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# New perspectives on the roots of real polynomials of degree $n$ and number theory.

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**Abstract**

In this work we obtained some results on the real and complex roots of real polynomials of the form $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where the $a_k$ are real numbers for $k \in \{0, \ldots, n-1\}$, we also obtained results on linear Diophantine equations of the form $ax + by + cz = d$, where $a, b, c$ and $d$ are integers.

To obtain the desired results, the relation that states the following was used: for any real numbers $A$ and $B$, there exists a unique real number $\lambda$, such that $A + B = \lambda[A^2 + B^2]$. This result is appropriately linked to the object of study.

For the polynomials, optimal domains were obtained where the real or complex roots are found, without the use of higher calculus.

For the linear Diophantine equation, the desired solutions were obtained, which were found by establishing several links between the Diophantine equation and the relationship $A + B = \lambda[A^2 + B^2]$. Several examples of the results obtained are illustrated, which are intended to show the benefits of this proposal.

Furthermore, using the same technique used to solve the previous problems, we propose an alternative method to address Fermat's Last Theorem.

***Key Words***: *Roots of Polynomials, Diophantine equations, Functional Analysis, Fermat's Theorem.*

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# 1 Introduction

The roots of polynomials and Diophantine equations continue to be studied due to their wide applications in mathematics (numerical analysis), science, engineering, computer science, and various other areas. For example, in linear algebra they are used to determine the eigenvalues of operators and matrices. In control theory, the roots of a polynomial play an important role in determining the stability and behavior of a system.

In numerical analysis, finding the roots of a polynomial is a fundamental problem. Several algorithms have been developed to approximate these roots, such as Newton's method, the bisection method, Müler's method, Durand-Kerner method and others. These methods are essential in computational mathematics and scientific computing. Néwton's method may not converge if the initial estimate is too far from a root. Thus, some of these methods depend on the initial estimation interval with which these algorithms are applied. Of course, there are methods with global convergence such as Müler's method, however, almost all of these methods use differential calculus as a basis.

In this work, without the use of differential calculus, the approximate real or complex roots of polynomials of degree $n$ with real coefficients are achieved, using as a starting point the method presented in [3], [2] and [1]. Initial estimates of the roots of the polynomial are also given in a different way than traditional. This method consists of establishing a link between the study problem and the relationship that states: $\forall a_1, a_2, ..., a_n \in \mathbb{R}$ exists *lambda* satisfying the equality $a_1 + a_2 + \cdots + a_n = \lambda(a_1^2 + a_2^2 + \cdots + a_n^2)$.

Next, we present a general non-classical solution for the Diophantine equation of the form $Ax + By + Cz = D$. These types of equations are crucial due to their various applications in number theory, algorithms and computing, as well as in geometry and topology.

We conclude by offering a novel perspective on Fermat's Last Theorem.

# 2 Preliminaries

Let us consider the polynomial of degree $n$

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0.$$

By the fundamental theorem of algebra it is known that $p$ has $n$ real or complex roots, counting multiplicities. If the coefficients $a_0, a_1, \ldots, a_n$ are all real, then the complex roots appear in conjugate pairs, that is, in the form $c \pm di$, where $c, d \in \mathbb{R}$ and $i^2 = -1$. On the other hand, if the coefficients are complex, it is not necessary that the complex roots be related.

There are methods to determine roots of a polynomial. For example, among the analytical methods we have: the method of formulas (quadratic, cubic and quartic), the remainder theorem, the factor theorem and the rational root theorem. Among the numerical methods we have: Synthetic division by a simple factor, Synthetic division by a quadratic factor, Bairstow method, Laguerre method, Bernoulli method, Newton's method, Müler's method, Quotient difference algorithm, Graeffe square root method and Jenkins-Traub method.

On the other hand, computational methods (use of Mathematical Software) combine a variety of numerical and analytical methods to find the roots of polynomials efficiently and accurately. These methods are carefully implemented to handle the complexities of high-degree polynomials with complex coefficients. For example, the "Jenkins-Traub" method is an efficient and robust algorithm for finding all roots of a polynomial, both real and complex.

The following Theorem is one of the many results that give us how to take initial estimates (intervals) where the roots are located.

**Theorem 2.1.** *Given a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0, a_n \neq 0$, define the polynomials*

$$P(x) = |a_n| x^n + |a_{n-1}| x^{n-1} + \cdots + |a_1| x - |a_0|,$$
$$Q(x) = |a_n| x^n - |a_{n-1}| x^{n-1} - \cdots - |a_0|.$$

*According to Descartes' rules, $P(x)$ has exactly one positive real zero $r_1$ and $Q(x)$ has exactly one positive real zero $r_2$. Then all zeros $z$ of $p(x)$ are in the annular region*

$$r_1 \leqslant |z| \leqslant r_2$$

**Theorem 2.2.** *Let $a_1, \ldots, a_n$ be any real numbers, then there exists $\lambda \in \mathbb{R}$, such that*

$$\sum_{i=1}^{n} a_i = \lambda \sum_{i=1}^{n} a_i^2$$

*Proof.* Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x_1, ..., x_n) = \sum_{i=1}^{n} a_i x_i$, be defined by $f$ is a linear and continuous functional. Therefore

$$\mathbb{R}^n = \ker f \oplus (\ker f)^{\perp}. \tag{1}$$

Thus we have that:

$$\dim \mathbb{R}^n = \dim \operatorname{Im} f + \dim \ker f$$

$$n = 1 + \dim \ker f$$

Therefore $\dim \ker f = n - 1$
Thus, from (1) we have:

$$(1, 1, \ldots, 1) = \sum_{i=1}^{n-1} \lambda_i u_i + \lambda_n u_n \tag{2}$$

where $\{u_1, \ldots, u_n - 1\} \subset \ker f$ and $u_n \in (\ker f)^{\perp}$
From (2) and taking into account that $f$ is a linear functional, we have

$$f(1, \ldots, 1) = \lambda f(u_n)$$

$$\sum_{i=1}^{n} a_i = \lambda \sum_{i=1}^{n} a_i^2$$

$$\text{since } u_n = (a_1, \ldots, a_n) \in (\ker f)^{\perp}$$

Using this last relation, which must be linked with the function to be maximized and with the given restrictions. Below we show several problems that illustrate the given theory. □

# 3 About Polynomials

In this section we will find the roots of a polynomial of degree $n$ of the form

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, \tag{3}$$

with the condition that $a_1 \neq 0$ and $a_0 < 0$. Therefore the polynomial $P(x)$ can be written as $P(x) = Q(x) + a_1 x + a_0$, where

$$Q(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2. \tag{4}$$

Using the technique described in the article [2] and assuming that $x$ is a root we obtain

$$Q(x) + a_1 x = \lambda \left[ Q^2(x) + a_1^2 x^2 \right]. \tag{5}$$

From this last relationship we obtain

$$Q(x) - \frac{1}{2\lambda} = \frac{\sqrt{2} b_1}{2|\lambda|} \tag{6}$$

$$a_1 x - \frac{1}{2\lambda} = \frac{\sqrt{2} b_2}{2|\lambda|} \tag{7}$$

where $b_1^2 + b_2^2 = 1, b_1 = b_1(x), b_2 = b_2(x)$ y $\lambda = \lambda(x)$.
From the relations (5) and (6) we obtain

$$-2 \leqslant \lambda a_0 \leqslant 0. \tag{8}$$

**Theorem 3.1.** *Let* $P(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0$*, where* $a_0<0$ *y* $a_1>0$*, if* $p(x_0)=0$*. Then there exists* $\lambda>0$ *such that* $0<\lambda\leqslant -\frac{2}{a_0}$*,*

$$\begin{cases} -\dfrac{a_0}{2a_1}\leqslant x_0 \wedge \dfrac{1}{a_1x_0}<\lambda<-\dfrac{2}{a_0} \\ \vee \\ -\dfrac{a_0}{a_1}<x_0 \wedge 0<\lambda\leqslant -\dfrac{2}{a_0} \\ \vee \\ 0<x_0<-\dfrac{a_0}{2a_1} \wedge 0<\lambda<-\dfrac{2}{a_0} \end{cases}$$

*where* $x_0=-\frac{a_0}{2a_1}\pm\frac{1}{2a_1}\sqrt{-a_0^2-\frac{2a_0}{\lambda}}$.

*Proof.* From the relation (8) we obtain that

$$0<\lambda\leqslant -\frac{2}{a_0}. \tag{9}$$

Also from (5) we have

$$Q(x_0)\left[1-\lambda Q(x_0)\right]=a_1x_0\left[\lambda a_1x_0-1\right]. \tag{10}$$

From the relation (10) and being $Q(x_0)=-a_1x_0-1$ we obtain

$$(-a_1x_0-a_0)\left[1-\lambda(-a_1x_0-a_0)\right]=a_1x_0\left[\lambda a_1x_0-1\right]. \tag{11}$$

From this equation (11), the following cases are derived

i) Let $-a_1x_0-a_0>0 \wedge 1+\lambda(a_1x_0+a_0)>0$, from this we have

$$\begin{aligned} &a_1x_0>0 \wedge \lambda a_1x_0-1>0 \\ &\vee \\ &a_1x_0<0 \wedge \lambda a_1x_0-1<0. \end{aligned}$$

If $-a_1x_0-a_0\geqslant a_1x_0$, then

$$1+(a_1x_0+a_0)\lambda<\lambda a_1x_0-1,$$

which is absourd $-2>\lambda a_0,\ a_0<0$ y $\lambda>0$.

If $-a_1x_0-a_0\leqslant a_1x_0$, then

$$1+(a_1x_0+a_0)\lambda>\lambda a_1x_0-1.$$

The relationship $a_1x_0<0$ is false, since it is observed that $-a_1x_0-a_0<0$ which implies that $0<a_0$, which is false.

Therefore we have

$$-\frac{a_0}{2a_1}\leqslant x_0 \ \wedge\ \frac{1}{a_1x_0}<\lambda<-\frac{2}{a_0}$$

ii) Let $-a_1x_0-a_0<0$ and $1+\lambda(a_1x_0+a_0)<0$ which implies that

$$\begin{aligned} &a_1x_0>0 \ \wedge\ \lambda a_1x_0-1>0 \\ &\vee \\ &a_1x_0<0 \ \wedge\ \lambda a_1x_0-1<0. \end{aligned}$$

The relationship in item (ii) is impossible, since $a_1x_0+a_0>0$ and $\lambda>0$ imply $\lambda(a_1x_0+a_0)>0$ and since $1+\lambda(a_1x_0+a_0)<0$ we have that $1<0$, which is false.

iii) Let $-a_1x_0 - a_0 < 0 \ \wedge \ 1 + \lambda(a_1x_0 + a_0) > 0$, from this we have

$$\begin{aligned} &a_1x_0 < 0 \wedge \ \lambda a_1x_0 - 1 > 0 \\ &\vee \\ &a_1x_0 > 0 \wedge \ \lambda a_1x_0 - 1 < 0. \end{aligned}$$

Since $-a_1x_0 - a_0 < a_1x_0$ y $1 + \lambda(a_1x_0 + a_0) > \lambda a_1x_0 - 1$ we have

$$-\frac{a_0}{\hat{a}_1} < x_0 \wedge \ \lambda < \frac{1}{a_1x_0} \ \wedge \ \lambda \leqslant -\frac{2}{a_0},$$

that is,

$$-\frac{a_0}{a_1} < x_0 \wedge \ 0 < \lambda \leqslant -\frac{1}{a_0}.$$

iv) Let $-a_1x_0 - a_0 > 0 \wedge 1 + \lambda(a_1x_0 + a_0) < 0$, from this we have

$$\begin{aligned} &a_1x_0 < 0 \ \wedge \ \lambda a_1x_0 - 1 > 0 \\ &\vee \\ &a_1x_0 > 0 \ \wedge \ \lambda a_1x_0 - 1 < 0. \end{aligned}$$

If $-a_1x_0 - a_0 > a_1x_0$ y $1 + \lambda(a_1x_0 + a_0) < \lambda a_1x_0 - 1$, which is an impossible relationship.
If $-a_1x_0 - a_0 > a_1x_0$ y $1 + \lambda(a_1x_0 + a_0) > \lambda a_1x_0 - 1$, which implies that

$$0 < x_0 < -\frac{a_0}{2a_1} \ \wedge \ 0 < \lambda < -\frac{2}{a_0}.$$

In addition to the relation (10) we obtain

$$x_0 = -\frac{a_0}{2a_1} \pm \frac{1}{2a_1}\sqrt{-a_0^2 - \frac{2a_0}{\lambda}}, \ \ 0 < \lambda \leqslant -\frac{2}{a_0}.$$

□

**Example 3.1.** *Find the roots of the polynomial of degree five*

$$P(x) = x^5 + x^4 + x^3 + x^2 + x - 2$$

**Solution 3.1.** *We have that $a_0 = -2$, $a_1 = 1$ from the relation (i) we obtain*

$$1 \leqslant x_0 \wedge \frac{1}{x_0} < \lambda < 1.$$

*From the relation ( iii ) we obtain*

$$2 < x_0 \wedge 0 < \lambda \leqslant 1$$

*From the relation ( iv ) we obtain*

$$0 < x_0 < 1 \wedge 0 < \lambda < 1$$

*The real root is*

$$x_0 = 1 \pm \frac{1}{2}\sqrt{-4 + \frac{4}{\lambda}} = 1 \pm \sqrt{-1 + \frac{1}{\lambda}},$$

*furthermore $x_0 < \frac{1}{\lambda}$. Analyzing the variation of $\lambda$ and $x_0$ we have that $\lambda$ must be very close to one. Taking $\lambda = 0.95$ then*

$$x_0 = 1 - \frac{\sqrt{19}}{19}$$

*would be an approximate root.*

**Remark 3.1.** *A connection between the given polynomial*

$$x^n + a_{n-1}x^{n-1} + \cdots + a_j x^j + \cdots + a_1 x + a_0 = 0 \tag{12}$$

*and the relation*

$$\sum_{k=1}^{N} A_k = \lambda \sum_{k=1}^{N} A_k^2 \tag{13}$$

*can give us other information to find the real or complex roots, in that sense we choose* $N = 2$.

$$A_1 = x^n + a_{n-1}x^{n-1} + \cdots + a_{j+1}x^{j+1} + a_{j-1}x^{j-1} + \cdots + a_2x^2 + a_1 x \tag{14}$$

$$A_2 = a_j x^j. \tag{15}$$

*Substituting the relations (14) and (15) into (13), we obtain*

$$-a_0 = a_j \left[a_j^2 x^{2j} + (-a_0 - a_j x^j)^2\right]. \tag{16}$$

*From the relation (16) we obtain*

$$x^j = -\frac{a_0}{2a_j} \pm \frac{1}{2a_j}\sqrt{-a_0^2 - 2\frac{a_0}{\lambda_j}}. \tag{17}$$

*Assuming that* $a_0 > 0$, *we have that* $\lambda_j < 0$, *therefore from the relation (17) we obtain*

$$-\frac{2}{a_0} \leqslant \lambda_j < 0. \tag{18}$$

*Let's define*

$$c_j := -a_0^2 - \frac{2a_0}{\lambda_j}. \tag{19}$$

*From the relationship (19) completing squares we have*

$$(\lambda_j c_j)^2 + (a_0\lambda_j + 1)^2 = 1. \tag{20}$$

*Parameterizing the relationship (20) we have*

$$\lambda_j c_j = \frac{1 - t_j^2}{1 + t_j^2}, \quad \lambda_j a_0 + 1 = \frac{2t_j}{1 + t_j^2}, \quad t_j \in [-1, 1]. \tag{21}$$

*From the relation (21) we obtain*

$$c_j = \frac{1 + t_j}{1 - t_j} a_0, \ t_j \in [-1; 1\rangle. \tag{22}$$

*From the relation (22) we obtain in (17) the following*

$$x^j = \frac{a_0}{a_j} \times \frac{t_j}{1 - t_j}, \quad t_j \in [-1; 1\rangle, \quad j \in \{1, \ldots, n\}. \tag{23}$$

*or*

$$x^j = -\frac{a_0}{a_j} \times \frac{1}{1 - t_j}, \quad t_j \in [-1; 1\rangle, \quad j \in \{1, \ldots, n\}. \tag{24}$$

*From the relation (18) we have that if* $\lambda_j \in \langle -\infty, \frac{-2}{a_0}\rangle \cup \langle 0, +\infty\rangle$, *then* $x^j$ *is a complex number.*
*In the relation (17) let's set*

$$d_j^2 := a_0^2 + 2\frac{a_0}{\lambda j}. \tag{25}$$

*From the relation (25) we obtain the following*

$$\frac{d_j \lambda_j}{a_0\lambda_j + 1} = \frac{1 - t_j^2}{1 + t_j^2}, \quad \frac{1}{a_0\lambda_j + 1} = \frac{2t_j}{1 + t_j^2}. \tag{26}$$

*Using the relation (26) we obtain*

$$d_j = \frac{a_0(1+t_j)}{1-t_j} \tag{27}$$

*Using the relation (27) in (17) we obtain*

$$x^j = \frac{a_0}{2a_j}\left(-1+\left(\frac{1+t_j}{1-t_j}\right)i\right), \quad t_j \in [-1,1\rangle,\ a_0 > 0,\ j \in \{1,\dots,n\} \tag{28}$$

$$x^j = \frac{a_0}{2a_j}\left(-1-\left(\frac{1+t_j}{1-t_j}\right)i\right), \quad t_j \in [-1,1\rangle,\ a_0 > 0,\ j \in \{1,\dots,n\}. \tag{29}$$

**Remark 3.2.** *Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where $a_0 < 0$. Let*

$$A := x^n + a_{n-1}x^{n-1} + \cdots + a_{j+1}x^{j+1} + a_{j-1}x^{j-1} + \cdots + a_2x^2 + a_1x \tag{30}$$

$$B := a_jx^j. \tag{31}$$

*Using the relationship*

$$A + B = \lambda_j[A^2 + B^2]. \tag{32}$$

*Working in the same way as what was done in the Observation (3.1) we obtain the following*

$$0 < \lambda_j \leqslant -\frac{2}{a_0}, \tag{33}$$

*and also*

$$x^j = \frac{a_0}{a_j}\left(\frac{t_j}{1-t_j}\right), \quad t_j \in [-1,1\rangle,\ a_0 < 0,\ j \in \{1,\dots,n\} \tag{34}$$

$$x^j = \frac{a_0}{a_j}\left(-\frac{1}{1-t_j}\right), \quad t_j \in [-1,1\rangle,\ a_0 < 0,\ j \in \{1,\dots,n\}. \tag{35}$$

*For $\lambda_j \in \langle -\infty, 0\rangle \cup \langle -\frac{2}{a_0}, +\infty\rangle$ we obtain complex $x^j$, which has the next way*

$$x^j = \frac{a_0}{2a_j}\left(-1+\left(\frac{1+t_j}{1-t_j}\right)i\right), \quad t_j \in [-1,1\rangle,\ a_0 < 0,\ j \in \{1,\dots,n\} \tag{36}$$

$$x^j = \frac{a_0}{2a_j}\left(-1-\left(\frac{1+t_j}{1-t_j}\right)i\right), \quad t_j \in [-1,1\rangle,\ a_0 < 0,\ j \in \{1,\dots,n\}. \tag{37}$$

**Example 3.2.** *Find the complex and complex roots of the polynomial $P(x) = x^5 + x + 1$.*

**Solution 3.2.** *As*

$$x^j = -\frac{a_0}{a_j}\left(\frac{1}{1-t_j}\right) \quad or \quad x^j = \frac{a_0}{a_j}\left(\frac{t_j}{1-t_j}\right) \tag{38}$$

*where $t_j \in [-1,0\rangle \cup \langle 0,1\rangle$. In the relation (38) for $j = 1$ we have, $a_0 = 1$, $a_1 = 1$, then we obtain*

$$x = -\left(\frac{1}{1-t_j}\right) \quad or \quad x = \frac{t_j}{1-t_j}. \tag{39}$$

*For $t_1 = \frac{1}{2}$ we obtain in (39)*

$$x = -2 \quad or \quad x = 1. \tag{40}$$

*For $t_1 = -\frac{1}{2}$ we obtain in (39)*

$$x = -\frac{2}{3} \quad or \quad x = \frac{1}{3}. \tag{41}$$

*From the relationship (40) and (41) we obtain that* $x = \frac{-2}{3}$ *is a reasonable value.*

*For* $t_1 = -\frac{1}{4}$ *we have*

$$x = -\left(\frac{1}{1+\frac{1}{4}}\right) = -\frac{4}{5}. \tag{42}$$

*For* $t_1 = -\frac{3}{4}$ *we have*

$$x = -\frac{4}{7}. \tag{43}$$

*An approximate root is given for* $x = \dfrac{-\frac{2}{3}-\frac{4}{5}}{2} = -\dfrac{11}{15}$. *Only the midpoints of the intervals* $\langle -1, -\frac{1}{2}\rangle$ *have been taken.*

*For complex roots we have*

$$x^j = -\frac{a_0}{2a_j}\left(1 + \frac{(1+t_j)}{(1-t_j)}i\right), t_j \in [-1,0\rangle \cup \langle 0,1\rangle. \tag{44}$$

*or*

$$x^j = -\frac{a_0}{2a_j}\left(1 - \frac{(1+t_j)}{(1-t_j)}i\right), t_j \in [-1,0\rangle \cup \langle 0,1\rangle. \tag{45}$$

*For* $j = 1$, $a_0 = 1$, $a_1 = 1$ *and* $t_1 = \frac{1}{2}$ *we have*

$$x = -\frac{1}{2}(1+3i) \quad or \quad x = -\frac{1}{2}(1-3i). \tag{46}$$

*For* $t_1 = \frac{1}{4}$ *we have*

$$x = -\frac{1}{2} + \frac{5}{6}i \quad or \quad x = -\frac{1}{2} - \frac{5}{6}i. \tag{47}$$

*For* $t_1 = -\frac{1}{2}$ *we have*

$$x = -\frac{1}{2} + \frac{1}{6}i \quad or \quad x = -\frac{1}{2} - \frac{1}{6}i. \tag{48}$$

*For* $t_1 = -\frac{1}{4}$ *we obtain*

$$x = -\frac{1}{2} + \frac{3}{10}i \quad or \quad x = -\frac{1}{2} - \frac{3}{10}i. \tag{49}$$

*From the relations (46), (47), (48) , (49) when evaluating the polynomial with these values, it is observed that the best approximate root is given by relation (47).*

*For the last two complex roots of the given polynomial, we have to*

*i) For* $j = 5$, $a_5 = 1$, $a_0 = 1$ *and* $t_5 = 0$ *in the relation (44) we have*

$$x^5 = -\frac{1}{2}\left(1 + \left(\frac{1+0}{1-0}\right)i\right), \quad t_5 \in [-1,0\rangle \cup \langle 0,1]$$

*Using the formula*

$$\sqrt[n]{z} = \sqrt[n]{|z|}\left[\cos\left(\frac{\theta+2\pi k}{n}\right) + i\,\mathrm{sen}\left(\frac{\theta+2\pi k}{n}\right)\right], \quad k \in \{0,1,\ldots,n-1\}.$$

*For* $k = 0$, *the first fifth root would be* $x_0 = -0.9330294851 - 0.0025579597i$. *So on, for* $k = 3$, $x_3 = 0.6597539956 + 0.6597539956i$, *this value is a good approximation for the complex root.*

*ii) The approximation given in item i) can be improved by varying the parameter* $t_5$. *For example, taking midpoints of the intervals* $[-1,0\rangle$ *or* $\langle 0,1]$.

# 4 About Number Theory

In this work we propose a new method to solve the Diophantine equation

$$Ax + By + Cz = D \tag{50}$$

In (50) we will assume that $D > 0$ and $D > A, D > B, D > C$ and $A, B, C$ and $D$ without common factors. The technique consists of establishing a link between equation (50) and the relationship

$$\sum_{k=1}^{n} a_k = \lambda \sum_{k=1}^{n} a_k^2 \tag{51}$$

Where $a_k$ are any real numbers and $n$ is a natural number greater than or equal to 2 and $\lambda$ is unique. In this case we use $n = 2$

$$a_1 + a_2 = \lambda_1 \left(a_1^2 + a_2^2\right). \tag{52}$$

Let $a_1 = Ax + By; a_2 = Cz$, from the relation (52) and (50) we obtain

$$D = \lambda_1 \left[(Ax + By)^2 + C^2z^2\right] = \lambda_1 \left[(D - Cz)^2 + C^2z^2\right]. \tag{53}$$

From the relation (53) we obtain the following

$$z = \frac{D}{2C} \pm \frac{1}{2C}\sqrt{-D^2 + \frac{2D}{\lambda_1}}, \text{ where } 0 < \lambda_1 \leqslant \frac{2}{D}. \tag{54}$$

Let

$$-D^2 + \frac{2D}{\lambda_1} = T_1^2. \tag{55}$$

From the relation (55) the following is obtained

$$(\lambda_1 T_1)^2 + (\mathrm{D}\lambda_1 - 1)^2 = 1 \tag{56}$$

From the relation (56) two possibilities are obtained

**(A)** When

$$D\lambda_1 - 1 = \frac{1 - t_1^2}{1 + t_1^2} \text{ and } \lambda_1 T_1 = \frac{2t_1}{1 + t_1^2}, \tag{57}$$

where $|t_1| \leqslant 1$.

Let $t_1 = \dfrac{m_1}{n_1}$ where $m_1$ and $n_1$ are prime numbers relative to each other. From the relation (57) we obtain

$$\lambda_1 = \frac{2n_1^2}{D\left(m_1^2 + n_1^2\right)}, T_1 = \frac{m_1 D}{n_1}. \tag{58}$$

From the relation (58) and (54) we obtain

$$z = \frac{D}{2n_1 C}\left(n_1 + m_1\right) \quad \text{or} \quad z = \frac{D}{2n_1 C}\left(n_1 - m_1\right). \tag{59}$$

For $z = \frac{D}{2n_1 C}\left(n_1 + m_1\right)$ and the equation (50) we obtain

$$Ax + By = \frac{D}{2n_1}\left(2n_1 - n_1 - m_1\right) = \widetilde{D}. \tag{60}$$

Again we apply the relationship (51) by connecting it with (60).

Let $a = Ax; By = b$ we get

$$\widetilde{D} = \lambda_2 \left[(Ax)^2 + (By)^2\right] = \lambda_2 \left[(Ax)^2 + (\widetilde{D} - Ax)^2\right] \tag{61}$$

From the relation (61) we obtain

$$x = \frac{\widetilde{D}}{2A} \pm \frac{T_2}{2A}, \tag{62}$$

where

$$-\widetilde{D}^2 + \frac{2\widetilde{D}}{\lambda_2} = T_2^2. \tag{63}$$

From the relation (63) we obtain

$$(\lambda_2 T_2)^2 + \left(\widetilde{D}\lambda_2 - 1\right)^2 = 1. \tag{64}$$

From the relation (64) two possibilities arise

$(A_1)$

$$\widetilde{D}\lambda_2 - 1 = \frac{1 - t_2^2}{1 + t_2^2} \quad and \quad \lambda_2 T_2 = \frac{2t_2}{1 + t_2^2} \tag{65}$$

or

$(A_2)$

$$\widetilde{D}\lambda_2 - 1 = \frac{2t_2}{1 + t_2^2} \quad and \quad \lambda_2 T_2 = \frac{1 - t_2^2}{1 + t_2^2}, \tag{66}$$

where $|t_2| \leqslant 1$ and $t_2 = \dfrac{m_2}{n_2}$ with $m_2$ and $n_2$ numbers relative cousins to each other.

From the relation $(A_1)$ we obtain

$$\lambda_2 = \frac{2n_2^2}{\widetilde{D}\left(m_2^2 + n_2^2\right)}, \quad T_2 = \frac{m_2}{n_2}\widetilde{D} \tag{67}$$

From the relations (60), (62) and (67) we obtain

$$x = \frac{\widetilde{D}}{2A}\left(1 + \frac{m_2}{n_2}\right) = \frac{D}{2n_1}\left(n_1 - m_1\right) \times \frac{1}{2An_2}\left(n_2 + m_2\right) \tag{68}$$

or

$$x = \frac{D}{2n_1}\left(n_1 - m_1\right) \times \frac{1}{2An_2}\left(n_2 - m_2\right) \tag{69}$$

From the relation $A_2$) we obtain

$$\lambda_2 = \frac{\left(m_2 + n_2\right)^2}{\widetilde{D}\left(n_2^2 + m_2^2\right)}, \quad T_2 = \left(\frac{n_2 - m_2}{n_2 + m_2}\right)\widetilde{D}, \quad n_2 \neq -m_2. \tag{70}$$

From the relations (60), (62) and (70) we obtain

$$x = \frac{\widetilde{D}}{2A}\left(1 + \frac{n_2 - m_2}{n_2 + m_2}\right) = \frac{D}{2n_1}\left(n_1 - m_1\right) \times \frac{1}{2A\left(n_2 + m_2\right)}\left(2n_2\right) \tag{71}$$

or

$$x = \frac{\widetilde{D}}{2A}\left(1 - \frac{n_2 - m_2}{n_2 + m_2}\right) = \frac{D}{2n_1}\left(n_1 - m_1\right) \times \frac{2m_2}{2A\left(n_2 + m_2\right)}\left(2n_2\right). \tag{72}$$

From case A), for $z = \frac{D}{2n_1 C}(n_1 - m_1)$ we obtain in the equation (50)

$$Ax + By = \frac{D}{2n_1}(n_1 + m_1) = \tilde{\tilde{D}}. \tag{73}$$

Again we apply the relationship (51) by linking it with the relationship (73) we obtain. Let $a = Ax$, $b = By$ we obtain

$$\tilde{\tilde{D}} = \tilde{\lambda}_2 \left[(Ax)^2 + (By)^2\right] = \tilde{\lambda}_2 \left[(Ax)^2 + (\tilde{\tilde{D}} - Ax)^2\right] \tag{74}$$

From the relation (74) we obtain

$$x = \frac{\tilde{\tilde{D}}}{2A} \pm \frac{\tilde{T}_2}{2A}, \tag{75}$$

where

$$-\tilde{\tilde{D}}^2 + \frac{2\tilde{\tilde{D}}}{\tilde{\lambda}_2} = \tilde{T}_2^2 \tag{76}$$

Repeating the process carried out from the relation (64) to the relation (72) we obtain

$$\tilde{T}_2 = \frac{\widetilde{m}_2}{\tilde{n}_2}\tilde{\tilde{D}}, \tag{77}$$

where $\widetilde{m}_2$ and $\tilde{n}_2$ are relative primes.
From the relations (75) and (77) obtain the following solution

$$x = \frac{\tilde{D}}{2A}\left(1 + \frac{\tilde{m}_2}{\tilde{n}_2}\right) = \frac{D}{4An_1\tilde{n}_2}(n_1 + m_1)(\tilde{n}_2 + \widetilde{m}_2) \tag{78}$$

or

$$x = \frac{\tilde{D}}{2A}\left(1 - \frac{\tilde{m}_2}{\tilde{n}_2}\right) = \frac{D}{4An_1\tilde{n}_2}(n_1 + m_1)(\tilde{n}_2 - \widetilde{m}_2). \tag{79}$$

For the relation similar to the relation (77) we have

$$\tilde{T}_2 = \frac{(\tilde{n}_2 - \tilde{m}_2)}{(\tilde{n}_2 + \tilde{m}_2)}\tilde{\tilde{D}}, \quad \tilde{n}_2 \neq \tilde{m}_2. \tag{80}$$

Using the relationship (80) and (75) we obtain the following solution

$$x = \frac{\tilde{\tilde{D}}}{2A}\left(1 + \frac{\tilde{n}_2 - \tilde{m}_2}{\tilde{n}_2 + \tilde{m}_2}\right) = \frac{D(n_1 + m_1)}{4An_1} \times \frac{2\tilde{n}_2}{(\tilde{n}_2 + \tilde{m}_2)} \tag{81}$$

or

$$x = \frac{\tilde{\tilde{D}}}{2A}\left(1 - \frac{\tilde{n}_2 - \tilde{m}_2}{\tilde{n}_2 + \tilde{m}_2}\right) = \frac{D(n_1 + m_1)}{4An_1} \times \frac{2\widetilde{m}_2}{(\tilde{n}_2 + \widetilde{m}_2)} \tag{82}$$

**(B)** For this case we have the following situations

$$D\lambda_1 - 1 = \frac{2t_1}{1 + t_1^2}, \quad \lambda_1 T_1 = \frac{2t_1}{1 + t_1^2}. \tag{83}$$

From the relation (83) we obtain

$$T_1 = \left(\frac{n_1 - m_1}{n_1 + m_1}\right) D. \tag{84}$$

From the relation (84) y (54) we obtain

$$z=\frac{D}{2C}\left(1+\frac{n_1-m_1}{n_1+m_1}\right)=\frac{D}{2C}\times\frac{2n_1}{(n_1+m_1)} \tag{85}$$

or

$$z=\frac{D}{2C}\left(1-\frac{(n_1-m_1)}{n_1+m_1}\right)=\frac{D}{2C}\times\frac{2m_1}{(n_1+m_1)} \tag{86}$$

After replacing (85) in (50) or replacing (86) in (50), the same steps are performed, from the relation (60) until the relation (82).

From the relation (85) and the relation (50) we have

$$Ax+By=D-\frac{D}{2}\times\frac{2n_1}{n_1+m_1}=D\left(1-\frac{n_1}{n_1+m_1}\right)=\frac{D\,(m_1)}{n_1+m_1}=\tilde{\tilde{\tilde{D}}} \tag{87}$$

After the relation (87) the solutions are obtained

$$x=\frac{\tilde{\tilde{\tilde{D}}}}{2A}\left(1+\frac{\tilde{\tilde{m}}_2}{\tilde{\tilde{n}}_2}\right)=\frac{Dm_1}{2A\,(n_1+m_1)}\times\frac{\tilde{\tilde{n}}_2+\tilde{\tilde{m}}_2}{\tilde{\tilde{n}}_2}$$

or

$$x=\frac{Dm_1}{2A\,(n_1+m_1)}\times\frac{\tilde{\tilde{n}}_2-\tilde{\tilde{m}}_2}{\tilde{\tilde{n}}_2}.$$

Similar for the other case see (70)

$$x=\frac{Dm_1}{2\,(n_1+m_1)\,A}\left(1+\frac{\tilde{\tilde{n}}_2-\tilde{\tilde{m}}_2}{\tilde{\tilde{n}}_2+\tilde{\tilde{m}}_2}\right)=\frac{Dm_1}{2A\,(n_1+m_1)}\times\frac{2\tilde{\tilde{n}}_2}{\tilde{\tilde{n}}_2+\tilde{\tilde{m}}_2}$$

or

$$x=\frac{Dm_1}{2\,(n_1+m_1)\,A}\left(1+\frac{\tilde{\tilde{n}}_2-\tilde{\tilde{m}}_2}{\tilde{\tilde{n}}_2+\tilde{\tilde{m}}_2}\right)=\frac{D\,(m_1)}{2A\,(n_1+m_1)}\times\frac{2\tilde{\tilde{m}}_2}{\tilde{\tilde{n}}_2+\tilde{\tilde{m}}_2}.$$

From the relation (86) and the relation (50) the following is obtained:

$$Ax+By=D-\frac{D}{2}\times\frac{2n_1}{(n_1+m_1)}=D\left(1-\frac{n_1}{n_1+m_1}\right)=\frac{Dn_1}{m_1+n_1}=D^* \tag{88}$$

Then we have the following solutions

$$x=\frac{Dn_1}{2A\,(n_1+m_1)}\times\left(\frac{\tilde{\tilde{\tilde{n}}}_2+\tilde{\tilde{\tilde{m}}}_2}{\tilde{\tilde{\tilde{n}}}_2}\right)$$

or

$$x=\frac{Dn_1}{2A\,(n_1+m_1)}\times\left(\frac{\tilde{\tilde{\tilde{n}}}_2-\tilde{\tilde{\tilde{m}}}_2}{\tilde{\tilde{\tilde{n}}}_2}\right).$$

Similar for the other case, see (70)

$$x=\frac{Dn_1}{2A\,(n_1+m_1)}\times\frac{2\tilde{\tilde{\tilde{n}}}_2}{\tilde{\tilde{\tilde{n}}}_2+\tilde{\tilde{\tilde{m}}}_2}$$

or

$$x=\frac{Dn_1}{2A\,(n_1+m_1)}\times\frac{2\tilde{\tilde{\tilde{m}}}_2}{\tilde{\tilde{\tilde{n}}}_2+\tilde{\tilde{\tilde{m}}}_2}$$

So we have the following solutions

$S_1)$

$$z=\frac{D\left(n_1+m_1\right)}{2n_1C};\ x=\frac{D\left(n_1-m_1\right)}{4An_1n_2}\left(n_2+m_2\right);\ y=\frac{D}{4Bn_1n_2}\left(n_2-m_2\right)\left(n_1-m_1\right)$$

$S_2)$

$$z=\frac{D\left(n_1+m_1\right)}{2n_1C};x=\frac{D}{4An_1n_2}\left(n_1-m_1\right)\left(n_2-m_2\right);y=\frac{Dm_2}{4Bn_1n_2}\left(n_2+m_2\right)\left(n_1-m_1\right)$$

$S_3)$

$$z=\frac{D\left(n_1+m_1\right)}{2n_1C};\ x=\frac{D\left(n_1-m_1\right)n_2}{2n_1A\left(n_2-m_2\right)};\ y=\frac{Dm_2\left(n_1-m_1\right)}{2Bn_1\left(n_2+m_2\right)}$$

$S_4)$

$$z=\frac{D\left(n_1+m_1\right)}{2n_1C};x=\frac{D\left(n_1-m_1\right)m_2}{2n_1A\left(n_2-m_2\right)};y=\frac{Dn_2\left(n_1-m_1\right)}{2Bn_1\left(n_2+m_2\right)}$$

$S_5)$

$$z=\frac{D\left(n_1-m_1\right)}{2n_1C};x=\frac{D\left(n_1+m_1\right)\left(\tilde{n}_2+\tilde{m}_2\right)}{4An_1\tilde{n}_2};y=\frac{D}{4Bn_1\tilde{n}_2}\left[\left(n_1+m_1\right)\left(\tilde{n}_2-\tilde{m}_2\right)\right]$$

$S_6)$

$$z=\frac{D\left(n_1-m_1\right)}{2n_1C};x=\frac{D\left(n_1+m_1\right)\left(\tilde{n}_2-\tilde{m}_2\right)}{4An_1\tilde{n}_2};y=\frac{D}{4Bn_1\tilde{n}_2}\left[\left(n_1+m_1\right)\left(\tilde{n}_2+\tilde{m}_2\right)\right]$$

$S_7)$

$$z=\frac{D\left(n_1-m_1\right)}{2n_1C};x=\frac{D\left(n_1+m_1\right)}{2An_1\left(\tilde{n}_2+\tilde{m}_2\right)}\tilde{n}_2;y=\frac{D\left(n_1+m_1\right)\tilde{m}_2}{2An_1\left(\tilde{n}_2+\tilde{m}_2\right)}$$

$S_8)$

$$z=\frac{D\left(n_1-m_1\right)}{2n_1C};x=\frac{D\left(n_1+m_1\right)\tilde{m}_2}{2An_1\left(\tilde{n}_2+\tilde{m}_2\right)};y=\frac{D\left(n_1+m_1\right)\tilde{n}_2}{2An_1\left(\tilde{n}_2+\tilde{m}_2\right)}$$

$S_9)$

$$z=\frac{Dn_1}{C\left(n_1+m_1\right)};x=\frac{Dm_1}{2A\left(n_1+m_1\right)}\frac{\left(\tilde{\tilde{n}}_2+\tilde{\tilde{m}}_2\right)}{\tilde{\tilde{n}}_2};y=\frac{Dm_1\left(\tilde{\tilde{n}}_2-\tilde{\tilde{m}}_2\right)}{2B\left(n_1+m_1\right)\tilde{\tilde{n}}_2}$$

$S_{10})$

$$z=\frac{Dn_1}{C\left(n_1+m_1\right)};x=\frac{Dm_1}{2A\left(n_1+m_1\right)}\frac{\left(\tilde{\tilde{n}}_2-\widetilde{\widetilde{m}}_2\right)}{\tilde{\tilde{n}}_2};y=\frac{Dm_1\left(\tilde{\tilde{n}}_2+\widetilde{\widetilde{m}}_2\right)}{2B\left(n_1+m_1\right)\tilde{\tilde{n}}_2}$$

$S_{11})$

$$z=\frac{Dn_1}{C\left(n_1+m_1\right)};x=\frac{Dm_1}{A\left(n_1+m_1\right)}\frac{\tilde{\tilde{n}}_2}{\left(\tilde{\tilde{n}}_2+\widetilde{\widetilde{m}}_2\right)};y=\frac{Dm_1\widetilde{\widetilde{m}}_2}{B\left(n_1+m_1\right)\left(\tilde{\tilde{n}}_2+\widetilde{\widetilde{m}}_2\right)}$$

$S_{12})$

$$z=\frac{Dn_1}{C\left(n_1+m_1\right)};x=\frac{Dm_1}{A\left(n_1+m_1\right)}\frac{\tilde{\tilde{n}}_2}{\left(\tilde{\tilde{n}}_2+\widetilde{\widetilde{m}}_2\right)};y=\frac{Dm_1\tilde{\tilde{n}}_2}{B\left(n_1+m_1\right)\left(\tilde{\tilde{n}}_2+\widetilde{\widetilde{m}}_2\right)}$$

$S_{13})$

$$z=\frac{Dm_1}{C\left(n_1+m_1\right)};x=\frac{Dn_1}{2A\left(n_1+m_1\right)}\frac{\left(\tilde{\tilde{\tilde{n}}}_2+\tilde{\tilde{\tilde{m}}}_2\right)}{\tilde{\tilde{\tilde{n}}}_2};y=\frac{Dn_1\left(\tilde{\tilde{\tilde{n}}}_2-\tilde{\tilde{\tilde{m}}}_2\right)}{2B\left(n_1+m_1\right)\tilde{\tilde{\tilde{n}}}_2}$$

$S_{14})$

$$z=\frac{Dm_1}{C\left(n_1+m_1\right)};x=\frac{Dn_1}{2A\left(n_1+m_1\right)}\frac{\left(\tilde{\tilde{\tilde{n}}}_2-\tilde{\tilde{\tilde{m}}}_2\right)}{\tilde{\tilde{\tilde{n}}}_2};y=\frac{Dn_1\left(\tilde{\tilde{\tilde{n}}}_2+\tilde{\tilde{\tilde{m}}}_2\right)}{2B\left(n_1+m_1\right)\tilde{\tilde{\tilde{n}}}_2}$$

$S_{15})$

$$z=\frac{Dm_1}{C\left(n_1+m_1\right)};x=\frac{Dn_1}{A\left(n_1+m_1\right)}\frac{\tilde{\tilde{\tilde{n}}}_2}{\left(\tilde{\tilde{\tilde{n}}}_2+\tilde{\tilde{\tilde{m}}}_2\right)};y=\frac{Dn_1\tilde{\tilde{\tilde{m}}}_2}{B\left(n_1+m_1\right)\left(\tilde{\tilde{\tilde{n}}}_2+\tilde{\tilde{\tilde{m}}}_2\right)}$$

$S_{16})$

$$z=\frac{Dm_1}{C\left(n_1+m_1\right)};x=\frac{Dn_1}{A\left(n_1+m_1\right)}\frac{\tilde{\tilde{\tilde{m}}}_2}{\left(\tilde{\tilde{\tilde{n}}}_2+\tilde{\tilde{\tilde{m}}}_2\right)};y=\frac{Dn_1\tilde{\tilde{\tilde{n}}}_2}{B\left(n_1+m_1\right)\left(\tilde{\tilde{\tilde{n}}}_2+\tilde{\tilde{\tilde{m}}}_2\right)}$$

Where the pairs of numbers $\left(n_1,m_1\right),\left(n_2,m_2\right),\left(\tilde{n}_2,\tilde{m}_2\right),\left(\tilde{\tilde{n}}_2,\tilde{\tilde{m}}_2\right),\left(\tilde{\tilde{\tilde{n}}}_2,\tilde{\tilde{\tilde{m}}}_2\right)$ are relative cousins to each other.

# 5 About Fermat's Last Theorem

**Theorem 5.1.** *The equation $x^n+y^n=z^n$ has no positive integer solution for $n\geqslant 3$, $n\in\mathbb{N}$.*

*Proof.* Suppose there are positive integer solutions $x,y,z$ where $M.C.D(x,y)=1$, $M.C.D(x,z)=1$, $M.C.D(y,z)=1$ satisfying

$$x^n+y^n=z^n. \tag{89}$$

Let

$$A:=x^n,\ B:=y^n,\ C:=z^n. \tag{90}$$

Let's use the relationship

$$A+B+C=\lambda[A^2+B^2+C^2]. \tag{91}$$

From the relations (89), (90) and (91) we obtain

$$2z^n=\lambda[x^{2n}+y^{2n}+z^{2n}]. \tag{92}$$

From the relation (92) we have

$$x^n=\frac{1}{\lambda}c_1,\ y^n=\frac{1}{\lambda}c_2,\ z^n-\frac{1}{\lambda}=\frac{1}{\lambda}c_3, \tag{93}$$

where $c_1^2+c_2^2+c_3^2=1$, $\lambda\in\mathbb{Q}^+$ and $c_1,c_2$ y $c_3\in\mathbb{Q}$. Using the relation (93) and (89) we obtain

$$c_1+c_2=c_3+1 \tag{94}$$

Using the relation $c_1^2 + c_2^2 + c_3^2 = 1$ and the relation given in (94) we obtain

$$c_1^2 + c_2^2 + c_1c_2 - c_1 - c_2 = 0. \tag{95}$$

Using the relationship (89), (93) and (95) we obtain

$$z^n \left( z^n - \frac{1}{\lambda} \right) = x^n y^n. \tag{96}$$

Since $\lambda$ is positive rational, then $\lambda := \frac{M}{N}$, where

$$M.C.D(M, N) = 1. \tag{97}$$

From (97) and (96) we have

$$z^n \left( z^n - \frac{N}{M} \right) = x^n y^n. \tag{98}$$

From the relation (98) we have the first possibility

$$z = Mr, \ r \in \mathbb{N}. \tag{99}$$

Using (99) and (98) we have $r = 1$. Then we get

$$M^{2n} - N.M^{n-1} = x^n y^n. \tag{100}$$

From the relation (100) we obtain that this is absurd because for $z = M$, the right hand side is not divisible by $z$.

The second case in the relation (98) is when $M = z^n$, therefore from here we obtain

$$M^2 - N = x^n y^n. \tag{101}$$

Since $x^n = \dfrac{c_1}{\lambda}$ and $y^n = \dfrac{c_2}{\lambda}$, where $\lambda = \dfrac{M}{N}$, and as $c_1, \ c_2 \in \mathbb{Q}^+, \ 0 < c_1 \leqslant 1, \ 0 < c_2 \leqslant 1$, we can express $c_1$ and $c_2$ como $c_1 = \dfrac{m_1}{n_1}$ and $c_2 = \dfrac{m_2}{n_2}$, respectively, where $MCD(m_1, n_1) = 1$ y $MCD(m_2, n_2) = 1$. Therefore

$$x^n = \frac{m_1 N}{n_1 M}, \ \ y^n = \frac{m_2 N}{n_2 M}. \tag{102}$$

From the equation (102) it follows that, given that $MCD(x, y) = 1$, it is true that

$$N = n_1 n_2, \ \ m_1 = r_1 M, \ \ m_2 = r_2 M, \tag{103}$$

where $MCD(r_1, r_2) = 1$.

Therefore, substituting (103) into (102) we obtain

$$x^n = r_1 n_2 \ \text{ and } \ y^n = r_2 n_1. \tag{104}$$

Using the relation (104) and (103) in (101) we obtain

$$M^2 - n_1 n_2 = r_1 r_2 n_1 n_2. \tag{105}$$

From the relation (105) the only possibility is that

$$n_1 n_2 = 1. \tag{106}$$

From (106) we obtain $n_1 = n_2 = 1$, which implies that $c_1 = 1$, $c_2 = 1$ and the equation $c_1^2 + c_2^2 + c_3^2 = 1$ would be absurd. □

## 6 Conclusions

It is interesting to observe that an appropriate connection between the object of study and the relation $A_1 + \cdots + A_n = \lambda \sum_{k=1}^{n} A_k^2$ (where $A_k \in \mathbb{R}$, $k = 1$, $n \in \mathbb{N}$, and $\lambda \in \mathbb{R}$ is unique) allows for the resolution of classical problems. Remarkably, through this mechanism, it is possible to obtain approximate zeros, both real and complex, of a real monic polynomial of degree $N$, general solutions to the linear Diophantine equation in three variables, and a potential new approach to Fermat's Last Theorem. We still do not fully understand why this method works; one possible explanation is that the problem is transformed into an $n$-dimensional sphere, which is symmetric in relation to a coordinate system, making it easier to approach.


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