A from B. Therefore, Einstein and anyone else could not logically derive the Lorentz transformation from the two postulates. The propaganda of Einstein's achievement in such derivations is incorrect. The present findings build on the early studies (Ma 2004, 2013).

The four postulates needed for the derivation of the Lorentz transformation are 1) the principle of relativity; 2) the constancy of the speed of light; 3) motion has no impact on the length or spatial distance in the directions perpendicular to its velocity; 4) the coefficient of the spatial coordinate term such as x (or x') term equals that of the velocity-time term vt (or $v't'$) on the right-hand side of the spatial transformation equations. Without the third postulate, an infinite number of transformations satisfy the constancy of the speed of light and the principle of relativity. The Voigt transformation is one of them. Without the fourth postulate, an infinite number of transformations satisfy Einstein’s two postulates as well. Therefore, these four postulates are sufficient conditions for the Lorentz transformation.

It is easy to show that the Lorentz transformation is also a sufficient condition for the four postulates. The transformation from *K* to *K’* has the same form as that from *K’* to *K*, satisfying the principle of relativity. It also ensures the constancy of the speed of light. The third and fourth postulates are contained in the transformation equations. Therefore, the Lorentz transformation is a sufficient and necessary condition for the four postulates, and the four postulates are also sufficient and necessary conditions for the Lorentz transformation.

Since the Lorentz transformation is not a necessary condition for the constancy of the speed of light (and the principle of relativity), nor are the constancy of the speed of light and the principle of relativity sufficient conditions for the Lorentz transformation, it seems inappropriate to raise the constancy of the speed of light to such a fundamental principle status in the theory of relativity as Einstein did. Instead of having Einstein’s two postulates as primary principles to (impossibly) derive the Lorentz transformation, it is more reasonable and logically valid to directly postulate the Lorentz transformation as the primary principles of the theory of relativity. The constancy of the speed of light should be secondary to the primary principles, i.e. a consequence of the Lorentz transformation. The four postulates, which include Einstein’s two postulates, are sufficient and necessary conditions for the Lorentz transformation, hence postulating the Lorentz transformation is equivalent to having the four postulates.

Besides being logically more valid, directly postulating the Lorentz transformation as the primary principle of the theory of relativity can also avoid conflicting statements on the speed of light between special relativity and general relativity. Although Einstein argued that it is impossible to measure the one-way speed of light in inertial systems and postulated the constancy of the speed of light, the theory of relativity has to face the awkward situation that the one-way speed of light can be measured in non-inertial systems (Sagnac 1913; Wang et al. 2003; Wang, Zheng, and Yao 2004; Wang 2005) and there is no inertial system in the universe. General relativity cannot assume the constancy of the speed of light because the one-way speed of light can be measured in non-inertial systems. Since the Lorentz transformation can produce the constancy of the speed of light and still works in general relativity, using the Lorentz transformation as the primary principle can avoid the contradiction between special relativity and general relativity. The constancy of the speed of light does not hold in general relativity because it is a secondary outcome and can be influenced by other factors.

Since Einstein (1905) first tried to derive the Lorentz transformation from his two postulates, many researchers and educators have provided various derivations from the two postulates or the principle of relativity alone. It is worth pointing out that Einstein's first expression of the constancy of the speed of light differs from the current understanding of it. It has the same description as that of classical physics, i.e. the speed of light does not depend on the velocity of the light source. The modern understanding is that the speed of light does not depend on the observer's velocity. The present study shows that all these derivations (Einstein 1905, 1952; Kittle, Knight, and Ruderman 1973; Lee and Kalotas 1975; Lévy-Leblond 1976; Field 1997; Coleman 2003b) are logically invalid as a deduction or inference. They either covertly introduced additional conditions (postulates), or manipulated the operations to obtain a desired equation from denominators equalling 0, and often did both. For example, Coleman (2003b) first introduced $v = v'$ (a postulate); second used $x' = 0$ and $x = vt$ to show $\alpha = b$ in the spatial transformation equation $x' = \alpha x - bvt$ (which is another postulate because *x* in this equation denotes more than $x = vt$); third imposed $x' = 0$ and $x = 0$ (which is true only when $t' = 0$ and $t = 0$) to obtain $-w = \frac{x}{t} = \frac{u+v}{1+uv\Omega}$, which arises due to $x/t = 0/0$; fourth excluded negative Ω on the ground of velocity addition singularity despite positive Ω also leads to velocity addition singularity (which is a logic mistake and can be considered a postulate).

The Lorentz transformation relates the coordinates of a moving or stationary object in one reference frame to those in another. The *x*-variable in the equation is not simply a variable, rather it can be any function of *t* besides *vt*, i.e. $x'(x(t), t) = \alpha_x(v)x(t) + b(v)vt$. Almost all derivations of the Lorentz transformation let $x'(x(t), t) = 0$ and $x(t) = vt$ to obtain $\alpha_x(v) = -b(v)$ or directly let $\alpha_x(v) = -b(v)$ without acknowledging that doing so is using an additional condition (postulate). When $x'(x(t), t) = 0$ and $x(t) = vt$, $\alpha_x(v) = -b(v)$; this result cannot be understood as $\alpha_x(v) = -b(v)$ for other function forms of $x(t)$ or for $x'(x(t), t) \neq 0$.

Expanding the applicability of $\alpha_x(v) = -b(v)$ from $x'(x(t), t) = 0$ and $x(t) = vt$ to other function forms of $x(t)$ or $x'(x(t), t) \neq 0$ is adding another postulate. Those researchers are self-deceiving to believe that $\alpha_x(v) = -b(v)$ obtained from $x'(x(t), t) = 0$ and $x(t) = vt$ applies to all function forms of $x(t)$without incurring another postulate, hence the Lorentz transformation can be derived from Einstein's two postulates or the principle of relativity alone. If their belief is true, the Lorentz transformation would be a necessary condition for the constancy of the speed of light. However, we have shown in Section 3 that an infinite number of transformations can result in the constancy of the speed of light, besides the Voigt-type transformations discussed in Section 2. Therefore, their derivation of $\alpha_x(v) = -b(v)$ from $x'(x(t), t) = 0$ and $x(t) = vt$ and its application to general functions are just adding another postulate by stealth. It is the same as postulating $\alpha_x(v) = -b(v)$ directly but logically not as sound.

The Poincaré transformation also needs four postulates to derive. The fact that initial values of spatial and temporal coordinates must be subtracted from the corresponding variables indicates that the Poincaré transformation handles coordinate differences rather than coordinates per se. The Lorentz transformation gives an impression of coordinate transformation because all initial values are 0. The transformation equation for the initial values measured in two reference frames with relative velocity does not exist. The Lorentz and Poincaré transformations need a reference point to transform coordinate differences, $dx' = (dx - vdt)/\sqrt{1 - v^2/c^2}$ and $dt' = (dt - vdx/c^2)/\sqrt{1 - v^2/c^2}$. In the Lorentz case, the reference point is when $x' = 0$ and $x = 0$ at the moment $t' = 0$ and $t = 0$. In the Poincaré case, the reference point is when $x' = x'_0$ coincides with $x = x_0$ at the moment $t' = t'_0$ coincides with $t = t_0$. Since the coordinates of the reference point have to be taken as given, the transformation equations cannot be applied to them. This is a remnant of what the Lorentz transformation was initially intended for, i.e. to address the invariance of the speed of light measured in different reference frames.

8. Conclusions

From our analysis in the present study, we can draw the following conclusions:

First, the principle of relativity and the constancy of the speed of light are not sufficient conditions for the Lorentz transformation, therefore the Lorentz transformation cannot be logically derived from Einstein's two postulates.

Second, the Lorentz transformation is a sufficient condition for the constancy of the speed of light, but not a necessary condition. The Voigt transformation and an infinite number of other transformations shown in this study can also result in the constancy of the speed of light. Since Lorentz transformation is not a necessary condition for the constancy of the speed of light and the principle of relativity, it cannot be logically derived from the latter two.

Third, a derivation should be a logical deduction or inference from premises to conclusion. To obtain the Lorentz transformation through deduction, at least four postulates are needed: 1) the principle of relativity; 2) the constancy of the speed of light; 3) motion has no impact on the length or spatial distance in the directions perpendicular to the velocity; 4) the coefficient of the spatial coordinate term such as x (or x') term equals that of the velocity-time term vt (or $v't'$) on the right-hand side of the spatial transformation equations.

Fourth, the Lorentz transformation is also a sufficient condition for the above four postulates, hence, the four postulates are the sufficient and necessary conditions for the Lorentz transformation. Therefore, it is more appropriate to postulate the Lorentz transformation directly than raise the constancy of the speed of light to its current status in special relativity. This can avoid the conflicting treatments of the speed of light in special relativity and general relativity.

Fifth, most existing derivations of the Lorentz transformation involve covertly stipulating the above third and fourth postulates. The usual practice is to let $x'(x(t), t) = 0$ and $x(t) = vt$ to obtain $\alpha_x(v) = -b(v)$ and pretend this constrained result $\alpha_x(v) = -b(v)$ being a general result for all function forms of $x(t)$ and for $x'(x(t), t) \neq 0$.

Sixth, the Lorentz transformation or its alike cannot be logically derived from the principle of relativity alone. Those "derivations" using only the principle of relativity usually not only covertly use additional postulates listed above but also make mathematical and logical mistakes.

Seventh, the common mathematical mistake in most existing derivations using Einstein's two postulates or the principle of relativity alone is to obtain a certain function/equation by letting $x'(x(t),t) = 0$ and $x(t) = vt$, and another function/equation by letting $x(x'(t'),t') = 0$, and then use these functions/equations to derive the desired equation. Since the spacetime point satisfying $x' = 0$ and $x = 0$ in their derivation settings is only $x' = 0$, $x = 0$, $t' = 0$, and $t = 0$, Coleman's $-w = \frac{x}{t} = \frac{u+v}{1+uv\Omega}$ is an artefact arising from $x/t = 0/0$.

Eighth, only $\Omega = 0$ can avoid velocity addition singularity in Coleman's $-w = \frac{u+v}{1+uv\Omega}$, which leads to the Galilean transformation.

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