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A trial-dependent N-player game

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Abstract

The trials in the classical N-player gamblers' problem are repeated independently until one or more players get bankrupt. In this modern era, everyone wants to earn something in a limited amount of time as well as doesn't lose all his/her amount. In this research, we present a game that is stopped when the number of trials first reaches the minimum of the initial budget set by the players. We executed this game for N players and determined the probability distribution of the fortune for both symmetric and asymmetric games. The exact expressions for the expected fortune and variance of the distribution are derived.

Keywords: N-player game; fixed trials; expected fortune; variance of the game.

1 Introduction

A classical N-player game is a fundamental problem in the history of probability theory. It is a sequence of independent trials and in each trial, a winner player A_i, who starts with an initial budget a_i ($a_i \in \mathbb{N}$) dollars, receives one dollar with probability p_i from each of other $N-1$ players. The main concern lies in the anticipation of the ruined time to complete this game until one or more players go bankrupt.

It is well known for the 2-player gambler's ruin problem as:

$$\mathbb{E}(T) = a_1 a_2,$$

when $p_1 = p_2 = 1/2$, and

$$\mathbb{E}(T) = \left(\frac{1}{p_1 - p_2}\right)\left(a_2 - (a_1 + a_2)\left(\frac{1 - (p_1/p_2)^{a_2}}{1 - (p_1/p_2)^{a_1+a_2}}\right)\right),$$

when $p_1 \neq p_2$, see, for example, Feller (1968).

In the early 1960s, the expected ruin time for more than two players was claimed to be an unsolved problem by Ogilvy (1962). After almost three decades, Sandell (1989) broke this stalemate and offered the following expression for a 3-player game through martingale and Optional Stopping theorem.

$$\mathbb{E}(T) = \frac{a_1 a_2 a_3}{a_1 + a_2 + a_3 - 2}.$$

The above formula is valid for a symmetric game only, and for the asymmetric game, the result is still waiting to be solved, but recently Hussain et al. (2021) suggested an approximation to solve it in the following way:

$$\mathbb{E}(T) \approx 1 + \frac{\lambda_1}{\lambda_1 - \lambda_2 - 2\lambda_3},$$

where,

$$\lambda_1 = (a_1 + 2)(a_2 - 1)(a_3 - 1)p_1 + (a_1 - 1)(a_2 + 2)(a_3 - 1)p_2 + (a_1 - 1)(a_2 - 1)(a_3 + 2)p_3,$$

$$\lambda_2 = (a_1 + 4)(a_2 - 2)(a_3 - 2)p_1^2 + (a_1 - 2)(a_2 + 4)(a_3 - 2)p_2^2 + (a_1 - 2)(a_2 - 2)(a_3 + 4)p_3^2,$$

and,

$$\lambda_3 = (a_1 + 1)(a_2 + 1)(a_3 - 2)p_1p_2 + (a_1 + 1)(a_2 - 2)(a_3 + 1)p_1p_3 + (a_1 - 2)(a_2 + 1)(a_3 + 1)p_2p_3.$$

There are several variants available in the literature about 3-player game. Engel (1993) devised the three-tower problem, a popular modification to the 3-player game. In the adjacent past, Hussain (2022) proposed a variant for the classical 3-player game by involving the interest only in a specific player, who is playing against the other two adversaries. He derived the expressions for ruin probability and expected time to ruin for both symmetric and asymmetric games. Recently, Diaconis and Ethier (2022) suggested another variant for the game of three players. In their variant, two players are chosen at random for each round and one dollar is transferred from loser to winner based on their probabilities. Eventually, one of the players is eliminated, and the game still continues as a simple 2-player game. For a heavy total budget, they have discussed six different approaches to solve this game, i.e. exact computation by using the Markov chain, arbitrarily precise computation by using Jacobi iteration, linear interpolation from exact probabilities, the Monte Carlo approach, regression on an independent chip model (ICM), and approximation by Brownian motion.

The literature about gamblers' ruin problem for $N > 3$ is even more limited. Chang (1995) used the same idea as Sandell (1989) and solved the 4-player game by using two different expressions. However, his expressions are not in closed form. In a similar vein, Cho (1996) derived an expected ruin time for a 5-player game but his expression is also restricted by the same issue. Finally, Rocha and Stern (1999, 2004) proposed generalizations of N-player game with an equal initial budget.

In contrast with the classical gambler's ruin problem, this research creates a new direction for the game. It is a pure stochastic version but stops the game when its trials reaches the minimum initial budget involved by players. A similar study but with a different perspective was conducted for the gambler's problem, where the game continues as usual until the number of trials reaches a given magnitude called cover time by Chong et al. (2000). Recently, Perotto et al. (2021) suggested a decision-based gambling process, where a player can quit the game at any round and the game should be stopped there. A player can quit the game before ruining with suggesting an optimal strategy to stop the game was studied by Ankomah et al. (2020).

The major goal of the suggested version is to include the players who have a lower chance of winning. The length of the traditional game makes it monotonous for the spectators at times. For instance, the predicted number of trials in a two-person game when players have 10 and 15 bucks in their hands, respectively, is 150. By employing the new game plan, we can increase audience interest and that of the weaker player, who has a smaller chance of winning or starts the game with fewer stakes. For instance, a player who enters the game with the lowest stakes of all the participants will only lose all of the stakes if and only if he/she is unsuccessful in each trial of the game.

In the rest of the article, we demonstrate the theoretical foundation of the newly devised game with the main results in Section 2. In the next Section 3, we valuate the proposed strategy by using numerical findings when three and ten players are involved in the game. In the last Section 4, we provide a summary of the study.

2 Theoretical foundations

2.1 Problem formulation

Consider a contest with N players, denoted A_i, $i = 1, 2, \cdots, N$, each with their own initial budget, $a_1, a_2, \cdots, a_N$, respectively. In each trial, the player A_i is chosen as the winner with the probability p_i, where $\sum_{i=1}^{N} p_i = 1$, and get a dollar from each of the remaining $N-1$ gamblers. Furthermore, we assume that the game is over when the trials reach a minimum of a_i, say, $a_i = \alpha$. To conduct the analysis for this game plan, we divide the players into two groups: G_1, which consists of the players with the smallest initial budget, i.e. α, and all other players with an initial budget greater than α, should be in the group G_2, which is also the complement of the group G_1. In other words, G_1 is a set of one, two, or up to N players based on their initial budget α, e.g. $G_1 = \{A_i\}$, whereas G_2 is a set of $N-1$ players excluding player A_i. Another example of groups is $G_1 = \{A_1, A_2, \cdots, A_N\}$ if they all start with the same budget, i.e. α, whereas $G_2 = \phi$. Also, let the initial budget of a player A_i, who belongs to group G_2, be β. Our goals in this game plan are to determine the expected fortune and variance of a player A_i who belongs to group G_1 or G_2. To solve it, we examine the game from two perspectives: the player A_i from group G_1 or the player A_i from group G_2, each with his/her own initial budget and winning probability.

2.2 The distribution of the fortune

Let $\omega = (x_1, x_2, \cdots, x_\alpha)$ be the sequence of α independent trials, where x_l, $\forall\ l \in \{1, 2, \cdots, \alpha\}$, follows as the following indicator variable:

$$X_l = \begin{cases} N-1; & p_i \\ -1; & 1-p_i. \end{cases}$$

For a given α, the sample space consists of all 2^α such sequences. Another way to define it as a stochastic process $\{\Theta_\alpha\}$, we set

$$\Theta_\alpha = \alpha + X_1 + X_2 + \cdots + X_\alpha,$$

where, $X_0 = \alpha$. In this case, the fortune distribution is binomial, so no success in α trials by a player A_i with a probability of $(1-p_i)^\alpha$, one success means the player will finish the game with a probability of $p_i(1-p_i)^{\alpha-1}$, and so on.

Proposition 1 *Let $\{\Theta_\alpha\}$ be a simple random walk that terminates after a fixed number of trials, i.e. α. Let y represents the number of successes, such that $y \in \{0, 1, \cdots, \alpha\}$. The expected fortune for a player A_i in group G_1 is then:*

$$\mathbb{E}(A_i) = \sum_{y=0}^{y=\alpha} Ny \binom{\alpha}{y} p_i^y (1-p_i)^{\alpha-y}, \tag{1}$$

and the variance of the fortune as:

$$\mathbb{V}(A_i) = \sum_{y=0}^{y=\alpha} \Big(Ny - \mathbb{E}(A_i)\Big)^2 \binom{\alpha}{y} p_i^y (1-p_i)^{\alpha-y}. \tag{2}$$

where, N is the number of players involved in the game.

Corollary 1.1 *For a symmetric game, the expected fortune equals the initial budget of the player A_i, who belongs to the group G_1. We have*

$$\mathbb{E}(A_i) = \alpha.$$

Corollary 1.2 *For a symmetric game, the variance of the fortune is equal to the (N-1) times of α. We have*

$$\mathbb{V}(A_i) = (N-1)\alpha.$$

Proposition 2 *Let $\{\Theta_\alpha\}$ be a simple random walk that stops after a fixed number of trials, i.e. α. Let z be the number of successes, such that $z \in \{0, 1, \cdots, \alpha\}$. Then the expected fortune for a player A_i in group G_2 is:*

$$\mathbb{E}(A_i) = \sum_{z=0}^{z=\alpha} \Big((\beta - \alpha) + Nz\Big) \binom{\alpha}{z} p_i^z (1-p_i)^{\alpha-z}, \tag{3}$$

and the variance of the fortune as:

$$\mathbb{V}(A_i) = \sum_{z=0}^{z=\alpha} \Big((\beta - \alpha) + Nz - \mathbb{E}(A_i)\Big)^2 \binom{\alpha}{z} p_i^z (1-p_i)^{\alpha-z}, \tag{4}$$

where, β is the initial budget by a player A_i, who belongs to group G_2.

Corollary 2.1 *In a symmetric game, the expected fortune equals the player's initial budget (for a player who belongs to the group G_2). We have*

$$\mathbb{E}(A_i) = \beta.$$

Corollary 2.2 *In a symmetric game, the variance of a player's fortune equals (N-1) times α. We have*

$$\mathbb{V}(A_i) = (N-1)\alpha.$$

3 Numerical investigation

Using the respective expressions, proposed in the preceding section, we examine the numerical findings for the newly devised approach to the N-player game in this section. Table 1 comprehends the expected fortune and variance of fortune for a 3-player game with any arbitrary initial budget set by the players. We present the results for both cases, i.e. symmetric and asymmetric games. For a symmetric game, someone can verify the results with propositions. In a similar vein, we extend our results for a 10-player game for both cases (symmetric and asymmetric) in Table 2 with providing any arbitrary initial budget by the respective players. One may also appreciate that the findings in Table 2 also agree with our propositions. Based on these numerical investigations, any contender can easily find his/her expected fortune and variability against it before entering the game.

Table 1: Expected fortune for 3-player game with symmetric ($p_i = 1/3,\ i = 1, 2, 3$) and asymmetric ($p_1 = 0.4, p_2 = 0.35, p_3 = 0.25$) games.

			Symmetric						Asymmetric					
a_1	a_2	a_3	$\mathbb{E}(A_1)$	$\mathbb{E}(A_2)$	$\mathbb{E}(A_3)$	$\mathbb{V}(A_1)$	$\mathbb{V}(A_2)$	$\mathbb{V}(A_3)$	$\mathbb{E}(A_1)$	$\mathbb{E}(A_2)$	$\mathbb{E}(A_3)$	$\mathbb{V}(A_1)$	$\mathbb{V}(A_2)$	$\mathbb{V}(A_3)$
5	5	8	5	5	8	10	10	10	6	5.25	6.75	10.8	10.2	8.4
5	7	9	5	7	9	10	10	10	6	7.25	7.75	10.8	10.2	8.4
8	7	10	8	7	10	14	14	14	9.4	7.35	8.25	15.1	14.3	11.8
10	10	10	10	10	10	20	20	20	12	10.5	7.5	21.6	20.5	16.9
10	12	10	10	12	10	20	20	20	12	12.5	7.5	21.6	20.5	16.9
15	14	12	15	14	12	24	24	24	17.4	14.6	9	25.9	24.6	20.3

Table 2: Expected fortune for 10-player game with symmetric ($p_i = 1/10,\ i = 1, 2, \cdots, 10$) and asymmetric ($p_1 = 0.04,\ p_2 = 0.05,\ p_3 = 0.06,\ p_4 = 0.08,\ p_5 = 0.09,\ p_6 = 0.10,\ p_7 = 0.12,\ p_8 = 0.13,\ p_9 = 0.15,\ p_{10} = 0.18$) games.

										$\mathbb{E}(A_1)$ $\mathbb{V}(A_1)$	$\mathbb{E}(A_2)$ $\mathbb{V}(A_2)$	$\mathbb{E}(A_3)$ $\mathbb{V}(A_3)$	$\mathbb{E}(A_4)$ $\mathbb{V}(A_4)$	$\mathbb{E}(A_5)$ $\mathbb{V}(A_5)$	$\mathbb{E}(A_6)$ $\mathbb{V}(A_6)$	$\mathbb{E}(A_7)$ $\mathbb{V}(A_7)$	$\mathbb{E}(A_8)$ $\mathbb{V}(A_8)$	$\mathbb{E}(A_9)$ $\mathbb{V}(A_9)$	$\mathbb{E}(A_{10})$ $\mathbb{V}(A_{10})$
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}						Symmetric				
5	5	5	7	7	8	10	10	12	15	5 45	5 45	5 45	7 45	7 45	8 45	10 45	10 45	12 45	15 45
7	8	9	10	11	12	13	14	15	16	7 63	8 63	9 63	10 63	11 63	12 63	13 63	14 63	15 63	16 63
10	9	11	9	12	13	14	9	10	12	10 81	9 81	11 81	9 81	12 81	13 81	14 81	9 81	10 81	12 81
															Asymmetric				
5	5	5	7	7	8	10	10	12	15	2 19.2	2.5 23.75	3 28.2	6 36.8	6.5 40.95	8 45	11 52.8	11.5 56.55	14.5 63.75	19 73.8
7	8	9	10	11	12	13	14	15	16	2.8 26.88	4.5 33.25	6.2 39.48	8.6 51.52	10.3 57.33	12 63	14.4 73.92	16.1 79.17	18.5 89.25	21.6 103.32
10	9	11	9	12	13	14	9	10	12	4.6 34.56	4.5 42.75	7.4 50.76	7.2 66.24	11.1 73.71	13 81	15.8 95.04	11.7 101.79	14.5 114.75	19.2 132.84

4 Summary

This research contributes to a new direction in the classical N-player gamblers' problem. In the devised game plan, the trials are fixed in advance up to the minimum of the initial budget, say α, involved by the players. By solving the game, we divide the players into two groups. The first group contains all those players who have α budget and the remaining players are in the second group. Then we're interested in calculating the expected fortune and variance of any player A_i (either from the first or second group). As per our knowledge, this type of game (i.e. to stop the game at minimum of initial fortune by players) has not been viewed in the literature of gamblers' problem. The legitimacy of the proposed structure of the game is established mathematically and numerically as well.

Declaration of competing interest

There is no conflict of interest declared by the author(s).

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