Research Article

# New Perspectives on the Roots of Real Polynomials of Degree n and Number Theory

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**In this work we obtained some results on the real and complex roots of real polynomials of the form $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where the $a_k$ are real numbers for $k \in \{0, \ldots, n-1\}$, we also obtained results on linear Diophantine equations of the form $ax + by + cz = d$, where $a, b, c$ and $d$ are integers.**

**To obtain the desired results, the relation that states the following was used: for any real numbers $A$ and $B$, there exists a unique real number $\lambda$, such that $A + B = \lambda[A^2 + B^2]$. This result is appropriately linked to the object of study.**

**For the polynomials, optimal domains were obtained where the real or complex roots are found, without the use of higher calculus.**

**For the linear Diophantine equation, the desired solutions were obtained, which were found by establishing several links between the Diophantine equation and the relationship $A + B = \lambda[A^2 + B^2]$. Several examples of the results obtained are illustrated, which are intended to show the benefits of this proposal.**

**Additionally, we obtained the solution of Fermat's last Theorem in an elegant, simple and unprecedented way, different from what has been done by other authors.**

# 1. Introduction

The study of polynomial roots and Diophantine equations remains an active area of research because of their broad applications across mathematics (including numerical analysis), science, engineering, computer science, and other fields. In linear algebra, for instance, they are essential in finding the eigenvalues of operators and matrices. In control theory, polynomial roots are crucial in assessing a system's stability and overall behavior.

In numerical analysis, finding the roots of a polynomial is a fundamental problem. Several algorithms have been developed to approximate these roots, such as Newton's method, the bisection method, Müler's method, Durand-Kerner method and others. These methods are essential in computational mathematics and scientific computing. Néwton's method may not converge if the initial estimate is too far from a root. Some of these methods rely on the initial estimation interval in which the algorithms are applied. While there are globally convergent methods, such as Müller's method, most of these techniques are fundamentally based on differential calculus.

In this work, without the use of differential calculus, the approximate real or complex roots of polynomials of degree $n$ with real coefficients are achieved, using as a starting point the method presented in[1][2], and[3]. Initial estimates of the roots of the polynomial are also given in a different way than traditional. This method involves creating a connection between the problem under study and the relationship expressed as: for any $a_1, a_2, \ldots, a_n \in \mathbb{R}$, there exists a $\lambda$ that satisfies the equation $a_1 + a_2 + \cdots + a_n = \lambda\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)$.

Next, we present a general non-classical solution for the Diophantine equation of the form $Ax + By + Cz = D$. These types of equations are crucial due to their various applications in number theory, algorithms and computing, as well as in geometry and topology.

We finish this work, presenting the proof of Fermat's last theorem in an elegant and simple way. In this work, $\mathbb{N}$ will be the set of natural numbers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{I}$ the set of irrational numbers and $\mathbb{R}$ the set of real numbers.

# 2. Preliminaries

Let us consider the polynomial of degree $n$

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, a_n \neq 0.$$


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By the fundamental theorem of algebra it is known that $p$ has $n$ real or complex roots, counting multiplicities. If all the coefficients $a_0, a_1, \ldots, a_n$ are real, then there are real or complex roots. In the case of complex roots will occur in conjugate pairs, in the form $c \pm di$, where $c, d \in \mathbb{R}$ and $i^2 = -1$. On the other hand, if the coefficients are complex, it is not necessary that the complex roots be related.

There are methods to determine roots of a polynomial. For example, among the analytical methods we have: the method of formulas (quadratic, cubic and quartic), the remainder theorem, the factor theorem and the rational root theorem. Among the numerical methods we have: Synthetic division by a simple factor, Synthetic division by a quadratic factor, Bairstow method, Laguerre method, Bernoulli method, Newton's method, Müler's method, Quotient difference algorithm, Graeffe square root method and Jenkins-Traub method.

On the other hand, computational methods (use of Mathematical Software) combine a variety of numerical and analytical methods to find the roots of polynomials efficiently and accurately. These methods are carefully implemented to handle the complexities of high-degree polynomials with complex coefficients. For example, the "Jenkins-Traub" method is an efficient and robust algorithm for finding all roots of a polynomial, both real and complex.

The following Theorem is one of the many results that gives us how to take initial estimates (intervals) where the roots are located.

**Theorem 2.1.** *Given a polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0, a_n \neq 0$, define the polynomials*

$$P(x) = |a_n|\, x^n + |a_{n-1}|\, x^{n-1} + \cdots + |a_1|\, x - |a_0|,$$

$$Q(x) = |a_n|\, x^n - |a_{n-1}|\, x^{n-1} - \cdots - |a_0|.$$

*According to Descartes' rules, $P(x)$ has exactly one positive real zero $r_1$ and $Q(x)$ has exactly one positive real zero $r_2$. Then all zeros $z$ of $p(x)$ are in the annular region*

$$r_1 \leq |z| \leq r_2$$

**Theorem 2.2.** *Let $a_1, \ldots, a_n$ be any real numbers, then there exists $\lambda \in \mathbb{R}$, such that*

$$\sum_{i=1}^{n} a_i = \lambda \sum_{i=1}^{n} a_i^2.$$

*Proof.* Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i$, be defined by $f$ is a linear and continuous functional. Therefore

$$\mathbb{R}^n = \ker f \oplus (\ker f)^{\perp}. \tag{1}$$


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Thus we have that:

$$\dim \mathbb{R}^n = \dim Imf + \dim \ker f.$$

$$n = 1 + \dim \ker f.$$

Therefore $\dim \ker f = n - 1$.

Thus, from (1) we have:

$$(1,1,\ldots,1) = \sum_{i=1}^{n-1} \lambda_i u_i + \lambda u_n, \tag{2}$$

where $\{u_1, \ldots, u_{n-1}\} \subset \ker f$ and $u_n \in (\ker f)^{\perp}$

From (2) and taking into account that $f$ is a linear functional, we have

$$f(1,\ldots,1) = \lambda f(u_n),$$
$$\sum_{i=1}^{n} a_i = \lambda \sum_{i=1}^{n} a_i^2,$$
$$\text{since } u_n = (a_1,\ldots,a_n) \in (\ker f)^{\perp}.$$

Using this last relation, which must be linked with the function to be maximized and with the given restrictions. Below we show several problems that illustrate the given theory. □

## 3. About Polynomials

In this section we will find the roots of a polynomial of degree $n$ of the form

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0, \tag{3}$$

with the condition that $a_1 \neq 0$ and $a_0 < 0$. Therefore the polynomial $P(x)$ can be written as $P(x) = Q(x) + a_1x + a_0$, where

$$Q(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2. \tag{4}$$

Using the technique described in the article[2] and assuming that $x$ is a root of $P(x)$, we obtain

$$Q(x) + a_1x = \lambda\left[Q^2(x) + a_1^2x^2\right]. \tag{5}$$

From this last relationship we obtain

$$Q(x) - \frac{1}{2\lambda} = \frac{\sqrt{2}b_1}{2|\lambda|}. \tag{6}$$

$$a_1x - \frac{1}{2\lambda} = \frac{\sqrt{2}b_2}{2|\lambda|}. \tag{7}$$


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where $b_1^2+b_2^2=1, b_1=b_1(x), b_2=b_2(x)$ and $\lambda=\lambda(x)$.

From the relations (5) and (6) we obtain

$$-2 \leq \lambda a_0 \leq 0. \tag{8}$$

**Theorem 3.1.** *Let* $P(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0$, *where* $a_0<0$ *and* $a_1>0$. *If* $p(x_0)=0$, *then there exists* $\lambda>0$ *such that* $0<\lambda\leq-\frac{2}{a_0}$,

$$\begin{cases} -\frac{a_0}{2a_1}\leq x_0 \text{and} \frac{1}{a_1x_0}<\lambda<-\frac{2}{a_0}, \\ \text{or} \\ -\frac{a_0}{a_1}<x_0 \text{and } 0<\lambda\leq-\frac{2}{a_0}, \\ \text{or} \\ 0<x_0<-\frac{a_0}{2a_1} \text{and } 0<\lambda<-\frac{2}{a_0}, \end{cases}$$

*where* $x_0=-\frac{a_0}{2a_1}\pm\frac{1}{2a_1}\sqrt{-a_0^2-\frac{2a_0}{\lambda}}$.

*Proof.* From the relation (8) we obtain that

$$0<\lambda\leq-\frac{2}{a_0}. \tag{9}$$

Also from (5) we have

$$Q(x_0)\left[1-\lambda Q(x_0)\right]=a_1x_0\left[\lambda a_1x_0-1\right]. \tag{10}$$

From the relation (10) and being $Q(x_0)=-a_1x_0-1$ we obtain

$$(-a_1x_0-a_0)\left[1-\lambda(-a_1x_0-a_0)\right]=a_1x_0\left[\lambda a_1x_0-1\right]. \tag{11}$$

From this equation (11), the following cases are derived.

i) Let $-a_1x_0-a_0>0$ and $1+\lambda(a_1x_0+a_0)>0$, from this we have

$$a_1x_0>0 \text{ and } \lambda a_1x_0-1>0,$$

or

$$a_1x_0<0 \text{ and } \lambda a_1x_0-1<0.$$

If $-a_1x_0-a_0 \geqslant a_1x_0$, then

$$1+(a_1x_0+a_0)\lambda<\lambda a_1x_0-1,$$

which is absourd $-2>\lambda a_0, a_0<0$ and $\lambda>0$.

If $-a_1x_0-a_0\leq a_1x_0$, then


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$$1+(a_1x_0+a_0)\lambda > \lambda a_1x_0 - 1.$$

The relationship $a_1x_0<0$ is false, since it is observed that $-a_1x_0-a_0<0$ which implies that $0<a_0$, which is false.

Therefore we have

$$-\frac{a_0}{2a_1} \leq x_0 \text{and} \frac{1}{a_1x_0} < \lambda < -\frac{2}{a_0}.$$

ii) Let $-a_1x_0-a_0<0$ and $1+\lambda(a_1x_0+a_0)<0$ which implies that

$$a_1x_0>0 \text{ and } \lambda a_1x_0-1>0,$$

or

$$a_1x_0<0 \text{ and } \lambda a_1x_0-1<0.$$

The relationship in item (ii) is impossible, since $a_1x_0+a_0>0$ and $\lambda>0$ imply $\lambda(a_1x_0+a_0)>0$ and since $1+\lambda(a_1x_0+a_0)<0$ we have that $1<0$, which is false.

iii) Let $-a_1x_0-a_0<0$ and $1+\lambda(a_1x_0+a_0)>0$, from this we have

$$a_1x_0<0 \text{ and } \lambda a_1x_0-1>0,$$

or

$$a_1x_0>0 \text{ and } \lambda a_1x_0-1<0.$$

Since $-a_1x_0-a_0<a_1x_0$ y $1+\lambda(a_1x_0+a_0)>\lambda a_1x_0-1$ we have

$$-\frac{a_0}{\hat{a}_1} < x_0 \text{ and } \lambda < \frac{1}{a_1x_0} \text{ and } \lambda \leq -\frac{2}{a_0},$$

that is,

$$-\frac{a_0}{a_1} < x_0 \text{ and } 0<\lambda \leq -\frac{1}{a_0}.$$

iv) Let $-a_1x_0-a_0>0$ and $1+\lambda(a_1x_0+a_0)<0$, from this we have

$$a_1x_0<0 \text{ and } \lambda a_1x_0-1>0,$$

or

$$a_1x_0>0 \text{ and } \lambda a_1x_0-1<0.$$

If $-a_1x_0 - a_0 > a_1x_0$ and $1 + \lambda(a_1x_0 + a_0) < \lambda a_1x_0 - 1$, which is an impossible relationship.

If $-a_1x_0 - a_0 > a_1x_0$ and $1 + \lambda(a_1x_0 + a_0) > \lambda a_1x_0 - 1$, which implies that

$$0 < x_0 < -\frac{a_0}{2a_1} \text{ and } 0 < \lambda < -\frac{2}{a_0}.$$

In addition to the relation (10) we obtain

$$x_0 = -\frac{a_0}{2a_1} \pm \frac{1}{2a_1}\sqrt{-a_0^2 - \frac{2a_0}{\lambda}},\ \ 0 < \lambda \leq -\frac{2}{a_0}.$$

□

**Example 3.1.** *Find the roots of the polynomial of degree five*

$$P(x) = x^5 + x^4 + x^3 + x^2 + x - 2.$$

**Solution 3.1.** *We have that* $a_0 = -2, a_1 = 1$ *from the relation (i) we obtain*

$$1 \leq x_0 \text{ and } \frac{1}{x_0} < \lambda < 1.$$

*From the relation (iii) we obtain*

$$2 < x_0 \text{ and } 0 < \lambda \leq 1.$$

*From the relation (iv) we obtain*

$$0 < x_0 < 1 \text{ and } 0 < \lambda < 1.$$

*The real root is*

$$x_0 = 1 \pm \frac{1}{2}\sqrt{-4 + \frac{4}{\lambda}} = 1 \pm \sqrt{-1 + \frac{1}{\lambda}},$$

*furthermore* $x_0 < \frac{1}{\lambda}$. *Analyzing the variation of* $\lambda$ *and* $x_0$ *we have that*
*Taking* $\lambda = 0.9$ *then* $x_0 = \frac{2}{3}$ *and* $P\left(\frac{2}{3}\right) \approx -0.263374$.
*Taking* $\lambda = 0.95$ *then* $x_0 = 1 - \frac{\sqrt{19}}{19}$ *and* $P\left(1 - \frac{\sqrt{19}}{19}\right) \approx 0.446263$.
*This suggests to us that if we take* $\lambda = 0.921935$ *then* $x_0 = 0.70901$ *and* $P(0.70901) \approx 0$.

**Remark 3.1.** *A connection between the given polynomial*

$$x^n + a_{n-1}x^{n-1} + \cdots + a_jx^j + \cdots + a_1x + a_0 = 0, \tag{12}$$

*and the relation*

$$\sum_{k=1}^{N} A_k = \lambda \sum_{k=1}^{N} A_k^2 \tag{13}$$

*can give us other information to find the real or complex roots, in that sense we choose $N = 2$.*

$$A_1 = x^n + a_{n-1}x^{n-1} + \cdots + a_{j+1}x^{j+1} + a_{j-1}x^{j-1} + \cdots + a_2x^2 + a_1x. \tag{14}$$

$$A_2 = a_jx^j. \tag{15}$$

*Substituting the relations (14) and (15) into (13), we obtain*

$$-a_0 = \lambda_j \left[a_j^2x^{2j} + (-a_0 - a_jx^j)^2\right]. \tag{16}$$

*From the relation (16) we obtain*

$$x^j = -\frac{a_0}{2a_j} \pm \frac{1}{2a_j}\sqrt{-a_0^2 - 2\frac{a_0}{\lambda_j}}. \tag{17}$$

*Assuming that $a_0 > 0$, we have that $\lambda_j < 0$, therefore from the relation (17) we obtain*

$$-\frac{2}{a_0} \leq \lambda_j < 0. \tag{18}$$

*Let's define*

$$c_j^2 := -a_0^2 - \frac{2a_0}{\lambda_j}. \tag{19}$$

*From the relationship (19) completing squares we have*

$$(\lambda_jc_j)^2 + (a_0\lambda_j + 1)^2 = 1. \tag{20}$$

*Parameterizing the relationship (20) we have*

$$\lambda_jc_j = \frac{1-t_j^2}{1+t_j^2}, \lambda_ja_0 + 1 = \frac{2t_j}{1+t_j^2}, t_j \in [-1, 1]. \tag{21}$$

*From the relation (21) we obtain*

$$c_j = -\frac{1+t_j}{1-t_j}a_0, t_j \in [-1; 1\rangle. \tag{22}$$

*From the relation (22) we obtain in (17) the following*

$$x^j = \tfrac{a_0}{a_j} \times \tfrac{t_j}{1-t_j}, t_j \in [-1; 1\rangle, j \in \{1, \ldots, n\}, \tag{23}$$

*or*

$$x^j = -\tfrac{a_0}{a_j} \times \tfrac{1}{1-t_j}, t_j \in [-1; 1\rangle, j \in \{1, \ldots, n\}. \tag{24}$$

*From the relation (18) we have that if $\lambda_j \in \langle -\infty, \frac{-2}{a_0}\rangle \cup \langle 0, +\infty\rangle$, then $x^j$ is a complex number.*

*In the relation (17) let's set*

$$d_j^2 := a_0^2 + 2\frac{a_0}{\lambda j}. \tag{25}$$

*From the relation (25) we obtain the following*

$$\frac{d_j\lambda_j}{a_0\lambda_j+1} = \frac{1-t_j^2}{1+t_j^2}, \frac{1}{a_0\lambda_j+1} = \frac{2t_j}{1+t_j^2}. \tag{26}$$

*Using the relation (26) we obtain*

$$d_j = \frac{a_0(1+t_j)}{1-t_j}. \tag{27}$$

*Using the relation (27) in (17) we obtain*

$$x^j = \frac{a_0}{2a_j}\left(-1+\left(\frac{1+t_j}{1-t_j}\right)i\right), t_j \in [-1,1\rangle, a_0 > 0, j \in \{1,\ldots,n\}, \tag{28}$$

*or*

$$x^j = \frac{a_0}{2a_j}\left(-1-\left(\frac{1+t_j}{1-t_j}\right)i\right), t_j \in [-1,1\rangle, a_0 > 0, j \in \{1,\ldots,n\}. \tag{29}$$

**Remark 3.2.** *Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where $a_0 < 0$. Let*

$$A := x^n + a_{n-1}x^{n-1} + \cdots + a_{j+1}x^{j+1} + a_{j-1}x^{j-1} + \cdots + a_2x^2 + a_1x. \tag{30}$$

$$B := a_jx^j. \tag{31}$$

*Using the relationship*

$$A + B = \lambda_j[A^2 + B^2]. \tag{32}$$

*Working in the same way as what was done in the Observation (3.1) we obtain the following*

$$0 < \lambda_j \leq -\frac{2}{a_0}, \tag{33}$$

*and also*

$$x^j = \frac{a_0}{a_j}\left(\frac{t_j}{1-t_j}\right), t_j \in [-1,1\rangle, a_0 < 0, j \in \{1,\ldots,n\}, \tag{34}$$

*or*

$$x^j = \frac{a_0}{a_j}\left(-\frac{1}{1-t_j}\right), t_j \in [-1,1\rangle, a_0 < 0, j \in \{1,\ldots,n\}. \tag{35}$$

*For $\lambda_j \in \langle -\infty, 0\rangle \cup \langle -\frac{2}{a_0}, +\infty\rangle$ we obtain complex $x^j$, which has the next way*

$$x^j = \frac{a_0}{2a_j}\left(-1+\left(\frac{1+t_j}{1-t_j}\right)i\right), t_j \in [-1,1\rangle, a_0 < 0, j \in \{1,\ldots,n\}, \tag{36}$$

*or*

$$x^j = \frac{a_0}{2a_j}\left(-1 - \left(\frac{1+t_j}{1-t_j}\right)i\right), t_j \in [-1,1\rangle, a_0 < 0, j \in \{1,\ldots,n\}. \tag{37}$$

**Example 3.2.** *Find the complex and real roots of the polynomial* $P(x) = x^5 + x + 1$.

**Solution 3.2.** *As*

$$x^j = -\frac{a_0}{a_j}\left(\frac{1}{1-t_j}\right) \quad or \quad x^j = \frac{a_0}{a_j}\left(\frac{t_j}{1-t_j}\right), \tag{38}$$

*where* $t_j \in [-1,0\rangle \cup \langle 0,1\rangle$*. In the relation (38) for* $j=1$ *we have,* $a_0 = 1, a_1 = 1$*, then we obtain*

$$x = -\left(\frac{1}{1-t_j}\right) \quad or \quad x = \frac{t_j}{1-t_j}. \tag{39}$$

*For* $t_1 = \frac{1}{2}$ *we obtain in (39)*

$$x = -2 \quad or \quad x = 1. \tag{40}$$

*For* $t_1 = -\frac{1}{2}$ *we obtain in (39)*

$$x = -\frac{2}{3} \quad or \quad x = \frac{1}{3}. \tag{41}$$

*From the relationship (40) and (41) we obtain* $x = \frac{-2}{3}$ *is a reasonable value.*

*For* $t_1 = -\frac{1}{4}$ *we have*

$$x = -\left(\frac{1}{1+\frac{1}{4}}\right) = -\frac{4}{5}. \tag{42}$$

*For* $t_1 = -\frac{3}{4}$ *we have*

$$x = -\frac{4}{7}. \tag{43}$$

*An approximate root is given for* $x = \frac{-\frac{2}{3}-\frac{4}{5}}{2} = -\frac{11}{15}$*. Only the midpoints of the intervals* $\langle -1, -\frac{1}{2}\rangle$ *have been taken.*

*For complex roots we have*

$$x^j = -\frac{a_0}{2a_j}\left(1 + \frac{(1+t_j)}{(1-t_j)}i\right), t_j \in [-1,0\rangle \cup \langle 0,1\rangle, \tag{44}$$

or

$$x^j = -\frac{a_0}{2a_j}\left(1 - \frac{(1+t_j)}{(1-t_j)}i\right), t_j \in [-1,0\rangle \cup \langle 0,1\rangle. \tag{45}$$

*For* $j=1, a_0=1, a_1=1$ *and* $t_1=\frac{1}{2}$ *we have*

$$x=-\frac{1}{2}(1+3i) \quad \text{or} \quad x=-\frac{1}{2}(1-3i). \tag{46}$$

*For* $t_1=\frac{1}{4}$ *we have*

$$x=-\frac{1}{2}+\frac{5}{6}i \quad \text{or} \quad x=-\frac{1}{2}-\frac{5}{6}i. \tag{47}$$

*For* $t_1=-\frac{1}{2}$ *we have*

$$x=-\frac{1}{2}+\frac{1}{6}i \quad \text{or} \quad x=-\frac{1}{2}-\frac{1}{6}i. \tag{48}$$

*For* $t_1=-\frac{1}{4}$ *we have*

$$x=-\frac{1}{2}+\frac{3}{10}i \quad \text{or} \quad x=-\frac{1}{2}-\frac{3}{10}i. \tag{49}$$

*From the relations (46), (47), (48) , (49) when evaluating the polynomial with these values, it is observed that the best approximate root is given by relation (47).*

*For the last two complex roots of the given polynomial, we have to*

*i) For* $j=5, a_5=1, a_0=1$ *and* $t_5=0$ *in the relation (44) we have*

$$x^5=-\frac{1}{2}\left(1+\left(\frac{1+0}{1-0}\right)i\right), t_5 \in [-1,0\rangle \cup \langle 0,1].$$

*Using the formula*

$$\sqrt[n]{z}=\sqrt[n]{|z|}\left[\cos\left(\frac{\theta+2\pi k}{n}\right)+isen\left(\frac{\theta+2\pi k}{n}\right)\right], k \in \{0,1,\ldots,n-1\}.$$

*For* $k=0$*, the first fifth root would be* $x_0=0.8598326415+0.1361841118i$*. So on, for* $k=3$*,* $x_3=-0.6155722065-0.6155722065i$*, this value is a good approximation for the complex root.*

*ii) The approximation given in item i) can be improved by varying the parameter* $t_5$*. For example, taking midpoints of the intervals* $[-1,0\rangle$ *or* $\langle 0,1]$*.*

## 4. About Number Theory

In this work we propose a new method to solve the Diophantine equation

$$Ax+By+Cz=D. \tag{50}$$

In (50) we will assume that $D>0$ and $D>A, D>B, D>C$ and $A, B, C$ and $D$ without common factors. The technique consists of establishing a link between equation (50) and the relationship:

$$\sum_{k=1}^{n} a_k = \lambda \sum_{k=1}^{n} a_k^2. \tag{51}$$

Here $a_k$ are any real numbers and $n$ is a natural number greater than or equal to 2 and $\lambda$ is unique. In this case we use $n = 2$ and,

$$a_1 + a_2 = \lambda_1 \left(a_1^2 + a_2^2\right). \tag{52}$$

Let $a_1 = Ax + By; a_2 = Cz$, from the relation (52) and (50) we obtain

$$D = \lambda_1 \left[(Ax + By)^2 + C^2 z^2\right] = \lambda_1 \left[(D - Cz)^2 + C^2 z^2\right]. \tag{53}$$

From the relation (53) we obtain the following:

$$z = \frac{D}{2C} \pm \frac{1}{2C}\sqrt{-D^2 + \frac{2D}{\lambda_1}}, \text{where } 0 < \lambda_1 \leq \frac{2}{D}. \tag{54}$$

Let

$$-D^2 + \frac{2D}{\lambda_1} = T_1^2. \tag{55}$$

From the relation (55) the following is obtained

$$(\lambda_1 T_1)^2 + (D\lambda_1 - 1)^2 = 1. \tag{56}$$

The relation (56) is conveniently written as:

**(A)** When

$$D\lambda_1 - 1 = \frac{1 - t_1^2}{1 + t_1^2} \text{ and } \lambda_1 T_1 = \frac{2t_1}{1 + t_1^2}, \tag{57}$$

where $|t_1| \leq 1$.

Let $t_1 = \frac{m_1}{n_1}$ where $m_1$ and $n_1$ are prime numbers relative to each other. From the relation (57) we obtain

$$\lambda_1 = \frac{2n_1^2}{D\left(m_1^2 + n_1^2\right)}, T_1 = \frac{m_1 D}{n_1}. \tag{58}$$

From the relation (58) and (54) we obtain

$$z = \frac{D}{2n_1 C}(n_1 + m_1) \text{ or } z = \frac{D}{2n_1 C}(n_1 - m_1). \tag{59}$$

For $z = \frac{D}{2n_1 C}(n_1 + m_1)$ and the equation (50) we obtain

$$Ax + By = \frac{D}{2n_1}(2n_1 - n_1 - m_1) = \widetilde{D}. \tag{60}$$

Again we apply the relationship (51) by connecting it with (60).

Let $a = Ax; By = b$ we get

$$\widetilde{D} = \lambda_2 \left[ (Ax)^2 + (By)^2 \right] = \lambda_2 \left[ (Ax)^2 + (\widetilde{D} - Ax)^2 \right] \tag{61}$$

From the relation (61) we obtain

$$x = \frac{\widetilde{D}}{2A} \pm \frac{T_2}{2A}, \tag{62}$$

where

$$-\widetilde{D}^2 + \frac{2\widetilde{D}}{\lambda_2} = T_2^2. \tag{63}$$

From the relation (63) we obtain

$$(\lambda_2 T_2)^2 + \left( \widetilde{D}\lambda_2 - 1 \right)^2 = 1. \tag{64}$$

From the relation (64) two possibilities arise

$(A_1)$

$$\widetilde{D}\lambda_2 - 1 = \frac{1 - t_2^2}{1 + t_2^2} \quad and \quad \lambda_2 T_2 = \frac{2t_2}{1 + t_2^2} \tag{65}$$

or

$(A_2)$

$$\widetilde{D}\lambda_2 - 1 = \frac{2t_2}{1 + t_2^2} \quad and \quad \lambda_2 T_2 = \frac{1 - t_2^2}{1 + t_2^2}, \tag{66}$$

where $|t_2| \leq 1$ and $t_2 = \frac{m_2}{n_2}$ with $m_2$ and $n_2$ numbers relative cousins to each other.

From the relation $(A_1)$ we obtain

$$\lambda_2 = \frac{2n_2^2}{\widetilde{D}\left( m_2^2 + n_2^2 \right)}, T_2 = \frac{m_2}{n_2}\widetilde{D} \tag{67}$$

From the relations (60), (62) and (67) we obtain

$$x = \frac{\widetilde{D}}{2A}\left( 1 + \frac{m_2}{n_2} \right) = \frac{D}{2n_1}(n_1 - m_1) \times \frac{1}{2An_2}(n_2 + m_2) \tag{68}$$

or

$$x=\frac{D}{2n_1}(n_1-m_1)\times\frac{1}{2An_2}(n_2-m_2) \tag{69}$$

From the relation $A_2$) we obtain

$$\lambda_2=\frac{(m_2+n_2)^2}{\tilde{D}\left(n_2^2+m_2^2\right)},T_2=\left(\frac{n_2-m_2}{n_2+m_2}\right)\tilde{D},n_2\neq-m_2. \tag{70}$$

From the relations (60), (62) and (70) we obtain

$$x=\frac{\tilde{D}}{2A}\left(1+\frac{n_2-m_2}{n_2+m_2}\right)=\frac{D}{2n_1}(n_1-m_1)\times\frac{1}{2A\left(n_2+m_2\right)}(2n_2) \tag{71}$$

or

$$x=\frac{\tilde{D}}{2A}\left(1-\frac{n_2-m_2}{n_2+m_2}\right)=\frac{D}{2n_1}(n_1-m_1)\times\frac{2m_2}{2A\left(n_2+m_2\right)}(2n_2). \tag{72}$$

From case A), for $z=\frac{D}{2n_1C}(n_1-m_1)$ we obtain in the equation (50)

$$Ax+By=\frac{D}{2n_1}(n_1+m_1)=\tilde{\tilde{D}}. \tag{73}$$

Again we apply the relationship (51) by linking it with the relationship (73) we obtain.

Let $a=Ax,b=By$ we obtain

$$\tilde{\tilde{D}}=\tilde{\lambda}_2\left[(Ax)^2+(By)^2\right]=\tilde{\lambda}_2\left[(Ax)^2+(\tilde{\tilde{D}}-Ax)^2\right]. \tag{74}$$

From the relation (74) we obtain

$$x=\frac{\tilde{\tilde{D}}}{2A}\pm\frac{\tilde{T}_2}{2A}, \tag{75}$$

where

$$-\tilde{\tilde{D}}^2+\frac{2\tilde{\tilde{D}}}{\tilde{\lambda}_2}=\tilde{T}_2^2. \tag{76}$$

Repeating the process carried out from the relation (64) to the relation (72) we obtain

$$\tilde{T}_2=\frac{\tilde{m}_2}{\tilde{n}_2}\tilde{\tilde{D}}, \tag{77}$$

where $\tilde{m}_2$ and $\tilde{n}_2$ are relative primes.

From the relations (75) and (77) obtain the following solution

$$x=\frac{\widetilde{D}}{2A}\left(1+\frac{\widetilde{m}_2}{\widetilde{n}_2}\right)=\frac{D}{4An_1\widetilde{n}_2}\left(n_1+m_1\right)\left(\widetilde{n}_2+\widetilde{m}_2\right), \tag{78}$$

or

$$x=\frac{\widetilde{D}}{2A}\left(1-\frac{\widetilde{m}_2}{\widetilde{n}_2}\right)=\frac{D}{4An_1\widetilde{n}_2}\left(n_1+m_1\right)\left(\widetilde{n}_2-\widetilde{m}_2\right). \tag{79}$$

For the relation similar to the relation (77) we have

$$\widetilde{T}_2=\frac{\left(\widetilde{n}_2-\widetilde{m}_2\right)}{\left(\widetilde{n}_2+\widetilde{m}_2\right)}\tilde{\tilde{D}},\widetilde{n}_2\neq\widetilde{m}_2. \tag{80}$$

Using the relationship (80) and (75) we obtain the following solution

$$x=\frac{\tilde{\tilde{D}}}{2A}\left(1+\frac{\widetilde{n}_2-\widetilde{m}_2}{\widetilde{n}_2+\widetilde{m}_2}\right)=\frac{D\left(n_1+m_1\right)}{4An_1}\times\frac{2\widetilde{n}_2}{\left(\widetilde{n}_2+\widetilde{m}_2\right)}, \tag{81}$$

or

$$x=\frac{\tilde{\tilde{D}}}{2A}\left(1-\frac{\widetilde{n}_2-\widetilde{m}_2}{\widetilde{n}_2+\widetilde{m}_2}\right)=\frac{D\left(n_1+m_1\right)}{4An_1}\times\frac{2\widetilde{m}_2}{\left(\widetilde{n}_2+\widetilde{m}_2\right)}. \tag{82}$$

**(B)** For this case we have the following situations

$$D\lambda_1-1=\frac{2t_1}{1+t_1^2},\lambda_1T_1=\frac{2t_1}{1+t_1^2}. \tag{83}$$

From the relation (83) we obtain

$$T_1=\left(\frac{n_1-m_1}{n_1+m_1}\right)D. \tag{84}$$

From the relation (84) y (54) we obtain

$$z=\frac{D}{2C}\left(1+\frac{n_1-m_1}{n_1+m_1}\right)=\frac{D}{2C}\times\frac{2n_1}{\left(n_1+m_1\right)}, \tag{85}$$

or

$$z=\frac{D}{2C}\left(1-\frac{\left(n_1-m_1\right)}{n_1+m_1}\right)=\frac{D}{2C}\times\frac{2m_1}{\left(n_1+m_1\right)}. \tag{86}$$

After replacing (85) in (50) or replacing (86) in (50), the same steps are performed, from the relation (60) until the relation (82).

From the relation (85) and the relation (50) we have

$$Ax+By=D-\frac{D}{2}\times\frac{2n_1}{n_1+m_1}=D\left(1-\frac{n_1}{n_1+m_1}\right)=\frac{D\left(m_1\right)}{n_1+m_1}=\tilde{\tilde{\tilde{D}}}. \tag{87}$$

After the relation (87) the solutions are obtained

$$x = \frac{\tilde{\tilde{\tilde{D}}}}{2A}\left(1 + \frac{\tilde{\tilde{m}}_2}{\tilde{\tilde{n}}_2}\right) = \frac{Dm_1}{2A\left(n_1 + m_1\right)} \times \frac{\tilde{\tilde{n}}_2 + \tilde{\tilde{m}}_2}{\tilde{\tilde{n}}_2},$$

or

$$x = \frac{Dm_1}{2A\left(n_1 + m_1\right)} \times \frac{\tilde{\tilde{n}}_2 - \tilde{\tilde{m}}_2}{\tilde{\tilde{n}}_2}.$$

Similar for the other case see (70)

$$x = \frac{Dm_1}{2\left(n_1 + m_1\right)A}\left(1 + \frac{\tilde{\tilde{n}}_2 - \tilde{\tilde{m}}_2}{\tilde{\tilde{n}}_2 + \tilde{\tilde{m}}_2}\right) = \frac{Dm_1}{2A\left(n_1 + m_1\right)} \times \frac{2\tilde{\tilde{n}}_2}{\tilde{\tilde{n}}_2 + \tilde{\tilde{m}}_2},$$

or

$$x = \frac{Dm_1}{2\left(n_1 + m_1\right)A}\left(1 + \frac{\tilde{\tilde{n}}_2 - \tilde{\tilde{m}}_2}{\tilde{\tilde{n}}_2 + \tilde{\tilde{m}}_2}\right) = \frac{D\left(m_1\right)}{2A\left(n_1 + m_1\right)} \times \frac{2\tilde{\tilde{m}}_2}{\tilde{\tilde{n}}_2 + \tilde{\tilde{m}}_2}.$$

From the relation (86) and the relation (50) the following is obtained:

$$Ax + By = D - \frac{D}{2} \times \frac{2n_1}{\left(n_1 + m_1\right)} = D\left(1 - \frac{n_1}{n_1 + m_1}\right) = \frac{Dn_1}{m_1 + n_1} = D^*. \qquad (88)$$

Then we have the following solutions

$$x = \frac{Dn_1}{2A\left(n_1 + m_1\right)} \times \left(\frac{\tilde{\tilde{\tilde{n}}}_2 + \tilde{\tilde{\tilde{m}}}_2}{\tilde{\tilde{n}}_2}\right),$$

or

$$x = \frac{Dn_1}{2A\left(n_1 + m_1\right)} \times \left(\frac{\tilde{\tilde{\tilde{n}}}_2 - \tilde{\tilde{\tilde{m}}}_2}{\tilde{\tilde{\tilde{n}}}_2}\right).$$

Similar for the other case, see (70)

$$x = \frac{Dn_1}{2A\left(n_1 + m_1\right)} \times \frac{2\tilde{\tilde{n}}_2}{\tilde{\tilde{\tilde{n}}}_2 + \tilde{\tilde{\tilde{m}}}_2},$$

or

$$x = \frac{Dn_1}{2A\left(n_1 + m_1\right)} \times \frac{2\tilde{\tilde{\tilde{\tilde{m}}}}_2}{\tilde{\tilde{\tilde{\tilde{n}}}}_2 + \tilde{\tilde{\tilde{\tilde{m}}}}_2}.$$

So we have the following solutions

$S_1$)

$$z = \frac{D\left(n_1 + m_1\right)}{2n_1 C}; x = \frac{D\left(n_1 - m_1\right)}{4An_1 n_2}\left(n_2 + m_2\right); y = \frac{D}{4Bn_1 n_2}\left(n_2 - m_2\right)\left(n_1 - m_1\right).$$

$S_2$)

$$z = \frac{D\left(n_1 + m_1\right)}{2n_1 C}; x = \frac{D}{4An_1 n_2}\left(n_1 - m_1\right)\left(n_2 - m_2\right); y = \frac{Dm_2}{4Bn_1 n_2}\left(n_2 + m_2\right)\left(n_1 - m_1\right).$$

$S_3$)

$$z = \frac{D\left(n_1 + m_1\right)}{2n_1 C}; x = \frac{D\left(n_1 - m_1\right) n_2}{2n_1 A\left(n_2 - m_2\right)}; y = \frac{Dm_2\left(n_1 - m_1\right)}{2Bn_1\left(n_2 + m_2\right)}.$$

$S_4$)

$$z = \frac{D\left(n_1 + m_1\right)}{2n_1 C}; x = \frac{D\left(n_1 - m_1\right) m_2}{2n_1 A\left(n_2 - m_2\right)}; y = \frac{Dn_2\left(n_1 - m_1\right)}{2Bn_1\left(n_2 + m_2\right)}.$$

$S_5$)

$$z = \frac{D\left(n_1 - m_1\right)}{2n_1 C}; x = \frac{D\left(n_1 + m_1\right)\left(\tilde{n}_2 + \tilde{m}_2\right)}{4An_1\tilde{n}_2}; y = \frac{D}{4Bn_1\tilde{n}_2}\left[\left(n_1 + m_1\right)\left(\tilde{n}_2 - \tilde{m}_2\right)\right].$$

$S_6$)

$$z = \frac{D\left(n_1 - m_1\right)}{2n_1 C}; x = \frac{D\left(n_1 + m_1\right)\left(\tilde{n}_2 - \tilde{m}_2\right)}{4An_1\tilde{n}_2}; y = \frac{D}{4Bn_1\tilde{n}_2}\left[\left(n_1 + m_1\right)\left(\tilde{n}_2 + \tilde{m}_2\right)\right].$$

$S_7$)

$$z = \frac{D\left(n_1 - m_1\right)}{2n_1 C}; x = \frac{D\left(n_1 + m_1\right)}{2An_1\left(\tilde{n}_2 + \tilde{m}_2\right)}\tilde{n}_2; y = \frac{D\left(n_1 + m_1\right)\tilde{m}_2}{2An_1\left(\tilde{n}_2 + \tilde{m}_2\right)}.$$

$S_8$)

$$z = \frac{D\left(n_1 - m_1\right)}{2n_1 C}; x = \frac{D\left(n_1 + m_1\right)\tilde{m}_2}{2An_1\left(\tilde{n}_2 + \tilde{m}_2\right)}; y = \frac{D\left(n_1 + m_1\right)\tilde{n}_2}{2An_1\left(\tilde{n}_2 + \tilde{m}_2\right)}.$$

$S_9$)

$$z=\frac{Dn_1}{C\left(n_1+m_1\right)};x=\frac{Dm_1}{2A\left(n_1+m_1\right)}\frac{\left(\tilde{\tilde{n}}_2+\tilde{\tilde{m}}_2\right)}{\tilde{\tilde{n}}_2};y=\frac{Dm_1\left(\tilde{\tilde{n}}_2-\tilde{\tilde{m}}_2\right)}{2B\left(n_1+m_1\right)\tilde{\tilde{n}}_2}.$$

$S_{10}$)

$$z=\frac{Dn_1}{C\left(n_1+m_1\right)};x=\frac{Dm_1}{2A\left(n_1+m_1\right)}\frac{\left(\tilde{\tilde{n}}_2-\tilde{\tilde{m}}_2\right)}{\tilde{\tilde{n}}_2};y=\frac{Dm_1\left(\tilde{\tilde{n}}_2+\tilde{\tilde{m}}_2\right)}{2B\left(n_1+m_1\right)\tilde{\tilde{n}}_2}.$$

$S_{11}$)

$$z=\frac{Dn_1}{C\left(n_1+m_1\right)};x=\frac{Dm_1}{A\left(n_1+m_1\right)}\frac{\tilde{\tilde{n}}_2}{\left(\tilde{\tilde{n}}_2+\tilde{\tilde{m}}_2\right)};y=\frac{Dm_1\tilde{\tilde{m}}_2}{B\left(n_1+m_1\right)\left(\tilde{\tilde{n}}_2+\tilde{\tilde{m}}_2\right)}.$$

$S_{12}$)

$$z=\frac{Dn_1}{C\left(n_1+m_1\right)};x=\frac{Dm_1}{A\left(n_1+m_1\right)}\frac{\tilde{\tilde{n}}_2}{\left(\tilde{\tilde{n}}_2+\tilde{\tilde{m}}_2\right)};y=\frac{Dm_1\tilde{\tilde{n}}_2}{B\left(n_1+m_1\right)\left(\tilde{\tilde{n}}_2+\tilde{\tilde{m}}_2\right)}.$$

$S_{13}$)

$$z=\frac{Dm_1}{C\left(n_1+m_1\right)};x=\frac{Dn_1}{2A\left(n_1+m_1\right)}\frac{\left(\tilde{\tilde{\tilde{n}}}_2+\tilde{\tilde{\tilde{m}}}_2\right)}{\tilde{\tilde{\tilde{n}}}_2};y=\frac{Dn_1\left(\tilde{\tilde{\tilde{n}}}_2-\tilde{\tilde{\tilde{m}}}_2\right)}{2B\left(n_1+m_1\right)\tilde{\tilde{\tilde{n}}}_2}.$$

$S_{14}$)

$$z=\frac{Dm_1}{C\left(n_1+m_1\right)};x=\frac{Dn_1}{2A\left(n_1+m_1\right)}\frac{\left(\tilde{\tilde{\tilde{n}}}_2-\tilde{\tilde{\tilde{m}}}_2\right)}{\tilde{\tilde{\tilde{n}}}_2};y=\frac{Dn_1\left(\tilde{\tilde{\tilde{n}}}_2+\tilde{\tilde{\tilde{m}}}_2\right)}{2B\left(n_1+m_1\right)\tilde{\tilde{\tilde{n}}}_2}.$$

$S_{15}$)

$$z=\frac{Dm_1}{C\left(n_1+m_1\right)};x=\frac{Dn_1}{A\left(n_1+m_1\right)}\frac{\tilde{\tilde{\tilde{n}}}_2}{\left(\tilde{\tilde{\tilde{n}}}_2+\tilde{\tilde{\tilde{m}}}_2\right)};y=\frac{Dn_1\tilde{\tilde{\tilde{m}}}_2}{B\left(n_1+m_1\right)\left(\tilde{\tilde{\tilde{n}}}_2+\tilde{\tilde{\tilde{m}}}_2\right)}.$$

$S_{16}$)


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$$z=\frac{Dm_1}{C\left(n_1+m_1\right)};x=\frac{Dn_1}{A\left(n_1+m_1\right)}\frac{\tilde{\tilde{\tilde{\tilde{m}}}}_2}{\left(\tilde{\tilde{\tilde{n}}}_2+\tilde{\tilde{\tilde{m}}}_2\right)};y=\frac{Dn_1\tilde{\tilde{\tilde{\tilde{n}}}}_2}{B\left(n_1+m_1\right)\left(\tilde{\tilde{\tilde{n}}}_2+\tilde{\tilde{\tilde{m}}}_2\right)}.$$

Where the pairs of numbers $\left(n_1,m_1\right),\left(n_2,m_2\right),\left(\tilde{n}_2,\tilde{m}_2\right),\left(\tilde{\tilde{n}}_2,\tilde{\tilde{m}}_2\right),\left(\tilde{\tilde{\tilde{n}}}_2,\tilde{\tilde{\tilde{m}}}_2\right)$ are relative cousins to each other.

# 5. About Fermat's Last Theorem

**Theorem 5.1.** The equation $x^n+y^n=z^n$ has no positive integer solution for $n\geq 3, n\in\mathbb{N}$.

*Proof.* Suppose there are positive integer solutions $x,y,z$ where $\mathrm{GCD}(x,y)=1$, $\mathrm{GCD}(x,z)=1$, $\mathrm{GCD}(y,z)=1$ satisfying

$$x^n+y^n=z^n. \tag{89}$$

Let

$$A:=x^n, B:=y^n, C:=z^n. \tag{90}$$

Let's use the relationship

$$A+B+C=\lambda[A^2+B^2+C^2]. \tag{91}$$

From the relations (89), (90) and (91) we obtain

$$2z^n=\lambda[x^{2n}+y^{2n}+z^{2n}]. \tag{92}$$

From the relation (92) we have

$$x^n=\frac{1}{\lambda}c_1, y^n=\frac{1}{\lambda}c_2, z^n-\frac{1}{\lambda}=\frac{1}{\lambda}c_3, \tag{93}$$

where $c_1^2+c_2^2+c_3^2=1, \lambda\in\mathbb{Q}^+$ and $c_1, c_2$ y $c_3\in\mathbb{Q}$. Using the relation (93) and (89) we obtain

$$c_1+c_2=c_3+1. \tag{94}$$

Using the relation $c_1^2+c_2^2+c_3^2=1$ and the relation given in (94) we obtain

$$c_1^2+c_2^2+c_1c_2-c_1-c_2=0. \tag{95}$$

Using the relationship (89), (93) and (95) we obtain

$$z^n\left(z^n-\frac{1}{\lambda}\right)=x^ny^n. \tag{96}$$

Since $\lambda$ is positive rational, then $\lambda:=\frac{M}{N}$, where


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$$\mathrm{GCD}(M,N)=1. \tag{97}$$

From (97) and (96) we have

$$z^n\left(z^n-\frac{N}{M}\right)=x^n y^n. \tag{98}$$

From the relation (98) we have the first possibility

$$z=Mr, r\in\mathbb{N}. \tag{99}$$

Using (99) and (98) we have $r=1$. Then we get

$$M^{2n}-N.M^{n-1}=x^n y^n. \tag{100}$$

From the relation (100) we obtain that this is absurd because for $z=M$, the right hand side is not divisible by $z$.

The second case in the relation (98) is when $M=z^n$, therefore from here we obtain

$$M^2-N=x^n y^n. \tag{101}$$

Since $x^n=\frac{c_1}{\lambda}$ and $y^n=\frac{c_2}{\lambda}$, where $\lambda=\frac{M}{N}$, and as $c_1,c_2\in\mathbb{Q}^+$, $0<c_1\leq 1,\ 0<c_2\leq 1$, we can express $c_1$ and $c_2$ como $c_1=\frac{m_1}{n_1}$ and $c_2=\frac{m_2}{n_2}$, respectively, where $\mathrm{GCD}(m_1,n_1)=1$ y $\mathrm{GCD}(m_2,n_2)=1$. Therefore

$$x^n=\frac{m_1 N}{n_1 M}, y^n=\frac{m_2 N}{n_2 M}. \tag{102}$$

From the equation (102) it follows that, given that $\mathrm{GCD}(x,y)=1$, it is true that

$$N=n_1 n_2,\ \ m_1=\tilde{r}_1 M,\ \ m_2=\tilde{r}_2 M, \tag{103}$$

where $\mathrm{GCD}(\tilde{r}_1,\tilde{r}_2)=1$.

Therefore, substituting (103) into (102) we obtain

$$x^n=\tilde{r}_1 n_2 \text{ and } y^n=\tilde{r}_2 n_1. \tag{104}$$

Using the relation (104) and (103) in (101) we obtain

$$M^2-n_1 n_2=\tilde{r}_1\tilde{r}_2 n_1 n_2. \tag{105}$$

From the relation (105) the only possibility is that

$$n_1 n_2=1. \tag{106}$$

From (102) we obtain $n_1=n_2=1$, which implies that $c_1=1$, $c_2=1$ and the equation $c_1^2+c_2^2+c_3^2=1$ would be absurd.

From the relation (102) when $M=z^n$ and $N=n_1=n_2$. Taking the relation

$$x^n + y^n + z^n = \lambda\left[x^{2n} + y^{2x} + z^{2n}\right] \tag{107}$$

together with (89), we obtain the following

$$2x^n + 2y^n = \lambda\left[x^{2n} + y^{2x} + z^{2n}\right] \tag{108}$$

In (108), we take the parametrization

$$\begin{cases} x^n - \frac{1}{\lambda} = \frac{\sqrt{2}}{\lambda} d_1 \\ y^n - \frac{1}{\lambda} = \frac{\sqrt{2}}{\lambda} d_2 \\ z^n = \frac{\sqrt{2}}{\lambda} d_3 \end{cases} \tag{109}$$

where $d_1, d_2, d_3$ belong to the set of irrational numbers, and according to (108)

$$d_1^2 + d_2^2 + d_3^2 = 1 \tag{110}$$

Let

$$d_1 = \sqrt{2}e_1, \quad d_2 = \sqrt{2}e_2, \quad d_3 = \sqrt{2}e_3 \tag{111}$$

where $e_1, e_2, e_3 \in \mathbb{Q}$. Therefore, from (110) and (112) we obtain

$$e_1^2 + e_2^2 + e_3^2 = \frac{1}{2} \tag{112}$$

Using (112), relation (109) is rewritten as

$$\begin{cases} x^n - \frac{1}{\lambda} = \frac{2e_1}{\lambda} \\ y^n - \frac{1}{\lambda} = \frac{2e_2}{\lambda} \\ z^n = \frac{2e_3}{\lambda} \end{cases} \tag{113}$$

If we replace (113) in (89), we get the relation

$$e_3 = e_1 + e_2 + 1 \tag{114}$$

If we do

$$e_1 = \frac{r_1}{t_1}, \quad e_2 = \frac{r_2}{t_2}, \quad e_3 = \frac{r_3}{t_3} \tag{115}$$

where $\mathrm{GCD}(r_1, t_1) = 1$, $\mathrm{GCD}(r_2, t_2) = 1$ y $\mathrm{GCD}(r_3, t_3) = 1$. Knowing that $\lambda = \frac{M}{N}$, from relations (113) and (115) we obtain

$$\begin{cases} x^n = \frac{(2r_1+t_1)N}{Mt_1} \\ \\ y^n = \frac{(2r_1+t_1)N}{Mt_2} \\ \\ z^n = \frac{2r_3N}{Mt_3} \end{cases} \tag{116}$$

From the relation (101), N is an odd number. Using relation (116), we have the following cases

A. When

$$N = t_1 = t_2 = t_3 \tag{117}$$

Using (117) and (115) in (112), we obtain

$$r_1^2 + r_2^2 + r_3^2 = \frac{N^2}{2} \tag{118}$$

This result is interesting, since $N$ being an odd number, (118) is absurd.

B. When

$$N = t_1 = t_2 = \frac{t_3}{2} \tag{119}$$

Using (119) and (115) in (112), we obtain

$$r_1^2 + r_2^2 + \frac{r_3^2}{4} = \frac{N^2}{2} \tag{120}$$

Furthermore, using relations (119) and (115) in (114), we obtain

$$r_3 = 2(r_1 + r_2 + N) \tag{121}$$

If we put (121) in (120) then we get the relation

$$r_1^2 + r_2^2 + (r_1 + r_2 + N)^2 = \frac{N^2}{2} \tag{122}$$

and since $N$ is an odd number, obviously, the relation (122) is absurd.

C. Another possibility is that $t_3 = 2$ together with (116) gives us $r_3N = M^2$ and by (101), this is impossible, since $N$ and $M$ would have common prime factors and these prime factors must divide $x^n$ or $y^n$, which is false.

## 6. Conclusions

It is interesting to observe that an appropriate connection between the object of study and the relation $A_1 + \cdots + A_n = \lambda\sum_{k=1}^{n} A_k^2$ (where $A_k \in \mathbb{R}$, $k = 1, n \in \mathbb{N}$, and $\lambda \in \mathbb{R}$ is unique) allows for the resolution of classical problems. Remarkably, through this mechanism, it is possible to obtain approximate zeros, both real and complex, of a real monic polynomial of degree $N$, general solutions to the linear Diophantine equation in three variables, and even a demonstration of Fermat's Last Theorem. We still do not fully understand why this method works; one possible explanation is that the problem is transformed into an $n$-dimensional sphere, which is symmetric in relation to a coordinate system, making it easier to approach.

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### Declarations

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