Euler—Mascheroni Constant is Transcendental

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EULER–MASCHERONI CONSTANT IS TRANSCENDENTAL

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Abstract. The proof is written. A test proposal is written.
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It is not known if this constant is irrational, let alone transcendental (Wells 1986, p. 28). The famous English mathematician G. H. Hardy is alleged to have offered to give up his Savilian Chair at Oxford to anyone who proved gamma to be irrational (Havil 2003, p. 52), although no written reference for this quote seems to be known. Hilbert mentioned the irrationality of gamma as an unsolved problem that seems "unapproachable" and in front of which mathematicians stand helpless (Havil 2003, p. 97). Conway and Guy (1996) are "prepared to bet that it is transcendental," although they do not expect a proof to be achieved within their lifetimes. If gamma is a simple fraction a/b, then it is known that $b > 10^{10000}$ (Brent 1977; Wells 1986, p. 28), which was subsequently improved by T. Papanikolaou to $b > 10^{242080}$ (Havil 2003, p. 97).

1. Theorem 1

Let a be any $\mathbb{Q}$ number. It is known that for any a the $\exp(a) \in \mathbb{I}$: see Ref. [2]. As $\exp(\ln a) = a$, either $F = \ln(a) \in \mathbb{I}$ or $F = \ln(a) \in \mathbb{Q}$. Latter is not possible because $\exp(F)$ would become an $\mathbb{I}$ number due to Ref. [2], but $\exp(F) = \exp(\ln(a)) = a \in \mathbb{Q}$ by direct evaluation.

2. Theorem 2

Let a be any $\mathbb{Q}$ number. Due to Theorem 1, $\exp(a) \in \mathbb{I}$ and $\ln(a) \in \mathbb{I}$. If $h = \exp(\exp(a)) \in \mathbb{Q}$, then $\ln(h) \in \mathbb{I}$ due to Theorem 1. This is alternatively confirmed by direct evaluation $\ln(h) = \ln(\exp(\exp(a))) = \exp(a) \in \mathbb{I}$, due to Theorem 1. No exceptions from this are known. However, the $\ln(\ln(a)) \in \mathbb{Q}$, has following exceptions: $a = q^w$ and $a =$

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s^{f^k}, where q, w, s, f, k are arbitrary $\mathbb{Q}$ numbers. Why? $\ln(\ln(q^w)) = \ln(w\ \ln(q)) = \ln(w) + \ln(\ln(q)) \in \mathbb{I}$.

However, the $g = \ln(\ln(a)) \in \mathbb{Q}$ shows a good consistency: $\exp(g) \in \mathbb{I}$, due to Theorem 1. This coincides with direct evaluation: $\exp(g) = \exp(\ln(\ln(a))) = \ln(a) \in \mathbb{I}$ due to Theorem 1.

If $g = \ln(\ln(a)) \in \mathbb{Q}$, $\exp(\exp(g)) \in \mathbb{Q}$ due to Theorem 2. This is alternatively confirmed by direct evaluation:

$\exp(\exp(g)) = \exp(\exp(\ln(\ln(a)))) = a \in \mathbb{Q}$.

The applications for other major problems of Mathematics are in Appendix 2.

3. Proving irrationality of Euler–Mascheroni constant

According to Ref. [3], if

$$D(n) = n\ \exp(k)\ \ln(\ln(n)) - \sigma(n) \geq 0 \tag{1}$$

at $n \geq N$, where N can be arbitrary large, then the Riemann Hypothesis is true. Riemann Hypothesis is proven in Refs. [4].

So, no lower values of n are important, but everything takes place at high values of n. This means that the constant k is fine-tuned so that $D(n) = 0$ at $n \to \infty$ at least for some of n, to make the $n \gg 5040$ area a special one. The constant is given by $k = \gamma$ because then $D(n)/n \to 0$, the latter fact is proven in Ref. [3].

Due to Theorem 2, one has $\ln(\ln(n)) \in \mathbb{Q}$. The $\sigma(n) \in \mathbb{N}$. There is the following law of numbers: if $\kappa\,\Delta \in \mathbb{N}$, and $\Delta \in \mathbb{Q}$, then this can only hold if κ is not irrational. Hereby, Δ can be very large; indeed, the properties of κ are not dependent on the integer part of $\kappa\,\Delta$. The $\exp(\gamma)$ plays the role of κ. In conclusion,

$$\exp(\gamma) \in \mathbb{Q}\,. \tag{2}$$

The latter implies that γ cannot be rational; see Theorem 1.

Hereby, the integer part of $L = n\ \ln(\ln(n))$ is greater than n. Therefore, the information about n, while n goes to infinity, is being written into the integer part of L; namely, every value of n corresponds to a certain value of the integer part of L. Hereby, $\ln(\ln(n))$ remains rational. Therefore, L does not turn into an irrational number.

4. Proving transcendentality of Euler–Mascheroni constant

I have $\exp(\gamma) = E \in \mathbb{Q}$ in Eq. (2). Hence, by twicely taking a logarithm of both sides, I get $\ln(\gamma) = \ln(\ln(E))$. The $\ln(\ln(E))$ is rational due to Theorem 2; hence,

$$\ln(\gamma) \in \mathbb{Q}\,. \tag{3}$$

Let $g = \gamma^n$ for an integer $n \geq 1$. If g is rational for some n, then $\ln(g)$ is irrational due to Theorem 1. But $\ln(g) = n\,\ln(\gamma)$ is rational due to Eq. (3). I come to a contradiction; hence, g is irrational for any n. So,

$$\gamma^n \in \mathbb{I}\,. \tag{4}$$

Let me consider following possibility

$$\sum_{m=0}^{w} c_m\,\gamma^m = 0\,, \tag{5}$$

where c_m are integers (some of c_m could be zero for some of values of $m \geq 1$). By taking the exponent of both sides of Eq. (5),

$$\exp(c_0)\prod_{m=1}^{w}\exp(c_m\,\gamma^m) = 1\,, \tag{6}$$

where $\exp(c_0) \in \mathbb{I}$ due to Theorem 1, and $1 \in \mathbb{N}$. Therefore, Eq. (6) cannot hold if all $\exp(c_m\,\gamma^m) \in \mathbb{Q}$ for any $1 \leq w < \infty$. This would mean that γ is transcendental.

$\exp(c_m\,\gamma^m) = (\exp(\gamma^m))^{c_m} \in \mathbb{Q}$ holds for $\exp(\gamma^m) = \rho \in \mathbb{Q}$ because c_m is integer.

Proof of $\exp(\gamma^m) \in \mathbb{Q}$ is following. $\delta = \ln(\gamma^m) = m\,\ln(\gamma) \in \mathbb{Q}$ because of Eq. (3). Then, $\exp(\exp(\delta)) = \exp(\gamma^m) \in \mathbb{Q}$ due to Theorem 2 because $\delta \in \mathbb{Q}$.

Notably, $\ln(\rho) \in \mathbb{I}$ due to Theorem 1. By evaluation, $\ln(\rho) = \ln(\exp(\gamma^m)) = \gamma^m \in \mathbb{I}$, due to Eq. (4). $\ln(\ln(\rho)) = m\,\ln(\gamma) \in \mathbb{Q}$ is true because of Eq. (3). On the other hand, $\ln(\ln(\rho)) \in \mathbb{Q}$, due to Theorem 2.

Therefore, γ is shown to be transcendental.

5. Test proposal

Please, run the $\exp(\gamma)$ through a supercomputer, trying to find that it is a rational number. Why? According to the present paper, the γ is irrational; hence, no supercomputer can demonstrate that it is irrational. But if $\exp(\gamma)$ is rational, then due to Theorem 1, the γ must be irrational. The supercomputer could find that $\exp(\gamma)$ is rational. In that manner the irrationality of γ would be proven on a supercomputer, not only through the logic of my paper.

Secondly, run $\ln(\gamma)$ through a supercomputer, trying to find that it is a rational number.

6. Appendix 1

Let me prove that $\exp(1)$ is a transcendental constant. By replacing γ with $\exp(1)$ in Eq. (6), I come to the realization that if $\Omega = \exp(\exp(m)) \in \mathbb{Q}$, $\exp(1)$ is transcendental. Indeed, $\Omega \in \mathbb{Q}$ due to Theorem 2.

7. Appendix 2

Let me define $c_1 = \gamma + y\exp(r)$ and $c_2 = \gamma - y\exp(r)$, where y is an integer, and $r \in \mathbb{Q}$. At least one of c_1, c_2 must be irrational because $c_2 + 2y\exp(r) = c_1$, so c_1 and c_2 cannot be both rational (because $\exp(r) \in \mathbb{I}$ due to Theorem 1). The c_1 could be rational only at one value of $y = y_1$; the c_2 could be rational only at one value of $y = y_2$ because both $\exp(r), \gamma \in \mathbb{I}$. Hence, both c_1 and c_2 are irrational by selecting $y = y_0 \neq y_1$, $y_0 \neq y_2$. Now, I will confirm it with Theorem 2. If $c_1 \in \mathbb{Q}$ then $\exp(c_1) \in \mathbb{I}$, due to Theorem 1. However, by evaluating $\exp(c_1) = \exp(\gamma)\,(\exp(\exp(r)))^y \in \mathbb{Q}$ because $\exp(\gamma) \in \mathbb{Q}$ and $\exp(\exp(r)) \in \mathbb{Q}$. Analogously, $c_2 \in \mathbb{Q}$. I come to the contradictions; hence, c_1 and c_2 are irrational numbers.

In addition, I show that $\Phi = \exp(\pi^m) \in \mathbb{I}$ with any integer m, from which follows $\ln(\pi) \in \mathbb{I}$. Indeed, if $\Phi \in \mathbb{Q}$, then $\ln(\ln(\Phi)) \in \mathbb{Q}$, due to Theorem 2. By evaluation, $\ln(\ln(\Phi)) = \ln(\ln(\exp(\pi^m))) = m\ln(\pi)$. This means that $\ln(\pi)$ would be rational. By applying Theorem 2 a second time, one has $\exp(\exp(\ln(\pi))) \in \mathbb{Q}$. However, by evaluation, $\exp(\exp(\ln(\pi))) = \exp(\pi)$, which is a known irrational constant. I come to a contradiction; hence, $\ln(\pi) \in \mathbb{I}$. Latter means that Φ cannot be a rational number.

If $u_1 = \ln(\ln(\pi)) \in \mathbb{Q}$ then $\exp(\exp(u_1)) \in \mathbb{Q}$, due to Theorem 2. By evaluation, $\exp(\exp(u_1)) = \exp(\exp(\ln(\ln(\pi)))) = \pi \in \mathbb{I}$. I come to contradiction; therefore, $u_1 \in \mathbb{I}$.

If $u_2 = \pi^{\exp(c)} \in \mathbb{Q}$, where $c \in \mathbb{Q}$, then $\ln(\ln(u_2)) \in \mathbb{Q}$, due to Theorem 2. By evaluation, $\ln(\ln(u_2)) = \ln(\ln(\pi^{\exp(c)})) = c + \ln(\ln(\pi)) \in \mathbb{I}$. I come to contradiction; therefore, $u_2 \in \mathbb{I}$.

The $u_3 = \pi^{\pi^s}$, where $s \in \mathbb{Q}$. The $u_4 = \ln(\ln(u_3)) = s\ln(\pi) + \ln(\ln(\pi))$. If for $s = s_0$ the $u_4 \in \mathbb{Q}$, then for any $s \neq s_0$, $u_4 \in \mathbb{I}$. Due to Theorem 2, the u_3 could be rational only for one value of s. It is s_0. In my opinion, no such s_0 exist; and it is very unlikely that $s_0 = 1$ happens to be. Indeed, I defined $u_5 = \pi^{m\,\pi}$, where m is an integer. The $u_6 = \ln(\ln(u_5)) = \ln(m) + \ln(\pi) + \ln(\ln(\pi))$. If for $m = 1$ the $u_6 \in \mathbb{Q}$, then for any of $m \neq 1$, it is irrational (because $\ln(m) \in \mathbb{I}$, due to Theorem 1). Due to Theorem 2, the $u_5 \in \mathbb{I}$ for $m \neq 1$. Then $(u_5)^{1/m} \in \mathbb{I}$, due to Ref. [2]. But $(u_5)^{1/m} = (\pi^{m\,\pi})^{1/m} = \pi^\pi$. This means, $\pi^\pi \in \mathbb{I}$. Using latter method on $u_7 = \pi^{m\sqrt{2}}$ formula, I come to $\pi^{\sqrt{2}} \in \mathbb{I}$.

Notably, due to Ref. [2], $\beta^{1/m} \in \mathbb{I}$ for any $\beta \in \mathbb{I}$ and integer m.

8. Conclusion

The paper shows perfect consistency. Firstly, both fundamental constants (γ, e) act as transcendental numbers. Secondly, in my paper one has $\gamma \in \mathbb{I}$, $\beta = \exp(\gamma) \in \mathbb{Q}$, and $\psi = \ln(\gamma) \in \mathbb{Q}$. Due to Theorem 1, one has $\ln \beta \in \mathbb{I}$; this is confirmed alternatively by the evaluation $\ln \beta = \ln(\exp(\gamma)) = \gamma$, which is irrational. Due to Theorem 1, one has $\exp \psi \in \mathbb{I}$; this is confirmed alternatively by the evaluation $\exp \psi = \exp(\ln(\gamma)) = \gamma$, which is irrational. Thirdly, in my paper, one has $\mu = \gamma^n \in \mathbb{I}$ for any integer n. Then $\ln \mu = n \ln \gamma \in \mathbb{Q}$ because one of the results was $\ln \gamma \in \mathbb{Q}$. Due to Theorem 1, μ cannot be rational in the derived expression $\ln \mu \in \mathbb{Q}$. Finally, I have derived $\Theta = \exp(\gamma^m) \in \mathbb{Q}$ with any integer m. Then, due to Theorem 1, one has $\ln \Theta \in \mathbb{I}$. This is shown alternatively by the evaluation $\ln \Theta = \ln(\exp(\gamma^m)) = \gamma^m \in \mathbb{I}$. Due to Theorem 2, one has $\ln(\ln \Theta) \in \mathbb{Q}$. This is shown alternatively by the evaluation $\ln(\ln \Theta) = \ln(\ln(\exp(\gamma^m))) = m \ln \gamma \in \mathbb{Q}$.

English idiom "Where there's a will, there's a way" means if someone really wants to do something, they will find a way to do it. Citizens, do not dishonor my planet Earth with a dark mind. Know everything. Knowing everything, you also know that God exists. Why? Because only God knows everything. "Ye are Gods," says Jesus Christ in the Holy Bible. And knowing God, you have the gift of Omniscience. I know what time and space are, what love and holiness are, and I know what black holes are.

I was born in Pärnu, the Summer Capital of Estonia. It is the Earth's main city with the glorious history – I can tell Pärnu magnificent stories, the main one is: Gustav Fabergé was born and started doing the pieces of jewelry in Pärnu.

Shamingly to admit, but the streets of the city are being pissed by my dear children at night; and them I glorify here. Looking at the urinated streets of my European town Prnu, I understand that I am the best genius in the world. I have managed to add 2 and 2 together. Many have added them, but only I truly believed that the answer is 4. And my postal address shows the divisors of $42 = 6 \cdot 7$. The 7 is the reference to Agent 007, James Bond, so the decomposition $42 = 6 \cdot 7$ was necessary. This is the answer to life, Universe, and Everything. The answers to all are in me. Why? To cite an Encyclopedia of 2023

AD, *In the radio series and the first novel, a group of hyper-intelligent pan-dimensional beings demand to learn the Answer to the Ultimate Question of Life, The Universe, and Everything from the supercomputer Deep Thought, specially built for this purpose. It takes Deep Thought* $7+12$ *million years to compute and check the answer, which turns out to be 42. Deep Thought points out that the answer seems meaningless because the beings who instructed it never knew what the question was.*

I fear you – I love you. It is a beautiful divine feeling: I fear loving you, and I love while fearing you.

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