NEW APPROXIMATE SYMMETRY THEOREMS AND COMPARISONS WITH EXACT SYMMETRIES

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Abstract- Three new approximate symmetry theories are proposed. The approximate symmetries are contrasted with each other and with the exact symmetries. The theories are applied to nonlinear ordinary differential equations for which exact solutions are available. It is shown that from the symmetries, approximate solutions as well as exact solutions in some restricted cases can be retrievable. Depending on the specific approximate theory and the equations considered, the approximate symmetries may expand the Lie Algebra of the exact symmetries, may be a perturbed form of the exact symmetries or may be a subalgebra of the exact symmetries. Exact and approximate solutions are retrieved using the symmetries.

Keywords- Approximate Symmetry Theories, Lie Group Analysis, Group Invariant Solutions, Ordinary Differential Equations

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1. Introduction

Lie Group theory [1-3] is a systemized and unified approach in search of analytical solutions of differential equations. It is a generalized approach for finding solutions of especially nonlinear differential equations and has the capability of producing results obtained by other ad-hoc methods. Perturbation method [4] is another powerful technique employed in search of approximate symmetries for over a century. Attempts to combine these powerful techniques appeared in the literature. In case of perturbed equations, depending on the specific equation, the exact symmetries may not be sufficient to extract enough solutions. To extend the Lie Algebra and to construct further solutions, many approximate symmetry theories were proposed.

There are three main theories of approximate symmetries and a number of variants of these methods. The first method (Method I) is due to Baikov et al. [5,6] in which the symmetry generator is expanded in a perturbation series without expanding the depending variable. On the contrary, in the second method due to Fushchich and Shtelen [7] (Method II), the dependent variable is expanded in a perturbation series and the equations form a coupled system when separated with respect to orders. The approximate symmetry is then defined to be the exact symmetry of these coupled systems. In this method, since the number of dependent variables increase, the algebra for determining symmetries become rather involved. By assuming a linear unperturbed part and a nonlinear perturbed part for the differential equations, the hierarchical equations appearing in a separated block can be viewed as a linear non-homogenous equation with a known function for the non-homogenous part. This

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assumption reduces drastically the algebra and the approximate symmetries of the nonlinear perturbed equation corresponds to the exact symmetries of the linear non-homogenous equation [8, 9] (Method III). The three methods were contrasted with each other and the advantages and disadvantages were outlined by applying the methods to the potential Burgers equation [8], creeping flow equations of a second-grade fluid [8] and an ordinary differential equation with quadratic nonlinearity [9]. A more theoretical basis for the comparisons of Method I and Method II were later presented [10].

Many papers appeared in the literature applying the three methods to differential equations arising from mathematical physics. While a complete list of all work on the applications of the symmetry methods is beyond the scope of this study, a partial list will be given for the applications: Method I is applied in references [11-19], Method II in references [19-32] and Method III in references [33-35]. A Matlab package [36] was developed to symbolically compute approximate symmetries for all the three methods. Noetherian symmetries are another alternative to the conventional Lie Group symmetries which involve Lagrangians. Approximate Noether symmetries were also calculated for mathematical physics models [37-42]. Exterior calculus is the other alternative to the classical Lie Group methods for calculating symmetries. The pioneering work on the topic is due to Harrison and Estabrook [43] and later employed by others [44-47]. The approximate symmetry version of the exterior calculus approach was also presented [48, 49]. Approximate Homotopy Symmetry method is another approach developed in search of approximate symmetries [50-52].

In this work, three new approximate symmetry definitions are given for the first time. The exact symmetries and the approximate symmetries by the new three methods are contrasted with each other for sample ordinary differential equations whose exact solutions are known. Exact and approximate group invariant solutions are derived using the symmetries of each method. The new methods may extend the Lie Algebra, may be perturbed expansions of the exact symmetries, or maybe a subgroup of the exact symmetries depending on the method used and the specific equation considered. The approximate symmetries are capable of retrieving approximate solutions as well as exact solutions.

2. Approximate Symmetry Theories

Three new definitions for approximate symmetries will be given in this section for the first time. The definitions have some differences from each other which leads to different symmetry generators. To distinguish them from the Approximate Symmetry Theorems I-II and III discussed in the introduction, the new ones are numbered as IV-V and VI.

Approximate Symmetry Definition IV

For the k'th order perturbed nonlinear ordinary differential equation

$$F\left(x, y, y', y'', \ldots y^{(k)}, \varepsilon\right) = 0 \tag{2.1}$$

with ε being the perturbation parameter and the Lie Group transformation parameter, the first order approximate symmetry corresponds to

$$F|_{\varepsilon=0} + \varepsilon XF|_{\varepsilon=0} = 0 \tag{2.2}$$

where

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \mu\frac{\partial}{\partial \varepsilon} + \eta^1\frac{\partial}{\partial y_1} + \cdots + \eta^k\frac{\partial}{\partial y_k} + \mu\frac{\partial}{\partial \varepsilon} \tag{2.3}$$

is the approximate symmetry generator extended to k'th order with the group transformations

$$\begin{aligned}
x^* &= x + \varepsilon\xi(x,y,\varepsilon)\\
y^* &= y + \varepsilon\eta(x,y,\varepsilon)\\
y_1^* &= y_1 + \varepsilon\eta^1(x,y,y_1,\varepsilon)\\
&\vdots\\
y_k^* &= y_k + \varepsilon\eta^k(x,y,y_1,\dots,y_k,\varepsilon)\\
\mu^* &= \varepsilon\mu
\end{aligned} \tag{2.4}$$

where

$$y_k = y^{(k)}, \qquad \eta^k = \frac{D\eta^{k-1}}{Dx} - y_k\frac{D\xi}{Dx}, \quad \frac{D}{Dx} = \frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + y_2\frac{\partial}{\partial y_1} + \cdots + y_{k+1}\frac{\partial}{\partial y_k} \quad \square \tag{2.5}$$

Note that in determining the approximate symmetry generator, the whole block of (2.2) is used. In the case of exact symmetries, equation (2.2) separates into two equations and the Lie Group transformation parameter is different from the perturbation parameter.

A slightly different definition is suggested below as the Symmetry Definition V.

Approximate Symmetry Definition V

For the k'th order perturbed nonlinear ordinary differential equation

$$F\left(x,y,y',y'',\dots y^{(k)},\varepsilon\right) = 0 \tag{2.6}$$

with ε being the perturbation parameter and the Lie Group transformation parameter, the first order approximate symmetry corresponds to

$$XF|_{\varepsilon=0} = 0 \quad \text{when} \quad F|_{\varepsilon=0} = 0 \tag{2.7}$$

where

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \mu\frac{\partial}{\partial \varepsilon} + \eta^1\frac{\partial}{\partial y_1} + \cdots + \eta^k\frac{\partial}{\partial y_k} + \mu\frac{\partial}{\partial \varepsilon} \tag{2.8}$$

is the approximate symmetry generator extended to k'th order with the group transformations

$$\begin{aligned}
x^* &= x + \varepsilon\xi(x,y)\\
y^* &= y + \varepsilon\eta(x,y)\\
y_1^* &= y_1 + \varepsilon\eta^1(x,y,y_1)
\end{aligned} \tag{2.9}$$

$$\vdots$$

$$y_k^* = y_k + \varepsilon\eta^k(x, y, y_1, \dots, y_k)$$

$$\mu^* = \varepsilon\mu$$

where

$$y_k = y^{(k)}, \qquad \eta^k = \frac{D\eta^{k-1}}{Dx} - y_k\frac{D\xi}{Dx}, \quad \frac{D}{Dx} = \frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + y_2\frac{\partial}{\partial y_1} + \dots + y_{k+1}\frac{\partial}{\partial y_k} \qquad (2.10)$$

In the above version, the block, i.e. Eq. (2.2), is separated into two parts. It is still different from the exact symmetry definition, since the Lie Group transformation parameter is different from the perturbation parameter in the exact symmetry case. Also, in the exact case $F = 0$, whereas in this definition, the unperturbed equation satisfies the condition $F|_{\varepsilon=0} = 0$ which is merely an approximation of the original equation, namely the unperturbed equation itself. Note also that the infinitesimals $\xi(x, y)$ and $\eta(x, y)$ do not contain the perturbation parameter as an argument, while this is not the case for Approximate Symmetry Method IV.

A variant of the fourth definition may also be proposed where the Lie Group parameter is not the perturbation parameter.

Approximate Symmetry Definition VI

For the *k*'th order perturbed nonlinear ordinary differential equation

$$F\left(x, y, y', y'', \dots y^{(k)}, \varepsilon\right) = 0 \qquad (2.11)$$

with ε being the perturbation parameter and α being the Lie Group transformation parameter, the first order approximate symmetry corresponds to

$$F|_{\alpha=0} + \alpha XF|_{\alpha=0} = 0 \qquad (2.12)$$

where

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \eta^1\frac{\partial}{\partial y_1} + \dots + \eta^k\frac{\partial}{\partial y_k} \qquad (2.13)$$

is the approximate symmetry generator extended to *k*'th order with the group transformations

$$x^* = x + \alpha\xi(x, y, \varepsilon)$$

$$y^* = y + \alpha\eta(x, y, \varepsilon)$$

$$y_1^* = y_1 + \alpha\eta^1(x, y, y_1, \varepsilon) \qquad (2.14)$$

$$\vdots$$

$$y_k^* = y_k + \alpha\eta^k(x, y, y_1, \dots, y_k, \varepsilon)$$

where

$$y_k = y^{(k)}, \qquad \eta^k = \frac{D\eta^{k-1}}{Dx} - y_k\frac{D\xi}{Dx}, \quad \frac{D}{Dx} = \frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + y_2\frac{\partial}{\partial y_1} + \dots + y_{k+1}\frac{\partial}{\partial y_k} \qquad (2.15)$$

If the two terms in (2.12) are separated, then one obtains the exact symmetries. The idea here is not to separate the block in search of approximate symmetries. This definition indeed is not an approximate symmetry definition in the sense that it does not extend the Lie Algebra of the exact symmetries, rather produces a subgroup of the exact symmetries. It is included for comparison reasons and for outlining the importance of selecting the perturbation parameter as the Lie Group parameter as was done in definitions V and VI.

3. Approximate Symmetry Calculations

For a number of ordinary differential equations, symmetries corresponding to the three methods are calculated together with the exact symmetries (Table 1).

Table 1- Exact and Approximate Symmetries

Equation	Exact Symmetry	Approximate Symmetry IV	Approximate Symmetry V	Approximate Symmetry VI
$y' + \varepsilon y = 0$	Unsolvable $\eta_x + \varepsilon(\eta - y\eta_y + y\xi_x) - \varepsilon^2 \xi_y y^2 = 0$	$\xi = -\frac{\mu}{2}x^2 + a_1 x + a_2$ $\eta = \left(-\mu x + a_1 - \frac{1}{\varepsilon}\right) y + b$	$\xi = \xi(x, y)$ $\eta = -\mu xy + a(y)$	$\xi = a + be^{-\varepsilon x}$ $\eta =-(\varepsilon b e^{-\varepsilon x} + \frac{1}{\alpha})y + ce^{-\varepsilon x}$
$y' + e^{\varepsilon y} = 0$	Unsolvable $\eta_x - e^{\varepsilon y}(\eta_y - \xi_x - \varepsilon\eta) - \xi_y e^{2\varepsilon y} = 0$	$\xi = -\frac{\mu}{2}x^2 + a_1 x + a_2$ $\eta = \left(-\mu x + a_1 - \frac{1}{\varepsilon}\right) y - \frac{1}{\varepsilon}x + b_1$	$\xi = \xi(x, y)$ $\eta = -\xi + \mu\frac{y^2}{2} + a(x + y)$	$\xi = \frac{1}{\alpha}x + b$ $\eta = -\frac{1}{\varepsilon\alpha}$
$y'' + \varepsilon y'^2 = 0$	$\xi = (ax + b)e^{\varepsilon y} + cx^2 + dx + e$ $\eta = \left(fe^{-\varepsilon y} + \frac{c}{\varepsilon}\right)x + ge^{-\varepsilon y} + h + \frac{a}{\varepsilon}e^{\varepsilon y}$	$\xi = ax + b$ $\eta = \left(2a - \frac{1}{\varepsilon}\right)y + cx + d$	$\xi = (a_2 x + a_3)y + c_1 x^2 + b_1 x + b_2$ $\eta = (2a_2 - \mu)\frac{y^2}{2} + (c_1 x + c_2)y + d_1 x + d_2$	$\xi = \frac{1}{2\alpha}x + a$ $\eta = b$
$y'' - 2\varepsilon yy' = 0$	$\xi = ax + b$ $\eta = -ay$	$\xi = a_2 x + a_3$ $\eta = \left(2a_2 - \frac{1}{\varepsilon}\right)y + b_1 x + b_2$	$\xi = \left(\frac{\mu}{3}x^2 + a_2 x + a_3\right)y + c_1 x^2 + b_1 x + b_2$ $\eta = \left(\frac{2\mu}{3}x + a_2\right)y^2 + (c_1 x + c_2)y + d_1 x + d_2$	$\xi = ax + b$ $\eta = -ay$
$y'' - y + \varepsilon y^2 = 0$	$\xi = a$ $\eta = 0$	$\xi = a$ $\eta = -\frac{1}{\varepsilon}y + b_1 e^x + b_2 e^{-x}$	$\xi = (a_1 e^x + a_2 e^{-x})y + b_1 + b_2 e^{2x} + b_3 e^{-2x}$ $\eta = (a_1 e^x - a_2 e^{-x})y^2 + (c_1 + b_2 e^{2x} - b_3 e^{-2x})y$ $d_1 e^x + d_2 e^{-x}$	$\xi = 0$ $\eta = 0$
$y''' = \varepsilon f(y', y'')$	$\xi = a$ $\eta = b$	$\xi = a_1 x + a_2$ $\eta = \left(3a_1 - \frac{1}{\varepsilon}\right)y + b_1 x^2 + b_2 x + b_3$	$\xi = a_1 x^2 + a_2 x + a_3$ $\eta = (2a_1 x + a_2 + c)y + b_1 x^2 + b_2 x + b_3$	$\xi = 0$ $\eta = 0$

From the symmetries, for the specific problems considered, some conclusions can be given:

For first order equations;

- In case of exact symmetries, usually the determining equation for the infinitesimals cannot be separated and remains unsolvable, unless some further simplifying assumptions are made.
- On the contrary, the infinitesimals are solvable for the approximate symmetries.
- Among the symmetries, the richest symmetry corresponds to the approximate symmetry V case for first order equations

For the higher order equations;

- For the equation $y'' + \varepsilon y'^2 = 0$, while the exact and approximate symmetry V possess 8-parameter Lie Group transformations, the other symmetries possess less parameters.
- For the equation $y'' + \varepsilon y'^2 = 0$, if the exact symmetry is expanded in a Taylor series up to $O(\varepsilon)$, the approximate symmetry V result can be retrieved.
- For the last 3 equations, approximate symmetries IV and V are richer than the exact symmetries. For the equation $y'' - y + \varepsilon y^2 = 0$, while the exact symmetries are one parameter, the approximate symmetry IV contains 3 parameter and the approximate symmetry V contains 8 parameter Lie Group transformations.
- As a general rule, approximate symmetry VI is a subalgebra of exact symmetries if not equal.
- As a general rule, approximate symmetry V produces the richest symmetries among the approximate ones.

4. Solutions

Using the symmetries, group invariant solutions are constructed for the four problems and listed in Table 2. In the table, the exact and one-correction term approximate solutions of the problem are given first and the specific symmetries to retrieve the results are given. The equation to be solved is

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)} \quad . \tag{4.1}$$

Substituting the outcome to the original equation to satisfy it and then applying the initial conditions, the approximate and exact solutions are obtained.

Table 2- Group Invariant Solutions

Equation	Exact and Approximate Solutions	Exact Symmetry	Approximate Symmetry IV	Approximate Symmetry V	Approximate Symmetry VI
$y' + \varepsilon y = 0$, $y(0) = 1$	$y_e = e^{-\varepsilon x}$	Retrievable	$\xi = a_2$, $\eta = -\frac{1}{\varepsilon} y$	$\xi = b$, $\eta = y$	$\xi = be^{-\varepsilon x}$, $\eta = -\varepsilon be^{-\varepsilon x} y$
	$y_a = 1 - \varepsilon x$	Not directly retrievable	$\xi = a_2$, $\eta = b$	$\xi = 1$, $\eta = a$	Not directly retrievable
$y' + e^{\varepsilon y} = 0$, $y(0) = 0$	$y_e = -\frac{1}{\varepsilon}\ln(1 + \varepsilon x)$	Not directly retrievable	Not directly retrievable	Not directly retrievable	$\xi = \frac{1}{\alpha} x + b$, $\eta = -\frac{1}{\varepsilon\alpha}$
	$y_a = -x + \varepsilon \frac{x^2}{2}$	Not directly retrievable	$\xi = a_2$, $\eta = -\frac{1}{\varepsilon} x + b_1$	Not directly retrievable	Not directly retrievable
$y'' + \varepsilon y'^2 = 0$, $y(0) = 0$, $y'(0) = 1$	$y_e = \frac{1}{\varepsilon}\ln(1 + \varepsilon x)$	$\xi = dx + e$, $\eta = h$	$\xi = ax + b$, $\eta = d$	$\xi = b_1 x + b_2$, $\eta = d_2$	$\xi = \frac{1}{2\alpha} x + a$, $\eta = b$
	$y_a = x - \varepsilon \frac{x^2}{2}$	$\xi = e$, $\eta = \frac{c}{\varepsilon} x + h$	$\xi = b$, $\eta = cx + d$	$\xi = b_2$, $\eta = d_1 x + d_2$	Not directly retrievable
$y'' - 2\varepsilon y y' = 0$, $y(0) = 1$, $y'(0) = \varepsilon$	$y_e = \frac{1}{1 - \varepsilon x}$	$\xi = ax + b$, $\eta = -ay$	$\xi = a_2 x + a_3$, $\eta = \left(2a_2 - \frac{1}{\varepsilon}\right) y$	$\xi = b_1 x + b_2$, $\eta = c_2 y$	$\xi = ax + b$, $\eta = -ay$
	$y_e = 1 + \varepsilon x$	Not directly retrievable	$\xi = a_3$, $\eta = b_2$	$\xi = b_2$, $\eta = d_2$	Not directly retrievable

Regarding the retrieval of solutions, approximate symmetry IV and V performs better than the approximate symmetry VI in most of the cases. Approximate symmetry VI cannot produce approximate solutions for all the problems considered since it produces a subgroup of the exact symmetries. In most of the cases, the approximate symmetries lead to the exact solutions also. This is because of the fact that the dependent variable is not expanded in a perturbation series, a feature observed in Approximate Symmetry I theory due to Baikov et. al. [5, 6] also, which can be questioned from the perturbation theory point of view [8]. In contrast to this similarity, the main difference between the mentioned Method I [5, 6] and the approximate symmetry theories presented here is that the generator is expanded in a perturbation series in the former case while it is not expanded in a series in the ones presented here.

5. Concluding Remarks

Based on this study and the previous work on approximate symmetry theories [5-10], the following conclusions can be made

- If the goal is to produce only the approximate solutions, Method II [7] and Method III with a less algebra [8] is recommended, since those methods are more consistent with the perturbation theory and directly leads to the approximate solutions.
- If the goal is to produce both the approximate and exact solutions, Method I [5, 6] and Method IV and V presented in this study can be employed.
- Among the new three approximate methods, Method V is recommended for second and higher order equations most, since it leads to richer symmetries.
- For first order differential equations, however, Method IV leads to simpler and solvable symmetry infinitesimals than those of exact symmetry and Method V cases.
- Method VI corresponds to the subgroup of the exact symmetries which leads to the group invariant solutions.
- The work can be extended directly to include partial differential equations. A comparison of the symmetries and solutions for partial differential equations is a further research topic in the future.

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