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Exact Solutions of Kantowski-Sachs spacetimes in the Framework of Creation-Field Cosmology

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Abstract

The Kantowski-Sachs spacetimes in the context of creation field theory of Hoyle and Narlikar are investigated. Exact solutions are derived by considering an arbitrary creation field function that depend generally on space and time. Exploring the physical parameters of these solutions reveal that both the inflationary scenario and the accelerated expansion of the universe are possible in the framework of C-field cosmology even if no restriction is made on the C-field function. In order to avoid the Big Bang scenario, a non-singular solution was introduced. This solution indicates that a linear C-field function gives rise to a universe of constant energy density.

Key words: Kantowski-Sachs Spacetime; General Relativity; Einstein Field Equations; Creation Field Cosmology

1 Introduction

The Big Bang theory provides the most prevailing cosmological model that describe the evolution of our actual universe. The successful predictions of the model range from the expansion of space to the existence of the cosmic microwave background radiations. Despite its extraordinary success, the Lambda-CDM model has several shortcomings due to its failure to provide satisfactory answers to some of the puzzles arising in cosmology [1].
In particular, the model gives rise to a singularity in the past (at $t = 0$) and possibly one in the future. The singularity in the beginning of the universe raises many issues that are hard to explain in the framework of the Big Bang cosmology. For example, how to explain the origin of the universe? Where did the energy in the beginning of the universe come from? These questions, along

with others, remained without satisfactory answers until the present day. Other issues of concern, with the Big Bang model, involves the horizon and the flatness problems which could be explained through the introduction of inflationary cosmology [2]. Although the theory of cosmic inflation at the very beginning of the universe settled many issues in cosmology and theoretical physics, the physical field that is supposed to cause such inflation has not yet been discovered.

In order to address these shortcomings, several alternative theories have been proposed. In this context, one of the most interesting approaches to settle such issues is provided by the creation field theory of Hoyle and Narlikar [3]. In this theory, a field theoretic approach was adopted where a chargeless and massless scalar field is implemented to explain the creation of matter. This spinless non-massive C-field is postulated to provide a mechanism for the continuous creation of matter in the universe. Such approach enables one to avoid the spontaneous explosive creation of matter at $t = 0$ in the Lambda-CDM model by suggesting that matter could be created at the expense of the negative energy of the C-field. The elimination of both past and future singularities is another advantage of this approach. Furthermore, the horizon and the flatness problems could disappear from cosmology in the framework of Hoyle-Narlikar theory.

In literature, a considerable number of studies have been conducted to obtain exact solutions of Einstein field equations (EFEs) in the presence of a scalar field. In this regard, Bianchi type spacetimes have been investigated using the creation field theory [4]. The theory has also been implemented to study plane symmetric spacetimes [5]. Furthermore, Gates and his collaborators explored Kaluza-Klein dust filled universe in the context of creation field cosmology [6]. The Friedman metric was also investigated in this formalism, see for instance [7, 8, 9].

On the other hand, Kantowski-Sachs spacetime is an interesting solution of Einstein field equations describing a spatially homogeneous and anisotropic universe [10]. In fact, the Kantowski-Sachs metric is the simplest anisotropic model which makes it a good tool for studying the early epoch of the universe. In the context of General Relativity, the Kantowski-Sachs universes have been extensively studied in literature [11, 12, 13, 14]. Singh and Chaubey have extended these research by implementing the creation field theory to explore such universes[4]. To the best of our knowledge, most of the published works in creation field theory considered a creation field function depending only on the time variable. We believe, however, that considering a more general C-field function that depends on space and time will enable us to increase our insight of the potential effects of the presence of such creation field on the geometry and evolution of Kantowski-Sachs universes. Therefore, we consider, in this paper, a more general C-field function of the form $C = C(t, r)$ aiming to provide a better explanation of the puzzles arising in cosmology.

The plan of this paper is given as follows. In the next section, we introduce the creation field theory of Hoyle and Narlikar. Then, the Einstein field equations in the presence of creation field for the Kantowski-Sachs metric is obtained in section III where exact solution of the derived equations are presented. Finally, physical interpretation of the solution and concluding remarks are discussed in the last section.

2 Creation Field Theory

Recall that the Einstein field equations of General Relativity are given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T^{(m)}_{\mu\nu}, \tag{1}$$

where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the metric tensor, $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar while $\kappa = \frac{8\pi G}{C^4}$ is Einstein gravitational constant which can be taken, in standard unit, as unity ($\kappa = 1$) and $T^{(m)}_{\mu\nu}$ is the energy-momentum tensor associated with matter. Notice that the energy-momentum tensor for perfect fluid is given as

$$T^{\mu\nu(m)} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu}, \tag{2}$$

where both the energy density ρ and the pressure p are functions of t and r in general. In Hoyle-Narlikar theory, the EFEs are modified by adding an extra term to the right hand side representing the energy-momentum tensor associated with the C-field. Therefore, the modified field equation in this formalism reads

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -(T^{(m)}_{\mu\nu} + T^{(C)}_{\mu\nu}), \tag{3}$$

where the C-field term is given by the following equation:

$$T^{(C)}_{\mu\nu} = -f(C_\mu C_\nu - \frac{1}{2}g_{\mu\nu}C^\alpha C_\alpha). \tag{4}$$

In the above equation, f represents the C-field coupling constant and $C_\mu = \frac{dC}{dx^\mu}$. Thus, the conservation of energy in this formalism is modified as

$$T^{\mu\nu(m)}_{;\mu\nu} = -T^{(C)}_{;\mu\nu} = fC^\mu C^\nu, \tag{5}$$

where the semicolon (;) denotes covariant differentiation. Recall that in the framework of C-field theory, the expansion of the universe could be explained by the presence of negative energy density due to the C-field where such energy gives rise to repulsive gravitational field.

3 Solution of the Modified Field Equations

The line element of the Kantowski-Sachs metric is given by [10]

$$ds^2 = dt^2 - R^2(t)dr^2 - S^2(t)[d\theta^2 + Sin^2\theta d\phi^2]. \tag{6}$$

It is worth noting that the 0-1 component of the field equation (3), i.e $R_{01} - \frac{1}{2}Rg_{01} = -(T^{(m)}_{01} + T^{(C)}_{01})$ is applied to metric (6), one arrives at the conclusion that $C_tC_r = 0$ which gives rise to two cases, either C_t=0 or C_r=0 or both of

them are zero. We consider these cases in details in the following analysis.
Case I : $C_t = 0$ and $C_r \neq 0$
In this case, C is a function of r only. Applying the EFEs (3) to Kantowski-Sachs metric (6) for this case gives rise to the following system:

$$\frac{S'}{S^2} + 2\frac{R'S'}{RS} + \frac{1}{S^2} = \rho - \frac{1}{2} f C_r^2, \tag{7}$$

$$\frac{2S''}{S} + \frac{S'^2}{S^2} + \frac{1}{S^2} = f(\frac{1}{2} + \frac{1}{R^2}) C_r^2 - p, \tag{8}$$

$$\frac{R'S'}{RS} + \frac{S''}{S} + \frac{R''}{R} = \frac{1}{2} f C_r^2 - p, \tag{9}$$

where the dash represents differentiation with respect to t while the partial derivatives of the creation field function C were explicitly written. In order to solve the above system, we can subtract Eq.(9) from Eq.(8) and differentiate the resulting equation with respect to r to obtain $C_r C_{rr} = 0$. Hence, $C_{rr} = 0$ since we assumed that $C_r \neq 0$. Therefore, we have a linear C-field function, i.e

$$C(r) = \lambda r + \alpha, \tag{10}$$

where λ and α are real numbers. We can find solution to the field equations by inspection. For instance, if one assumes that $R = R_0$ and $S = Cosh(t)$ then it is straightforward to show that the field equations for this assumption gives rise to constant energy density and constant pressure where $\rho = 1 - R_0^2$ and $p = R_0^2 - 1$. A realistic physical solution requires that $R_0 \in (-1, 1)$ so that the energy density is positive and consequently the pressure will be negative. Also, for this solution to satisfy all the field equations (Eqs.(7) to (9)), we need to have $\lambda = \sqrt{\frac{2}{f}} R_0$. Therefore, the C-field function (10) can be re-written as

$$C(r) = \sqrt{\frac{2}{f}} R_0 r + \alpha. \tag{11}$$

The resulting metric for this solution is given by

$$ds^2 = dt^2 - R_0^2 dr^2 - Cosh^2(t)(d\theta^2 + Sin^2\theta d\phi^2), \tag{12}$$

and the corresponding physical parameters are shown in table 1 below.

The Metric ds^2	$dt^2 - R_0^2 dr^2 - Cosh^2(t)(d\theta^2 + Sin^2\theta d\phi^2)$.
Volume V	$R_0 Cosh^2(t)$
Scale Factor $a(t)$	$\sqrt[3]{R_0 Cosh^2(t)}$
Directional Hubble Parameters	$H_1 = 0, \qquad H_2 = H_3 = Tanh(t)$
Mean Hubble Parameter H	$\frac{2}{3} Tanh(t)$
Expansion Scalar θ	$2Sech(t)Sinh(t)$
Deceleration Parameter q	$\frac{1}{2} - \frac{3}{2} Coth^2(t)$

Table 1: Physical parameters of the first Solution (metric (12))

Case II : $C_r = 0$ **and** $C_t \neq 0$

In this case, C is a function of t only. Similarly, both the energy density and the pressure will be independent of r. The field equations (7) to (9) is reduced to the following system:

$$\frac{S'^2}{S^2} + 2\frac{R'S'}{RS} + \frac{1}{S^2} = \rho, \tag{13}$$

$$\frac{2S''}{S} + \frac{S'^2}{S^2} + \frac{1}{S^2} = -p, \tag{14}$$

$$\frac{R'S'}{RS} + \frac{S''}{S} + \frac{R''}{R} = -p. \tag{15}$$

We can now make further assumptions to obtain solutions for the above system. In particular, we will consider a solution in the form of power function and another solution in terms of exponential function as shown below.

Case II-1 Solution in terms of Power Functions

Assume $R = t^m$ and $S = t^n$. Then, subtracting Eq.(15) from Eq.(14) after substituting the assumed values of R and S leads to the following equation:

$$t^{2-2n} + 2n^2 - 2n - mn + n - m^2 + m = 0. \tag{16}$$

The above equation makes sense only if $n = 1$ and $m = \pm\sqrt{2}$. As a result, $S = t$ and $R = t^{\pm\sqrt{2}}$ giving rise to the following metric:

$$ds^2 = dt^2 - t^{\pm 2\sqrt{2}} dr^2 - t^2(d\theta^2 + Sin^2\theta d\phi^2). \tag{17}$$

Consequently, the energy density and pressure are given as

$$\rho = \frac{1}{2} f C'^2(t) + \frac{2 \pm 2\sqrt{2}}{t^2}, \tag{18}$$

$$p = \frac{1}{2} f C'^2(t) \mp \frac{2}{t^2}. \tag{19}$$

The parameters of the above solutions are given in table 2 below.

The Metric ds^2	$dt^2 - t^{\pm 2\sqrt{2}} dr^2 - t^2(d\theta^2 + Sin^2\theta d\phi^2)$.
Volume V	$t^{2\pm\sqrt{2}}$
Scale Factor $a(t)$	$t^{\frac{2\pm\sqrt{2}}{3}}$
Directional Hubble Parameters	$H_1 = \pm\frac{\sqrt{2}}{t}, \quad H_2 = H_3 = \frac{1}{t}$
Mean Hubble Parameter H	$\frac{2\pm\sqrt{2}}{3t}$
Expansion Scalar θ	$\frac{2\pm\sqrt{2}}{t}$
Deceleration Parameter q	$\frac{1\mp\sqrt{2}}{2\pm\sqrt{2}}$

Table 2: Physical parameters of the second Solution (metric (17))

Obviously, the equation of energy density (18) and pressure (19) are generally valid for arbitrary creation field function of time. However, we can determine the function $C(t)$ explicitly if we implement further assumption. For example, if we assume the energy density is constant, i.e $\rho = \rho_0$, then we can write the creation field function as

$$C(t) = \sqrt{\frac{2\rho_0}{f}t^2 - \frac{4 \pm 2\sqrt{2}}{f}} - \sqrt{\frac{4 \pm 2\sqrt{2}}{f}} Tan^{-1} \sqrt{\frac{2\rho_0}{4 \pm 2\sqrt{2}}t^2 - 1} + c_1, \quad (20)$$

which is an increasing function of time.

Case II-2 Solution in terms of Exponential Functions

We can develop a non-singular solution by implementing the exponential function. In particular, by assuming that $S = e^{At}$ and $R = e^{Bt}$ and substituting these values in the EFEs, one can easily reach the conclusion that: $A = 0$ and $B = \pm 1$. Consequently, $S = 1$ and $R = e^{\pm t}$ leading to the following metric:

$$ds^2 = dt^2 - e^{\pm 2t} dr^2 - (d\theta^2 + Sin^2\theta d\phi^2). \quad (21)$$

Hence, the energy density and pressure are equal and can be written as

$$\rho = p = \frac{1}{2} f C'^2(t) - 1, \quad (22)$$

The parameters corresponding to solution (21) are shown table 3.

The Metric ds^2	$dt^2 - e^{\pm 2t} dr^2 - (d\theta^2 + Sin^2\theta d\phi^2).$
Volume V	$e^{\pm t}$
Scale Factor $a(t)$	$e^{\pm \frac{t}{3}}$
Directional Hubble Parameters	$H_1 = \pm 1, \quad H_2 = H_3 = 0$
Mean Hubble Parameter H	$\pm \frac{1}{3}$
Expansion Scalar θ	$\frac{1}{2}$
Deceleration Parameter q	-1

Table 3: Physical parameters of the third Solution (metric (21))

Similar to the previous case, the creation field function is arbitrary but it can be determined explicitly if we assume the energy density is constant, i.e $\rho = \rho_0$. In this case, the C-function becomes

$$C(t) = \sqrt{\frac{2(\rho_0 + 1)}{f}} t + c_1, \quad (23)$$

which is linear and increasing.

Case III : $C_t = C_r = 0$
In this case, C is a pure constant, i.e $C = C_0$. One can find a solution for the EFEs (Eq.(7) to (9)) by implementing hyperbolic functions. In particular, if we assume $R = Sinh(t)$ and $S = Cosh(t)$, the resulting metric can be written as

$$ds^2 = dt^2 - Sinh^2(t)dr^2 - Cosh^2(t)(d\theta^2 + Sin^2\theta d\phi^2). \tag{24}$$

This assumption also gives rise to constant energy density and constant pressure where $\rho = 3$ and $p = -3$. The physical parameters of the obtained solution are given in table 4.

The Metric ds^2	$dt^2 - Sinh^2(t)dr^2 - Cosh^2(t)(d\theta^2 + Sin^2\theta d\phi^2)$.
Volume V	$Sinh(t)Cosh^2(t)$
Scale Factor $a(t)$	$\sqrt[3]{Sinh(t)Cosh^2(t)}$
Directional Hubble Parameters	$H_1 = Cosh(t), \quad H_2 = H_3 = Tanh(t)$
Mean Hubble Parameter H	$\frac{Coth(t)+2Tanh(t)}{3}$
Expansion Scalar θ	$Coth(t) + 2Tanh(t)$
Deceleration Parameter q	$\frac{9}{2Sinh^{2/3}(t)Cosh^{4/3}(t)}[Cosh^2(t) + Tanh^2(t) - 6.5]$

Table 4: Physical parameters of the fourth Solution (metric (24))

4 Discussion

Kantowski-Sachs spacetimes in the framework of creation field cosmology of Hoyle and Narlikar have been considered where four interesting solutions were obtained. In this context, our calculations show that the corresponding C-field can either be a function of time or a function of space but cannot depend on both variables simultaneously. In particular, the creation field can only admit a linear function of r but it has no restriction when expressed in terms of the time variable as it permits an arbitrary function of time in general. However, while the energy density and pressure, given by Eqs(18), (19) and (22), involve arbitrary C-field function of t, our calculations interestingly show that if the energy density is assumed to be constant, then the corresponding C-field is restricted to increasing functions of time which is in agreement with the results obtained by Hoyle and Narlikar [15]
The first solution which correspond to metric (12) admits an inflationary scenario if $R_0 \in (0, 1)$ since the scale factor in this interval is increasing function of time. Furthermore, the deceleration parameter is always negative which indicates an accelerating universe. It is also worth noting that the relation between the constant energy density and the constant pressure for this solution is given as $p = -\rho$. Recall that this equation of state, in the context of Big Bang cosmology, describes a universe dominated by dark energy. However, in the creation

field theory, dark energy is not needed to produce negative pressure.
Similarly, the second solution (17) leads to inflationary scenario and can describe both an accelerating and decelerating universes depending on the metric coefficient R. In particular, the deceleration parameter q is negative for the metric $ds^2 = dt^2 - t^{2\sqrt{2}}dr^2 - t^2(d\theta^2 + Sin^2\theta d\phi^2)$ and positive for the metric $ds^2 = dt^2 - t^{-2\sqrt{2}}dr^2 - t^2(d\theta^2 + Sin^2\theta d\phi^2)$. Hence, the former metric describes an accelerating universe while the later characterises a decelerating one. Furthermore, the relation between energy density and pressure is given by

$$p = \rho - \frac{4 \pm \sqrt{2}}{t^2}, \tag{25}$$

which imply that the pressure and energy density tend to be equal as t approaches infinity. This indicates that at the far future, the equation of state (25) describes a universe filled with stiff-fluid.
On the other hand, the third solution (21) was free of singularities and thus does not require a big bang scenario. This means that the universe has always existed and consequently there is no need for a mechanism of spontaneous creation as in the Lambda-CDM model. The evolution of the scale factors of both metrics (21) indicates that the inflationary scenario is possible for the first metric $ds^2 - e^t dr^2 - (d\theta^2 + Sin^2\theta d\phi^2)$ in the period $t \in (0, \infty)$ while inflation takes place for the other metric $ds^2 = dt^2 - e^{-t}dr^2 - (d\theta^2 + Sin^2\theta d\phi^2)$ in the period $t \in (-\infty, 0)$. Furthermore, the pressure and energy density are always equal for this solution giving rise to a universe of stiff fluid.
Finally, the fourth solution (24) was constructed by assuming a constant C-field function. This solution characterises a universe with inflationary scenario and admits an equation of state of the form $p = -\rho$. Interestingly, the universe described in this case gives rise to a scenario in which the expansion is accelerating for a short period of time after the big bang then decelerating afterwards as the deceleration parameter q is generally positive and can be negative only for small t (roughly $t < 1.5$).

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