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INTEGER TOPOLOGICAL PROOF OF DIRICHLET'S THEOREM

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Abstract. Closure of Golomb's topology over the composite numbers provides a substantial condition for the infinitude of prime numbers in relatively prime arithmetic progressions.

1. Introduction

Arithmetic progressions of the form $a\mathbb{N}+b$ with coprime coefficients contains infinitely many prime numbers as it was proven by Dirichlet back in 1837 [1], we use a few properties of Golomb's topology [2] over the integers $\mathbb{Z}$ by applying a similar Furstenberg's approach on the infinitude of Primes [3] to provide another proof of Dirichlet's result.

Recall that Golomb's topology takes as a basis the collection of all sets $p\mathbb{Z}+q$ with relatively prime coefficients (p,q) while in the classical definition the topology is based on the positive integers, this is crucial because otherwise it will appear to be a discrete topology since a few basic properties are required in order to confirm our point which also requires it to be a profinite topological group, you may also notice it's a regular space [4] as appears from the first property of 2.0.0.1 and later in 2.0.1
Relatively prime arithmetic progression can be expressed analytically as $S(p,q)=p\mathbb{Z}+q,\ \gcd(p,q)=1$ with $q\notin S^*(p,q)=p\mathbb{Z}^*+q$ where we notate $\mathbb{Z}^*=\mathbb{Z}\setminus\{0\}$ along with that introduce the isomorphic arithmetic progression $s(n)=pn+q$ and its image $S(p,q)$ where the set of prime-generating numbers is $s_p=s^{-1}\big(S(p,q)\cap\mathbb{P}\big)$ and its complement $\mathbb{Z}\setminus s_p=s^{-1}\big(S(p,q)\setminus\mathbb{P}\big)$

2. Infinitude of prime numbers in arithmetic progressions

Let us recall a few notable properties of Golomb's topology

Lemma 2.0.0.1 (Closure of $S(p,q)$ and finite sets)**.** *Golomb's topology endows two simple properties.*

(1) *Every relatively prime arithmetic progression $S(p,q)$ is clopen.*
(2) *Any finite set is closed but not open.*

Proof. The first property is due to the fact that $S(p,q)$ is the complement of a union of other arithmetic progressions $S\big(p,\mathbb{N}_p\setminus q\big)$, secondly it is obvious that a finite set P cannot be open, it's however closed as it will appear that $S^*(p,0)$ for each $p\in P$ is open since for any $z\in\mathbb{Z}^*$ we will find $a\in\mathbb{Z}$ such that $S(a,pz)\subset S^*(p,0)$. □

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The following part uses the basic topological properties of the product set $S(p,1)S(p,q)$ in order to show that the set of positive prime numbers $\mathbb{P} \ni 1$ has infinitely many elements to be found in a relatively prime arithmetic progression.

Theorem 2.0.1 (Closure of $S^*(s(n),n)$ in $\mathbb{Z} \setminus s_p$)**.** *There is a closure $s_{cl(n)}$ of a relatively prime arithmetic progression $S^*(s(n),n)$ in $\mathbb{Z} \setminus s_p$ for $c = \max s_p$*

Proof. The required closure conforms to the identity $S^*(s(n),n) \subseteq s_{cl(n)} \subseteq \mathbb{Z} \setminus s_p$ then it must be obvious that $S^*(s(n),n) \subseteq S(c, \mathbb{N}_c \setminus s_p) \cup S(a,b)$ since it's clearly disjoint from s_p with some fitting coprime numbers a, b.

Those numbers a, b can be found via the intersection

$$S^*(s(n),n) \cap S(c, s_p) \subseteq S(lcm(c, s(n)), s(n)\mathbb{Z}_h^* + n)$$

where $h = \frac{lcm(s(n),c)}{s(n)}$ and $\mathbb{Z}_h^* = \mathbb{Z} \cap [-h,h] \cap \mathbb{Z}^*$ implying

$$(a,b) = (lcm(c,s(n)), s(n)\mathbb{Z}_h^* + n)$$

but heed that whenever $lcm(s(n),c) = s(n)$ the index h is necessarily $h = \frac{lcm(s(n),c)}{c}$ if $b = c\mathbb{Z}_h + s_p$ and hence $s_{cl(n)}$ must exist. □

Our next main result relies on the statement that there is such closure $s_{cl(n)}$ as required.

Theorem 2.0.2 (Infinitude of primes in arithmetic progressions)**.** *There are infinitely many prime numbers in relatively prime arithmetic progressions.*

Proof. Assume the finitude of prime numbers in $S(p,q)$ implying that the corresponding finite primes generating set s_p cannot be open 2.0.0.1.

The non-prime product set $S^*(p,1)\big(S(p,q) \cap \mathbb{P}\big) \subset S(p,q) \setminus \mathbb{P}$ can be excluded in the following way

$$S_{cl}(p,q) = S(p,q) \setminus \mathbb{P} \setminus S_0(p,1)\big(S(p,q) \cap \mathbb{P}\big)$$

It's possible to show that $s^{-1}(S_{cl}(p,q))$ must be clopen since its complement is as deduced shortly below

$$\begin{aligned} \mathbb{Z} \setminus s^{-1}(S_{cl}(p,q)) &= s^{-1}\big(S(p,q) \setminus S_{cl}(p,q)\big) \\ &= s^{-1}\big(S(p,1)\big(S(p,q)\big)\big) \\ &= \bigcup_{s(n) \in \mathbb{P}} S(s(n),n) \end{aligned}$$

Following the closure $s_{cl(n)}$ of $S^*(s(n),n)$ in $\mathbb{Z} \setminus s_p$ due to 2.0.1 which is clopen for any $s(n) \in \mathbb{P}$ we conclude that $\mathbb{Z} \setminus s_p$ is clopen since it is a finite union of all such $s_{cl(n)}$ and of $s^{-1}(S_{cl}(p,q))$, however, that's contradictory to our initial argument of s_p hence there must be infinitely many prime numbers in $S(p,q)$. □

References

[1] Dirichlet, P. G. L. (1837), "Beweis des Satzes, dass jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthalt" [Proof of the theorem that every unbounded arithmetic progression, whose first term and common difference are integers without common factors, contains infinitely many prime numbers], Abhandlungen der Koniglichen Preusischen Akademie der Wissenschaften zu Berlin, 48: 45-71

[2] Golomb, S. W. (1959). A Connected Topology for the Integers. The American Mathematical Monthly, 66(8), 663-665. doi:10.2307/2309340

[3] Furstenberg, H. (1955). On the Infinitude of Primes. The American Mathematical Monthly, 62(5), 353-353. doi:10.2307/2307043

[4] Szczuka, Paulina. (2014). Regular open arithmetic progressions in connected topological spaces on the set of positive integers. Glasnik Matematicki. 49. 13-23. doi:10.3336/gm.49.1.02.