Research Article

Statistical Mechanics of Cosmological Systems Taking into Account Negative Temperatures

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In the paper author considers a new method for constructing quasi-equilibrium thermodynamics of a system of gravitationally interacting particles with the modified (by including a cosmological term) Newton potential. Since the dynamics of each pair of particles changes fundamentally with increasing distance between them, The work uses a technique previously used for similar behavior of vortex structures by L. Onsager . In this case, the concept of negative temperature in the system associated with the definition of the kinetic temperature of A.A. Vlasov. We constructed the approximating expressions for the configuration integral and the full statistical sum of the canonical ensemble in the case of negative temperatures. A methodology for studying quasi-equilibrium manifolds in the system under study is proposed.

1. Introduction

The emergence of large-scale structures of small dimensions in cosmological models from the point of view of the multiparticle dynamics of gravitationally interacting "elementary" substructure (representing stars, star clusters, galaxies, etc.) Currently, most scientific publications describe it based on models, representing the development methods proposed by Ya.B. Zeldovich[1][2] (the "pancake" theory or "walls"). This approach has its distinctive feature initially a simple mathematical apparatus based on taking into account the preferred directions of weakening that arose in the process evolution in an astrophysical system density fluctuations. The results based on this assumption modeling evolutionary dynamics for an ensemble of substructures allow, in principle, to obtain an externally reliable picture of the structure of the observed parts of the Universe; however, this requires the introduction of far from ordinary assumptions that allow the mentioned picture to look consistent. Without the mentioned assumptions, the existing observed definite ordering of the biscaled structure cosmological structures (and, accordingly, the bimodal Hubble flow) has a number of aspects that are difficult explain from a physical point of view at relatively recent stages of genesis these structures.

The existence of a specific spectrum of stochastic disturbances and the introduced a priori ordered distribution in the space of hydrodynamic parameters of the medium to describe the formation of macrostructures is a highly unlikely combination of independent physical conditions. It is reasonable to raise the question of the causal determinacy of the widespread, large-scale distribution of this set of conditions; after all, in essence, self-construction of low-dimensional cosmological

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structures by damping disturbances in selected directions (due to the presence of unexplained reasons), at certain scales can be characterized as an extremely strange and artificial looking scenario. As an alternative to the probabilistic nature of the local genesis of macro-objects, distributed by default — without a clear explanation of the reasons — to the entire observable part of the Universe, it is advisable to consider the mechanism of the completely causal formation of cosmological structures based on the analysis of the properties of the equation Vlasov-Poisson for gravity taking into account Einstein's anti–attractive term.

In this article we formulate approaches to constructing the quasi-equilibrium thermodynamics of a multiparticle system of gravitating particles in cosmological background with account of cosmological Λ–term repulsive effect.

2. Quasi-equilibrium statistical mechanics of a system of particles taking into account antigravity

We will consider equilibrium and near-equilibrium states of a cosmological system of gravitating particles (which we will understand as star clusters, galaxies, galaxy clusters, ...) in a bounded region of space $\Xi_3 \in \mathbb{R}^3$; we will assume by default that its sphere $\Xi_3 = (\{O\} \cup |\mathbf{x}| \leq R)$, $\partial\Xi_3 = (\{\mathbf{x}\}; |\mathbf{x}| = R)$. We will be interested in the dynamics of a system of N $(1 < N < \infty)$ particles (with equal masses m) with the Hamiltonian function

$$H_N = \sum_{\ell=1}^{4} \sum_{j=1}^{N} \mathfrak{H}_j^{(\ell)}, \mathfrak{H}_j^{(1)} \equiv \frac{\mathbf{p}_j^2}{2m}, \mathfrak{H}_j^{(3)} \equiv m\phi_j(\mathbf{x}_j), \mathfrak{H}_j^{(4)} \equiv \Upsilon(R, \mathbf{x}_j), \tag{1}$$

$$\mathfrak{H}_j^{(2)} \equiv \frac{1}{2} \sum_{i=1;j\neq i}^{N} m\Phi(|\mathbf{x}_i - \mathbf{x}_j|), m\Phi(|\mathbf{x}_i - \mathbf{x}_j|) = -\frac{\gamma m^2}{|\mathbf{x}_i - \mathbf{x}_j|} - \frac{mc^2\Lambda}{6}|\mathbf{x}_i - \mathbf{x}_j|^2, \tag{2}$$

where $\phi_j(\mathbf{x}_j)$ is the potential of the external gravitational field at point $\mathbf{x}_j$, $\Upsilon(R, \mathbf{x}_j)$ is the contribution to the potential energy of the j-th particle due to the influence of boundary conditions, $\Phi(r) \equiv \Phi^{(GN)}(r)$ is a modified Newton potential [3] [4], corresponding to a combination of gravity and antigravity between particles (the latter force arises taking into account the influence of the cosmological term), $r \equiv |\mathbf{x}_\ell - \mathbf{x}_{\ell'}|$ $(\ell, \ell' = 1, 2)$, $\mathbf{x}_j \in \mathbb{R}^3_{\mathbf{r}}$, $\mathbf{p}_j \in \mathbb{R}^3_{\mathbf{p}}$. Let us clarify the situation for the case of 2–particle interaction of particles mass M $(M = m(N-1) \approx mN$ for the non-relativistic Milne- McCrea cosmological model) and m. Unlike Newton's potential $\Phi^{(class.)}(r) = -\gamma M/r$, which is continuously increasing on interval $r \in (0, +\infty)$ $(\Phi^{(class.)} \in (-\infty, 0))$, generalized potential potential Newton $\Phi^{(GN)}(r) \equiv -\gamma M/r - \frac{1}{6}c^2\Lambda r^2$ has maximum $\Phi^{(GN)}_{(max)} = -\frac{\Lambda^{1/3}}{2}(3\gamma cM)^{2/3}$ at $r = r_c(M, m) = \frac{3^{1/3}}{c\Lambda}\left(\gamma Mc\Lambda^2\right)^{1/3}$.

The objective of this work is to construct statistical mechanics of a system of massive particles interacting with each other in accordance with potential (2). The situation there is significantly complicated by the fact in the case of the presence of classical gravitational attraction (standard Newtonian potential) between particles, the descriptions of systems using the formalism of microcanonical (μCE) and canonical (CE) ensembles may be nonequivalent[5]. This is due to the fact that the virial theorem in a self-gravitating system of particles leads to the conclusion that the total energy of this system will be negative ($E = -K$, where K we denoted the kinetic energy of particles); therefore, choosing as an example an isothermal sphere (of radius R) containing an ideal gas (with zero potential energy), we have $E = 4\pi R^3 p(R) - K = 4\pi R^3 p(R) T/\mathfrak{M} - 3M(R)k_B T/(2\mathfrak{M})$ ($\mathfrak{M} = M(R)/(4\pi\rho_0 R_0^3), \rho_0 \equiv \rho(r=0)$). Then the specific heat capacity $c_V = dE/dT < 0$ at $\rho_0/\rho(R) \in (32.1;\ 709)$, which contradicts positivity of the specific heat capacity in the

CE formalism (in this case, for $E < -0.335\gamma M^2/R$ there is no equilibrium state of the particle system, which leads to the so-called "gravothermal catastrophe"). For cosmological scales the problem statement is different, but the question of the consequences of special regions of thermodynamic potentials is very relevant.

Thus, it seems It is appropriate to consider separately the cases μCE, CE, as well as their connection with the kinetic description of the system based on a self-consistent field. A separate, extremely important aspect here is the possibility of identifying singularities and special points of a different type (leading to to a formal violation of standard thermodynamics due to the specific form of the interparticle interaction potential), and possible consequences in the form of a change in the local equation of state of matter with a change in the form (in phase space) of matter moving in dipole gravitational structures (flattening of clusters and formation of quasi-two-dimensional structures from them, such as void walls).

Based on the above physical premises, we will consider CE for particles with the Hamiltonian of gravitational interaction H_N. For simplicity, we will exclude the effect of the external gravitational field on the particles of the system, and temporarily ignore the influence of boundary conditions (they will be taken into account by a posteriori introduction dimensions of the system). The method of constructing thermodynamic potentials proposed in the work[6] (based on the introduction of Mayer functions[7] for pairs of interacting particles, small in norm), in the case under study is not directly applicable: the potential modulus $|\Phi^{(GN)}(r\to\infty)|\to\infty$, therefore replacing the exponent with a truncated expansion of the potential in the neighborhood zero for the integrand $\exp(-\Phi/T)\to 1-\Phi/T$ is invalid (such a replacement is acceptable for a series with a high rate of convergence, and for large values r must be considered as an asymptotic series). Therefore, apply the methods analysis of a weakly non-ideal gas without significant adjustments to the system under consideration particles is impossible.

Let us start with the "heuristic" approach (phenomenological establishment of the EOS) to the construction of the corresponding statistical mechanics. Using the virial theorem Clausius[8]

$$PV = \frac{1}{3}\sum_i \langle m_i \mathbf{v}_i^2\rangle - \frac{1}{3}\sum_{i,j;i>j}\langle r\phi(r)\rangle, -\frac{1}{2}\langle \mathbf{x}\cdot\mathbf{F}\rangle - \frac{1}{2}\langle \mathbf{x}'\cdot\mathbf{F}\rangle = -\frac{1}{2}\langle rm\phi(r)\rangle\Big|_{\phi(r)\equiv -d\Phi(r)/dr} \tag{3}$$

($\mathbf{F} = -\mathbf{F} = m\phi(|\mathbf{x}-\mathbf{x}'|)(\mathbf{x}-\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|$), can be obtained from the definition of the second virial term

$$\sum_{i,j}\langle rm\phi(r)\rangle = 2\pi N^2 V^{-1}\int_{\Xi_3} r^3 m\phi(r)\exp\Big(-m\Phi^{(GN)}(r)/T\Big)dr \tag{4}$$

And fact of that for $N(N-1)/2$ pairs (selected from systems of N particles) number those whose centers lie down V in the interval $[r, r+dr]$ there are $2\exp\left(-m\Phi/T\right)\pi N^2 r^2 dr/V$, immediately obtain the equation of state of the "medium" of gravitating (mega)particles (URS) in the system:

$$PV = NT\left(1 - 2\pi\frac{N}{V}\mathcal{J}(R)\right), \mathcal{J}(\epsilon, R) \equiv \int_{\epsilon\to 0}^{R\leq\infty} K(r;T)dr, \tag{5}$$

$$K(r;T) \equiv r^3\left(-\frac{\gamma m^2}{r^2} + \frac{mc^2\Lambda r}{3}\right)\exp\left(-\frac{m\Phi^{(GN)}(r)}{T}\right).$$

Obviously, the integral on the right-hand side has rather unusual properties: 1) it changes sign at the inflection point of the potential r_c, since at distances $r > r_c$ the repulsion in the pair of selected particles will prevail; 2) the values of the integral $\mathcal{J}(R)$ as a function of the upper limit grow without limit as $R\to\infty$; in this paper we are interested in large scales, therefore, we restrict ourselves to the standard regularization of the singularity by softening the potential as $r\to 0$:

$1/r \to \eta(r, r_0) \equiv 1/(r^2 + r_0^2)^{1/2}$, r_0 is an empirical term, forming the final scattering radius[6][5]. In the region to the right of the inflection point of the 2−particle potential, the equation of state of matter takes on an indefinite form, depending on the dimensions of the region Ξ_3. In this case, the same divergence will be characteristic of the configuration integral and of all thermodynamic potentials. the Ursell -Mayer formalism is possible in this case as well, if we assume that the temperature of the particles in the system is, for example, indefinite : the system, depending on the distance between particles is in formal contact with two thermostats $(\mathfrak{T}(T_1), \mathfrak{T}(T_2))$ — with $T = T_1 > 0$ for "small" distances, and with $T = T_2 < 0$ for "large" ones distances; accordingly, the canonical ensemble corresponding to the biscale system under study is the union of two subensembles $\bigcup_{i=1,2} \mathfrak{A}(\mathfrak{T}(T_i), N_i, V_i)$ (with extensive parameters N_i, V_i, $\sum_i N_i = N$, $\sum_i V_i = V$). Let's imagine integral $\mathcal{J}$ in additive form : $\mathcal{J}(R) = \mathcal{J}_1 + \mathcal{J}_2 \equiv \int_{\varepsilon}^{rc-0} [T_1] + \int_{rc+0}^{R} [T_2]$ (intervals integrations answer growing And descending branches 2−partial potential). Thus, with a change in the sign of the temperature, there is an actual replacement of the region of small distances (where the linearization of cluster factors−exponentials is standardly performed) to the region of " pseudo-infinite " distances. Then we can simplify the EOS by obtaining a phenomenological representation in the standard form of expansion in powers of density:

$$P = (N/V)T\,(1 + B_2(T)(N/V)), B_2(T) = -\frac{2\pi}{3T}\mathcal{J}(R), T \in \{T_1, T_2\}, \tag{6}$$

$$\mathcal{J}(\epsilon, R) \equiv \mathcal{J}(\epsilon, r_c) + \mathcal{J}(r_c, R) = \int_{\epsilon}^{r_c-0} K(r; T_1)dr + \int_{r_c+0}^{R} K(r; T_2)dr.$$

The approach used is based on a significant idealization of the situation, since it implies the presence of only two thermostats in the problem (this actually means the need isolation of two subsystems with different temperatures from each other, which imposes significant restrictions on the physical conditions in the cosmological system). However, the equation of state constructed is very important in itself, since it sheds light from an unusual angle on the basic thermodynamic properties of matter in a system of dimensions corresponding to cosmological distances; in particular, we note that the term with the second virial coefficient consists of an increasing (with increasing distance between the interacting particles), but limited above (due to the finite upper limit) of the term $\mathcal{J}_1$, and decreasing as $r \to \infty$ from a large but also finite value as $r = r_c$ (which is analogous to "small distances" in the statistical justification of standard thermodynamics of positive temperatures).

A completely expected consequence of applying the equation of state (6) to cosmological systems is the existence of a point ($r = r_c$) at which this EOS takes on its form for an ideal gas (since $(d\Phi^{(GN)}/dr)|_{r=r_c} = 0$, and it is quite unstable. This can be explained using the example of a one-dimensional model containing two mass formations (with masses $M_{1,2}$), and an intermediate region between them — an "interaction channel" with a linear size $d > r_c(M_1, M_2)$, containing a "strongly non-ideal" gas. In the simplest case $d \gtrsim 2r_c$ ($M_1 \sim M_2$) the regions of high and low pressure near the critical point should lead to the formation of an interaction oppositely directed shock waves (superposition of two discontinuities in the medium pressure) in the vicinity of this point. In the interaction channel, the region of superposition of pressure differences (under conditions of the Rankine−Hugoniot type) forms an almost flat structure (in which local inhomogeneities should be significantly smoothed out eigenvalues of the velocities of particles whose vectors are parallel to the channel axis, that is, a one-dimensional coherence of fluctuation damping arises, close to the scenario of Ya.B. Zeldovich[1]). Apparently, this mechanism can be applied to the description of the formation of voids between large clusters of baryonic or "dark" matter.

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However, it should be understood that the above-described method for obtaining the EOS is a heuristic example to demonstrate a very non-obvious properties of the thermodynamics of the medium on large astrophysical scales with explicit consideration of the property of mass repulsion when exceeding critical distance between them, due to the inclusion of the cosmological Einstein's term, and the corresponding modification of the law of gravity.

3. Mathematical aspects of the application of the canonical ensemble formalism in the system gravitationally interacting particles

In the previous paragraph, based on the virial theorem, an equation of state of matter in a system of gravitationally interacting particles was constructed . However, As already mentioned, for a real description of the statistical mechanics of multiparticle systems containing megaparticles , it is necessary to use more rigorous and physically correct mathematical apparatus. We have one at our disposal, and it is based on the formalism of the statistical sum, however the concept of negative absolute temperatures to compensate for growing interparticle potentials requires certain (very non-obvious) modifications; in this case, the expediency of using negative temperatures of the cosmological environment becomes obvious. In this case, of course, there arises the question of the possibility of applying the concept of an "equilibrium system" in the usual sense to such an environment. Further, we assume that, if necessary, one can consider the mathematical apparatus of kinetics as a justification for the properties of a gravitationally interacting system (in accordance with the modified presence antigravity (Newton's law) of particles.

The statistical integral of the canonical ensemble for an N-particle system of (mega)particles has the form (for the Hamiltonian we use expression (1) without an external field and taking into account boundary effects):

$$Z_N(T,V) = \frac{1}{N!\omega^{3N}} \int \exp\left(-\sum_{k=1}^{N} \frac{\mathfrak{H}_k^{(1)} + \mathfrak{H}_k^{(2)}}{T}\right) \prod_{j=1}^{N} d\mathbf{p}_j d\mathbf{x}_j, \tag{7}$$

where ω is a normalization factor. After formal integration over momenta, the last expression can be represented in the form

$$Z_N(T,V) = \frac{(2\pi m T)^{3N/2}}{\omega^{3N}} \frac{\mathfrak{C}_N(T,V)}{N!}, \mathfrak{C}_N(T,V) \equiv \int_0^R \prod_{i=1}^{N} \prod_{k=2,k>i}^{N} \exp\left(-\frac{m\Phi(|\mathbf{x}_i - \mathbf{x}_k|)}{T}\right) d^N\mathbf{x}. \tag{8}$$

This formula implicitly assumes a rapid decrease in kinetic energy with increasing momentum modulus: this is true for ordinary statistical systems. But is this really so? for cosmological systems? Verified observational data cast doubt on this (the dual structure of the Hubble parameter, which includes, along with the Friedmann expansion, dependence on antigravity forces). This issue will be discussed in detail later, but for now we point out that $N-$fold integration over the computational domain (a sphere of radius $R \leq \infty$) for a potential growing in modulus ($\Phi \sim r^2$ for large interparticle distances) leads to a divergence of the configuration integral if the temperature parameter $T > 0 \forall r \in [0; \infty)$. Therefore, it follows consider splitting this integral into two with $T = T_1 > 0$ and $T = T_2 < 0$: $\mathfrak{C}_N(T,V) = \mathfrak{C}_N^{(1)}(T_1, V_1) + \mathfrak{C}_N^{(2)}(T_2, V_2)$. In the region of small distances (in the case when the dominant interaction potential in the interparticle potential is is the classical Newtonian attraction, with a singularity at zero), we can introduce $2-$partial functions of non-ideality of the medium $f_{ik} = \exp(-\Phi_{ik}/T_1) - 1$. Then we can write

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$$\mathfrak{C}_N^{(1)}(T_1,V_1)=\int_0^{rc}\dots\int_0^{rc}(1+f_{12})(1+f_{13})\dots(1+f_{N_1-1,N_1})\prod_{k=1}^{N_1}d\mathbf{x}_k, \tag{9}$$

and since in this region of interparticle distances the exponential function under the integral can be is expanded in a Taylor series, then

$$\mathfrak{C}_N^{(1)}=\int(-m\Phi_{12}/T_1)(-m\Phi_{13}/T_1)\dots(-m\Phi_{N_1-1,N_1}/T)d^{N_1}\mathbf{x}.$$

It is easy to obtain approximate values of the first configuration integrals: $\mathfrak{C}_1^{(1)}=V_1$,

$$\mathfrak{C}_2^{(1)}(T_1,V_1)=4\pi V_1^2\int_0^{rc}r^2(1+\Phi(r)/T_1)dr=V_1^2\left(1+A_1(A_{r_0}+A_\Lambda)\right), A_1=\frac{3\gamma m^2}{2r_cT_1}, \tag{10}$$

$$A_{r_0}=A_0+(r_0^2/r_c^2)\ln((r_0/r_c)/(1+A_0)), A_\Lambda=r_c^3\Lambda c^2/(15\gamma m), A_0=(1+r_0^2/r_c^2)^{1/2}. \tag{11}$$

By inductions we get $\mathfrak{C}_N^{(1)}=V^{N_1}(1+A_1(A_{r_0}+A_\Lambda))^{N_1-1}$.

Let us now consider the term $\mathfrak{C}_N^{(2)}(T_2,V_2)$. Let us explain where the idea of possible negative temperature in the second subensemble comes from . To do this, it is necessary to turn to the kinetic description of a cosmological multi-particle system in a state of nonequilibrium. The Vlasov-Poisson system of equations for describing cosmological dynamics in a system of N particles of equal masses m (star clusters, galaxies, ...) can be represented in the following form; in this case we we assume that the system is considered in the domain $\Omega\subseteq\mathbb{R}^3$ space, and the size of the region can be tended to "physical infinity" (the volume $V(\Omega)$ is finite, but large enough to not take into account the reverse influence of the boundaries on the system):

$$\frac{\partial f(\mathbf{x},\mathbf{p},t)}{\partial t}_{*}+m^{-1}div_{\mathbf{x}}(\mathbf{p}f)+\hat{G}(f;f)=0, \hat{G}(f;f)=\mathbf{G}(f)\frac{\partial f}{\partial\mathbf{p}}, \tag{12}$$

$$\mathbf{G}(f)=-\nabla_{\mathbf{x}}\Phi(f), \Phi(f)=4\pi A\gamma\int\int m\Phi(|\mathbf{x}-\mathbf{x}'|)f(\mathbf{x}',\mathbf{v}',t_*)d\mathbf{x}'d\mathbf{p}', \tag{13}$$

where $f(\mathbf{x},\mathbf{p},t_*)$ is the distribution function (gravitationally interacting) particles, A — normalization factor for particle density, t_* — fixed moment of time. Under the condition of quasi-stationarity of processes we have $\partial f/\partial t\approx 0$. According to A.A. Vlasov, the main requirement that distinguishes temperature solutions (particle distributions) is statistical independence of the distribution of particle momenta from their distribution by coordinates [9]. The condition for maximum statistical independence is the following multiplicative representation stationary distribution function: $f(\mathbf{x},\mathbf{p})=\rho(\mathbf{x})\prod_{i=1,2,3}\Upsilon(p_i^2)$. Substituting this expression into the Vlasov-Poisson equation (12) gives

$$\sum_{i=1,2,3}p_i\left(\frac{\partial\rho_i}{\partial x_i}\Upsilon(v_i^2)-m^{-1}\frac{\partial\Phi}{\partial x}\frac{\partial\Upsilon(p_i^2)}{\partial p_i}\right)\prod_{j,k\neq i}\Upsilon(p_j^2)\Upsilon(p_k^2)=0. \tag{14}$$

Since the components of the momentum are independent of each other, it is possible to separate the variables in the last equation:

$$(\partial\rho/\partial x_i)\left(\partial\Phi/\partial x_i\right)^{-1}=\left(\partial\Upsilon(p_i^2)/\partial p_i\right)\left(\partial\Phi/\partial\Upsilon(p_i^2)\right)^{-1}=-1/T, \tag{15}$$

where T is the separation parameter ("kinetic temperature", since it determines the magnitude of the dispersion of the pulse spread). Since in our case $-d\Phi/dr>0$ when $r>r_c$, then the repulsion ($d\rho/dr<0$) corresponds to the value of $T<0$. Therefore, the kinetic temperature can be negative for large interparticle distances.

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Therefore, for an equilibrium canonical ensemble, the possibility of introducing the concept of a generalized temperature seems physically justified, for which the configuration integral — as well as the total statistical sum — are representable as improper convergent integrals under the conditions of the anti- Gibbs structure of the energy spectrum of particles of the subensemble . Obviously, $\mathfrak{C}_1^{(2)}(T_2,V_2)=4\pi(R^3-r_c^3)\equiv V_2$; the factor $\mathfrak{C}_2^{(2)}(T_2,V_2)$ (for the subensemble in contact with the thermostat at temperature $T_2<0$), can be roughly represented as

$$\mathfrak{C}_2^{(2)}(T_2,V_2)=\int_{r_c}^{R}\exp\left(-\alpha r^2\right)4\pi r^2dr\Big|_{\alpha=mc^2\Lambda/(6|T_2|)}= \tag{16}$$

$$=4\pi\left(\frac{r_c}{2\alpha}\exp\left(-\alpha r_c^2\right)-\frac{R}{2\alpha}\exp\left(-\alpha R^2\right)-\frac{\sqrt{\pi}\alpha^{-3/2}}{4}(erfc(\sqrt{\alpha}R)-erfc(\sqrt{\alpha}r_c))\right)$$

where $erfc(r)$ — additional error function. In this case, the above formula takes into account that for the integration range, the dominant in the integrand is is the factor $\propto\exp\left(\alpha r^2\right)$ (the absence of a significant dependence of the integral value on the factor $\propto\exp\left(\alpha/r\right)$ can easily can be established by analyzing the properties of Riemann sums in the exact configuration integral). The additional error function admits an asymptotic representation in Laplace form ($erfc(y)\sim\exp\left(-y^2\right)(1-1/(2y^2)+\ldots)$) [10], so that the considered part of the configuration integral takes the form

$$\mathfrak{C}_2^{(2)}(T_2,V_2)=\frac{r_c}{2\alpha}\exp\left(-\alpha r_c^2\right)-\frac{R}{2\alpha}\exp\left(-\alpha R^2\right)+\frac{1}{4\alpha^2r_c}\exp\left(-\alpha r_c^2\right)-\frac{1}{4\alpha^2R}\exp\left(-\alpha R^2\right)+ \tag{17}$$

$$+\frac{1}{8\alpha^3R^3}\exp\left(-\alpha R^2\right)-\frac{1}{8\alpha^3r_c^3}\exp\left(-\alpha r_c^2\right)\equiv\mathfrak{B}(r_c,R;\alpha(T_2)).$$

For an N-particle system, we have approximately $\mathfrak{C}_N^{(2)}(T_2,V_2)=V_2(\mathfrak{B}(r_c,R;\alpha))^{N-1}$. It is interesting that there is no direct dependence on the volume of the system. for high-order configuration integrals — this is due to the fact that for $r<r_c$ the region of significant influence of the Newtonian potential is concentrated in the ball $r<r_0$, and for at large values of distances between particles the influence of the potential only decreases (at $r\to r_c$ is actually negligible for individual particles), while for $r>r_c$ the force interaction between these same particles increases significantly; thus the factors in the form of volume when considering clusters of particles are replaced by "pseudo-volumes", whose values are nonlinearly compressed with increasing distance between the repulsive particles. Note that in principle it is advisable when detailing the consideration of the canonical ensemble to introduce an interaction zone $r_0\ll r<r_c$, where the repulsive potential should be effectively taken into account (that is, to move from one term of the Taylor series to the asymptotic series).

Now let us turn again to the full statistical integral. Since it is now clear to us that the temperature of the particles in the system is indefinite , we should correctly perform the integration over the momenta taking into account the presence of two sub-ensembles ($Z_N(T,V)=Z_N^{(1)}(T_1,V_1)+Z_N^{(2)}(T_2,V_2)$):

$$Z_N(T,V)=\frac{\left(2\pi mT_1\right)^{3N_1/2}}{\omega^{3N_1}N_1!}\mathfrak{C}_N^{(1)}(T_1,V_1)+\frac{\left(2\pi m|T_2|\right)^{3N_2/2}}{\omega^{3N_2}N_2!}erfi(p_{max}/\sqrt{2m|T_2|})\mathfrak{C}_N^{(2)}(T_2,V_2)\times \tag{18}$$

$$\times\left(erfi(R/\sqrt{2m|T_2|})-erfi(r_c/\sqrt{2m|T_2|})\right),\ \int_0^R\exp\left(\alpha r^2\right)dr=\frac{-\pi i}{2\sqrt{\alpha}}\cdot erf\left(i\sqrt{\alpha}R\right).$$

(mega)particles interacting via modified Newton gravitational potential in the cosmological system, using the formalism of the canonical ensemble. For this purpose, we introduced the concept of generalized temperature, associated with the behavior of the modified gravitational potential, taking into account influence of the cosmological lambda term.

4. Thermodynamic potentials in a system of (mega)particles and possible generalizations of the formalism

Once the partition function is known, we can derive the thermodynamic equations of state for the system. The Helmholtz free energy carries all the useful information about the system that the partition function carries, and these are connected by the relation $F_j(N_j, T_j, V_j) = -T_j \ln Z_N^{(j)}(T_j, V_j)$ $(j = 1, 2)$. For subsystems of particles with dominant attraction, after applying Stirling's formula we have

$$F_1(N_1, T_1, V_1) = -T_1 \ln\left(\frac{(2\pi m T_1)^{3N_1/2}}{\omega^{3N_1} N_1!} V_1^{N_1} (1 + A_1(A_{r_0} + A_\Lambda))^{N_1 - 1}\right) \approx \tag{19}$$

$$\approx -\frac{1}{2} T_1 \left(2N_1 \ln\left(A_1 A_{r_0} + A_1 A_\Lambda + 1\right) + 2N_1 \ln(V_1) - 6N_1 \ln(\omega) + 3N_1 \ln(T_1)\right.$$

$$\left. + 3N_1 \ln(m) + 3N_1 \ln(2) + 3N_1 \ln(\pi) - 2(N_1 \ln(N_1) - N_1) - 2\ln\left(A_1 A_{r_0} + A_1 A_\Lambda + 1\right)\right)$$

and for a subsystem of particles with repulsion dominance:

$$F_2(N_2, T_2, V_2) = -T_2 \ln\left(\frac{(2\pi m |T_2|)^{3N_2/2}}{2\omega^{3N_2} N_2!} erfi(p_{max}/\sqrt{2m|T_2|}) \times \right. \tag{20}$$

$$\left. \times V_2 \mathfrak{B}^{N_2 - 1} \left(erfi(R/\sqrt{2m|T_2|}) - erfi(r_c/\sqrt{2m|T_2|})\right)\right) \approx$$

$$\approx -\frac{1}{2} T_2 \left(3N_2 \ln(T_2) - 6N_2 \ln(\omega) + 3N_2 \ln(m) + 3N_2 \ln(2) + 3N_2 \ln(\pi) + 2N_2 \ln(\mathfrak{B}(r_c, R; \alpha)) + \right.$$

$$\left. + 2\ln(V_2) - 2\ln(\mathfrak{B}(r_c, R; \alpha)) - 2(N_2 \ln(N_2) - N_2) - 2\ln\left(erfi(R/\sqrt{2mT_2}) - erfi(r_c/\sqrt{2mT_2})\right)\right).$$

Accordingly, the entropies of the subsystems of the cosmological system of (mega)particles can also be obtained using the statistical integral in accordance with With by the relation $S = T\left(\partial(\ln Z_N^{(\dots)})/\partial T\right)_{N,V} + \ln Z_N^{(\dots)}$ (s taking into account fact $A_1 \sim T^{-1}$, $\eta(T_1) \equiv A_1(T_1)(A_{r_0} + A_\Lambda)$):

$$S_1 = -\frac{1}{2(\eta + T_1)} (6\eta N_1 \ln(\omega) - 3\eta N_1 \ln(m) - 3\eta N_1 \ln(2) - 3\eta N_1 \ln(\pi) - 2\eta N_1 \ln(\eta + T_1) - \tag{21}$$

$$-\eta N_1 \ln(T_1) - 2\eta N_1 \ln(V_1) + 6T_1 N_1 \ln(\omega) - 3T_1 N_1 \ln(m) - 3T_1 N_1 \ln(2) - 3T_1 N_1 \ln(\pi) -$$

$$-2T_1 N_1 \ln(\eta + T_1) - T_1 N_1 \ln(T_1) - 2T_1 N_1 \ln(V_1) - \eta N_1 + 2\eta \ln(\eta + T_1) -$$

$$-2\eta \ln(T_1) + 2\eta \ln(N_1!) - 3N_1 T_1 + 2T_1 \ln(\eta + T_1) - 2T_1 \ln(T_1) + 2T_1 \ln(N_1!) - 2\eta$$

Since in the expression $\mathfrak{B}(r_c, R; \alpha)$, which is included in the configuration part of the statistical sum, $\alpha = \alpha(T_2)$ (and the variable T_2 is also included limits of integral functions $erfi(\sqrt{\alpha} r)$), the explicit expression for S_2 is rather cumbersome, but can be obtained using elementary operations.

Now let's consider the pressure in both subensembles (for $r \gtrless r_c$). In standard thermodynamics ideal gas pressure is determined in accordance with the relation $P = -(\partial F/\partial V)_{N,T}$; however, for a system with interaction between particles it turns out that in fact in expression (19) for free energy the terms containing the value η implicitly, but depend significantly on the variable V (namely, the second virial coefficient itself is a nonlinear function of volume). Then formally from the

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relations derived above we have for the equation of state $P_1 = (N_1/V_1)T_1\left(1 - \varphi(A_{r_0}, A_\Lambda, T_1)\right)$ (after comparing the additional terms that appear in expression for S_1):

$$P_1 = \frac{N_1}{V_1}T_1\left(1 - \frac{(A_{r_0} + A_\Lambda)(3\gamma m^2)/(2r_c)}{T_1 + (A_{r_0} + A_\Lambda)(3\gamma m^2)/(2r_c)}\right) \tag{22}$$

For "distant" interacting particles, as noted earlier, the second virial factor in the equation of state is provided by the easily detectable formulas for the statistical sum of the dependence on the geometric dimensions of the second subsystem and the temperature in the vicinity of its formal boundaries. Thus,

$$P_2 \approx (N_2/V_2)T_2\left[1 + \left(-3T_2\left(r_c\exp\left(-\frac{mc^2\Lambda r_c^2/T_2}{6}\right) - R\exp\left(-\frac{mc^2\Lambda R^2/T_2}{6}\right) + \dots\right)\right.\right. \times \tag{23}$$
$$\left.\left. \times\left(mc^2\Lambda - 3r_cT_2\exp\left(-\frac{mc^2\Lambda r_c^2/T_2}{6}\right) + 3RT_2\exp\left(-\frac{mc^2\Lambda R^2/T_2}{6}\right) + \dots\right)^{-1}\right] .$$

We have considered the question of the canonical ensemble in the paradigm of quasi-equilibrium thermodynamics with negative temperatures. Interesting questions arise: 1) there are are there sets of values of thermodynamic potentials, as a result of the continuous change of which we will return to the starting point on the variety of potentials used? 2) Is there a priority direction of change of thermodynamic potentials for a cosmological system that uses the division into near and far interactions? The optimal formalism for answering these questions is the geometrization of the thermodynamic system and the introduction of a metric by constructing a fundamental tensor and obtaining the corresponding Christoffel coefficients. If the thermodynamic manifold is defined for us by means of the relation $S = S(T, V)$, then, for example, can be taken for the components of the metric tensor $g_{11} \equiv (\partial^2 S/\partial T^2)$, $g_{12} \equiv (\partial^2 S/\partial T\partial V)$, etc. In this case, the dynamic Euler-Lagrange equation arises, for which it is necessary to introduce a "geodetic parameter" τ in the biparametric entropy manifold , the meaning of which can be given a dynamic, time-related meaning. Therefore, Euler's equations will describe geodesic lines on a manifold, including closed (Carnot type cycles, since we have two heat reservoirs in the system).

In fact, using the thermodynamic description of cosmological systems, it is possible to carry out a very detailed description of processes in large-scale astrophysics, especially in areas where kinetic and hydrodynamic modeling is quite difficult.

5. Conclusion

The paper examines the mathematical formalism of constructing the thermodynamics of cosmological systems taking into account the possibility of introducing negative absolute (non-equilibrium) temperature. Since the dynamics of each pair of particles changes fundamentally as the distance between them increases, The work uses a technique previously used for similar behavior of vortex structures by L. Onsager . In this case, the concept of negative temperature in the system associated with the definition of the kinetic temperature of A.A. Vlasov. Constructed Approximation expressions for the configuration integral and the full statistical sum of the canonical ensemble in the case of negative temperatures. A methodology for studying quasi-equilibrium manifolds in the system under study is proposed.

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