Exact Solutions of Kantoweski-Sachs spacetimes in the Framework of Creation-Field Cosmology

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Abstract

The Kantoweski-Sachs spacetimes in the context of creation field theory of Hoyle and Narlikar are investigated. Exact solutions are derived by considering an arbitrary creation field function of two independent variables. Exploring the physical parameters of these solutions reveal that both the inflationary scenario and the accelerated expansion of the universe are possible in the framework of C-field cosmology even if no restriction is made on the C-field function. In order to avoid the Big Bang scenario, a non-singular solution was introduced. This solution indicates that a linear C-field function gives rise to a universe of constant energy density.

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1. Introduction

The Big Bang theory provides the most prevailing cosmological model that describe the evolution of our actual universe. The successful predictions of the model range from the expansion of space to the existence of the cosmic microwave background radiations. Despite its extraordinary success, the Lambda-CDM model has several shortcomings due to its failure to provide satisfactory answers to some of the puzzles arising in cosmology.

In particular, the model gives rise to a singularity in the past (at $t = 0$) and possibly one in the future. The singularity in the beginning of the universe raises many issues that are hard to explain in the framework of the Big Bang cosmology. For example, how to explain the origin of the universe? Where did the energy in the beginning of the universe come from? These questions, along with others, remained without satisfactory answers until the present day. Other issues of concern, with the Big Bang model, involves the horizon and the flatness problems which could be explained through the introduction of inflationary cosmology [1]. Although the theory of cosmic inflation at the very beginning of the universe settled many issues in cosmology and theoretical physics, the physical field that is supposed to cause such inflation has not yet been discovered.

In order to address these shortcomings, several alternative theories have been proposed. In this context, one of the most interesting approaches to settle such issues is provided by the creation field theory of Hoyle and Narlikar [2]. In this theory, a field theoretic approach was adopted where a chargeless and massless scalar field is implemented to explain the creation of matter. This approach enables one to avoid the spontaneous explosive creation of matter at $t = 0$ in the Lambda-CDM model by suggesting that matter could be created at the expense of the negative energy of the C-field. The elimination of both past and future singularities is another advantage of this approach. Furthermore, the horizon and the flatness problems could disappear from cosmology in the framework of Hoyle-Narlikar theory.

In literature, a considerable number of studies have been conducted to obtain exact solutions of Einstein field equations (EFEs) in the presence of a scalar field. In this regard, Bianchi type spacetimes have been investigated using the creation field theory [3]. The theory has also been implemented to study plane symmetric spacetimes[4]. Furthermore, Gates and his collaborators explored Kaluza-Klein dust filled universe in the context of creation field cosmology [5]. The Friedman metric was also investigated in this formalism, see for instance [6][7][8].

On the other hand, Kantoweski-Sachs spacetime is an interesting solution of Einstein field equations describing a spatially homogeneous and anisotropic universe. In fact, the Kantoweski-Sachs metric is the simplest anisotropic model which makes it a good tool for studying the early epoch of the universe. In the context of General Relativity, the Kantoweski-Sachs universes have been extensively studied in literature [8][9][10][11]. Singh and Chaubey have extended these research by implementing the creation field theory to explore such universes [3].

To the best of our knowledge, most of the published works in creation field theory considered a creation field function depending only on the time variable. We believe, however, that considering a more general C-field function that depends on space and time will enable us to increase our insight of the potential effects of the presence of such creation field on the geometry and evolution of Kantoweski-Sachs universes. Therefore, we consider, in this paper, a more general C-field function depending on two variables, i.e $C = C(t, r)$ aiming to provide a better explanation of the puzzles arising in cosmology without having to invoke the assumptions made to reconcile our observation with General Relativity.

The plan of this paper is given as follows. In the next section, we introduce the creation field theory of Hoyle and Narlikar. Then, the Einstein field equations in the presence of creation field for the Kantoweski-Sachs metric is obtained in section III. Exact solution of the derived equations are presented in section IV where physical interpretation of the solution are also discussed. Finally, concluding remarks are given in the last section.

2. Creation Field Theory

Recall that the Einstein field equations of General Relativity are given by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T^{(m)}_{\mu\nu}, \qquad (1)$$

where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the metric tensor, $T^{(m)}_{\mu\nu}$ is the energy-momentum tensor associated with matter while $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar and $\kappa = \frac{8\pi G}{C^4}$ is Einstein gravitational constant which can be taken, in standard unit, as unity ($\kappa = 1$). In Hoyle-Narlikar theory, the EFEs are modified by adding an extra term to the right hand side representing the energy-momentum tensor associated with the C-field. Therefore, the modified field equation in this formalism is given by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T^{(m)}_{\mu\nu} + T^{(C)}_{\mu\nu}, \qquad (2)$$

whereas the C-field term is given by the following equation:

$$T^{(C)}_{\mu\nu} = -f\left(C_i C_k - \frac{1}{2} g_{ik} C^{\alpha} C_{\alpha}\right), \qquad (3)$$

where f is the C-field coupling constant and $C_i = \frac{dC}{dx^i}$. Thus, the conservation of energy in this formalism is modified as

$$T^{(m)ik}_{ik} = -T^{(C)}_{ik} = f C^i C^k. \qquad (4)$$

Recall that in the framework of C-field theory, the expansion of the universe could be explained by the presence of negative energy density due to the C-field where such energy gives rise to repulsive gravitational field.

3. Modified EFEs of Kantoweski-Sachs Spacetime

The line element of the Kantoweski-Sachs metric is given by

$$ds^2 = dt^2 - R^2(t)dr^2 - S^2(t)[d\theta^2 + Sin^2\theta d\phi^2]. \qquad (5)$$

Recall that the energy-momentum tensor for perfect fluid is given as

$$T^m_{ij} = (\rho + p)u_i u_j - pg_{ij} = (\rho, -p, -p, -p), \qquad (6)$$

where ρ is the energy density and p is the pressure. Applying the modified field equations (2) to metric (5) leads to the following system:

$$\frac{S'}{S^2} + 2\frac{R'S'}{RS} + \frac{1}{S^2} = \frac{1}{2}f[C_t^2 - C_r^2] - \rho, \qquad (7)$$

$$\frac{2S''}{S} + \frac{S'^2}{S^2} + \frac{1}{S^2} = \frac{1}{2}f[C_t^2 - C_r^2] - p, \qquad (8)$$

$$\frac{R'S'}{RS} + \frac{S''}{S} + \frac{R''}{R} = \frac{1}{2}f[C_t^2 + C_r^2] - p, \qquad (9)$$

where the dash in the left hand side represents differentiation with respect to t while the partial derivatives in the right hand side were written explicitly since the function C depends on two variables t and r.

4. Solution of the Modified Field Equations

In order to obtain exact solutions of the system of field equations (Eqs.(7) to (9)), we differentiate (8) and (9) with respect to r to obtain the following equations:

$$0 = \frac{1}{2}f[2C_tC_{tr} - 2C_rC_{rr}] - p_r, \qquad (10)$$

$$0 = \frac{1}{2}f[2C_tC_{tr} + 2C_rC_{rr}] - p_r. \qquad (11)$$

Subtracting Eq.(10) from Eq.(11) leads to the following equation:

$$C_rC_{rr} = 0. \qquad (12)$$

Notice that Eq.(12) gives rise to two cases, either $C_{rr} = 0$ or $C_r = 0$. We discuss these cases in details below.

Case I : $C_{rr} = 0$ **with** $C_r \neq 0$

This case gives rise to a linear creation field function of the form

$$C(t, r) = a(t)r + b(t). \qquad (13)$$

Subtracting Eq.(8) from (7) yields

$$2\frac{R'S'}{RS} - 2\frac{S''}{S} = p - \rho. \qquad (14)$$

Now we can implement additional assumption to simplify the equations. For instance, if we assume the universe is filled with a stiff fluid we have $p = \rho$ which simplifies the above equation to

$$2\frac{R'S'}{RS} - 2\frac{S''}{S} = 0. \qquad (15)$$

Integrating Eq.(15) leads to the following relation between metric coefficients S and R:

$$R = S'. \qquad (16)$$

Also, if we subtract Eq.(9) from (8) and by using (16), we can write the resulting equation as

$$\frac{S'^2}{S^2} + \frac{1}{S^2} - \frac{S'''}{S'} = -fa^2(t). \qquad (17)$$

At this stage, we look for a particular solution for Eq.(17). By inspection, one possible solution is given by $S = Cosh(kt)$ leading to $a(t) = \sqrt{\frac{k^2-1}{f}}Sech(kt)$, where $k > 1$. Also, from Eq.(16) we conclude that $R = kSinh(kt)$. Consequently, the creation field function in this case becomes

$$C(t,r) = \sqrt{\frac{k^2-1}{f}}Sech(kt)r + b(t), \qquad (18)$$

and the energy density (and pressure) are given by

$$\rho = p = \frac{1}{2}f[(k\sqrt{\frac{k^2-1}{f}}Sech(kt)Tankh(kt))r + b'(t)]^2 + \frac{k^2-1}{2}Sech^2(kt) - 3k^2. \qquad (19)$$

In order to determine the function $b(t)$ in Eq.(19) explicitly , we need further assumption. In particular, if we assume the energy density is infinite at $t = 0$ and constant for large t, i.e $\rho \rightarrow \rho_0$ as $t \rightarrow \infty$ then we must have $b'^2(0) = \infty$ and $b'^2(\infty) = \rho_0$. By inspection, a possible function satisfying these two conditions is $b(t) = Int + \rho_0 t + c_2$. In this case, Eq.(19) can be written as

$$\rho = p = \frac{1}{2}f[(k\sqrt{\frac{k^2-1}{f}}Sech(kt)Tankh(kt))r) + \frac{1}{t} + \rho_0]^2 + \frac{k^2-1}{2}Sech^2(kt) - 3k^2, \qquad (20)$$

and the C-field function becomes

$$C(t,r) = \sqrt{\frac{k^2-1}{f}}Sech(kt)r + Int + \rho_0 t + c_2. \qquad (21)$$

The resulting metric for this case can be written as

$$ds^2 = dt^2 - k^2 Sinh^2(kt)dr^2 - Cosh^2(kt)(d\theta^2 + Sin^2\theta d\phi^2). \qquad (22)$$

The parameters of metric (22) are given in table 1 below.

The Metric ds^2	$dt^2 - k^2 Sinh^2(kt)dr^2 - Cosh^2(kt)(d\theta^2 + Sin^2\theta d\phi^2).$
Volume V	$kSinh(kt)Cosh^2(kt)$
Scale Factor $a(t)$	$\sqrt[3]{kSinh(kt)Cosh^2(kt)}$
Directional Hubble Parameters	$H_1 = kCosh(kt), \qquad H_2 = H_3 = kTanh(kt)$
Mean Hubble Parameter H	$\frac{kCoth(kt)+2kTanh(kt)}{3}$
Expansion Scalar θ	$kCoth(kt) + 2kTanh(kt)$
Deceleration Parameter q	$\frac{9}{2Sinh^{2/3}(kt)}[Cosh^2(kt) + Tanh^2(kt) - 6.5]$

Table 1. Physical parameters of Solution 1 (metric (22))

Notice that the scale factor of metric (22) increases with time indicating that inflationary scenario is possible. Furthermore, the creation field function C is increasing in time while its r-component becomes negligible as $t \rightarrow \infty$. It is also interesting to note that the decelerating parameter is positive only for small t (in particular if kt is roughly smaller that 0.5), then it becomes negative for larger t which indicates that the expansion of the universe was decelerating for a short time after the Big Bang then it started to accelerate afterwards. Finally, it is worth noting that the accelerated expansion of the universe will almost stop in this scenario in the far future since $q \rightarrow 0$ as $t \rightarrow \infty$.

Case II : $C_r = 0$

In this case, the field equations (7) to (9) is reduced to the following system:

$$\frac{S'^2}{S^2} + 2\frac{R'S'}{RS} + \frac{1}{S^2} = \frac{1}{2}fC'^2(t) - \rho, \qquad (23)$$

$$\frac{2S''}{S} + \frac{S'^2}{S^2} + \frac{1}{S^2} = \frac{1}{2}fC'^2(t) - p, \qquad (24)$$

$$\frac{R'S'}{RS} + \frac{S''}{S} + \frac{R''}{R} = \frac{1}{2} fC'^2(t) - p. \qquad (25)$$

We can now make further assumptions to obtain solutions for the above system. In particular, we will consider a solution in the form of power function and another solution in terms of exponential function as shown below.

Case II-1 Solution in terms of Power Functions

Assume $S = t^m$ and $R = t^n$. Then, subtracting Eq.(25) from Eq.(24) leads to

$$t^{2-2m} + m^2 + m(m-1) - mn - n(n-1) = 0. \qquad (26)$$

The above equation makes sense only if $m = 1$ and $n = \pm\sqrt{2}$. As a result, $S = t$ and $R = t^{\pm\sqrt{2}}$ giving rise to the following metric:

$$ds^2 = dt^2 - t^{\pm 2\sqrt{2}} dr^2 - t^2(d\theta^2 + Sin^2\theta d\phi^2). \qquad (27)$$

Consequently, the energy density and pressure are given as

$$\rho = \frac{1}{2} fC'^2(t) - \frac{2 \pm 2\sqrt{2}}{t^2}, \qquad (28)$$

$$p = \frac{1}{2} fC'^2(t) \mp \frac{2}{t^2}. \qquad (29)$$

The parameters of the above solutions are given in table 2 below.

The Metric ds^2	$dt^2 - t^{\pm 2\sqrt{2}} dr^2 - t^2(d\theta^2 + Sin^2\theta d\phi^2).$
Volume V	$t^{2\pm\sqrt{2}}$
Scale Factor $a(t)$	$t^{\frac{2\pm\sqrt{2}}{3}}$
Directional Hubble Parameters	$H_1 = \pm\frac{\sqrt{2}}{t}, \quad H_2 = H_3 = \frac{1}{t}$
Mean Hubble Parameter H	$\frac{2\pm\sqrt{2}}{3t}$
Expansion Scalar θ	$\frac{2\pm\sqrt{2}}{t}$
Deceleration Parameter q	$\frac{1\mp\sqrt{2}}{2\pm\sqrt{2}}$

Table 2. Physical parameters of Solution 2 (metric (27))

Obviously, the scale factor of metric (27) increases with time giving rise to inflationary universe. The relation between energy density and pressure is given by

$$p = \rho - \frac{2}{t^2}, \qquad (30)$$

which imply that the value of the pressure will approach the value of energy density as time progresses. This means that at far future, the equation of state describes a stuff-fluid universe. Also, notice that the metric with positive power coefficient, i.e $ds^2 = dt^2 - t^{2\sqrt{2}}dr^2 - t^2(d\theta^2 + Sin^2\theta d\phi^2)$ is actually accelerating as it has a negative deceleration parameter whereas the universe with a negative power coefficient, i.e $ds^2 = dt^2 - t^{-2\sqrt{2}}dr^2 - t^2(d\theta^2 + Sin^2\theta d\phi^2)$, has a positive deceleration parameter giving rise to a decelerating universe. Finally, the solution corresponding to metric (27) is generally valid for arbitrary creation field function of time. However, we can determine the function $C(t)$ explicitly if we implement further assumption. For example, if we assume the energy density is constant, i.e $\rho = \rho_0$, then we can write the creation field function as

$$C(t) = \sqrt{\frac{2\rho_0}{f}t^2 - \frac{8\sqrt{2}+4}{f}} - \sqrt{\frac{8\sqrt{2}+4}{f}}\, Tan^{-1}\sqrt{\frac{\rho_0(2\sqrt{2}-1)}{14}t^2 - 1} + c, \qquad (31)$$

which is an increasing function of time.

Case II-2 Solution in terms of Exponential Functions

We can develop a non-singular solution by implementing the exponential function. In particular, by assuming that $S = e^{At}$ and $R = e^{Bt}$ and then if we subtract Eq.(25) from Eq.(24) one can easily reach the conclusion that: $A = 0$ and $B = \pm 1$. As a result, $S = 1$ and $R = e^{\pm t}$. Therefore, energy density and pressure are equal and can be written as

$$\rho = p = \frac{1}{2}fC'^2(t) - 1, \qquad (32)$$

The metric in this case is given as

$$ds^2 = dt^2 - e^{\pm 2t}dr^2 - (d\theta^2 + Sin^2\theta d\phi^2). \qquad (33)$$

The parameters corresponding to solution (33) is given in table 3 below.

The Metric ds^2	$dt^2 - e^{\pm 2t} dr^2 - (d\theta^2 + Sin^2\theta d\phi^2)$.
Volume V	$e^{\pm t}$
Scale Factor $a(t)$	$e^{\pm \frac{t}{3}}$
Directional Hubble Parameters	$H_1 = \pm 1, \quad H_2 = H_3 = 0$
Mean Hubble Parameter H	$\pm \frac{1}{3}$
Expansion Scalar θ	$\frac{1}{2}$
Deceleration Parameter q	-1

Table 3. Physical parameters of Solution 3 (metric (33))

According to the non-singular metric (33), the universe has always existed and consequently there is no need for a mechanism of spontaneous creation as in the Lambda-CDM model. The evolution of the scale factors of both metrics (33) indicates that the inflationary scenario is possible for the first metric $ds^2 - e^t dr^2 - (d\theta^2 + Sin^2\theta d\phi^2)$ in the period $t \in (0, \infty)$ while inflation takes place for the other metric $ds^2 = dt^2 - e^{-t}dr^2 - (d\theta^2 + Sin^2\theta d\phi^2)$ in the period $t \in (-\infty, 0)$. Notice that the mean Hubble parameter in this case is constant. Furthermore, the solution describes a universe of stiff fluid since the energy density and pressure are equal. Deceleration parameter is always negative indicating a universe of accelerating expansion. Similar to the previous case, the creation field function is arbitrary but it can be determined explicitly if we assume the energy density is constant, i.e $\rho = \rho_0$. In this case, the C-function becomes

$$C(t) = \sqrt{\frac{2(\rho_0 + 1)}{f}}\, t + c_1, \qquad (34)$$

which is linear and increasing.

5. Conclusion

In this work, Kantoweski-Sachs spacetimes in the framework of creation field cosmology of Hoyle and Narlikar has been considered where three interesting solutions have been obtained. While most of the researchers restricted their attention to a creation field function depending on time variable only, we addressed in this paper a more general function that depends on time and space. In this context, the C-field function (21) which corresponds to the first obtained metric (22) depends on t and r. However, for large values of t, the r component of the function C becomes negligible. While the first solution (22) describes an expanding universe with infinite energy density at initial moment, the density will approach a constant value as time progresses. Interestingly, the first solution gave rise to a scenario in which the expansion of the universe is decelerating for short period of time after the big bang then it will start to increase afterwards. In this scenario, the acceleration will never stop but it will be very small in the far future.

On the other hand, the second solution (27) can describe both an accelerating and decelerating universes depending on the sign of the power of the function R in the metric coefficient. While the relation (30) between energy density and pressure is generally not linear, it tends to a linear EoS for large t. Although the solution (27) is generally valid for arbitrary creation field function C, our calculation showed that a universe with constant energy density gives rise to an increasing C-field function as one can infer from Eq.(31).

Finally, the third solution (33) was free of singularities and thus does not require a big bang scenario. It is worth noting that inflationary scenario was possible for both metrics given by Eq.(33) but it took place in different periods of time. In particular, the metric with $R = e^t$ has inflationary scenario in the period $(0, \infty)$ while the metric with $R = e^{-t}$ has inflation in the period $(-\infty, 0)$. As in the previous solution, the third solution (33) is valid for arbitrary C-field function of time and it would give rise to a universe of constant energy density if the C-field function is linear.

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Ethical Approval

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