Modified Hawking radiation of Schwarzschild-like black hole in bumblebee gravity model

İzzet Sakallı*

AS245 Physics Department, Eastern Mediterranean University, Famagusta, 99628 North Cyprus via Mersin 10, Turkey

Esra Yörük†

AS324 Physics Department, Eastern Mediterranean University, Famagusta, 99628 North Cyprus via Mersin 10, Turkey

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Abstract

In this article, we study the Hawking radiation of the Schwarzschild black hole within the bumblebee gravity model (SBHBGM). Considering classical approaches involving Killing vectors and the standard Hamilton-Jacobi method, the Hawking radiation of SBHBGM is computed. The Painlevé-Gullstrand, ingoing Eddington-Finkelstein, and Kruskal-Szekeres coordinate systems are introduced as alternatives to the naive coordinates, providing insights into gravitational behavior around massive objects like black holes. Incorporating the Generalized Uncertainty Principle (GUP) into the Hamilton-Jacobi equation, a modified equation characterizing particle behavior near the event horizon is obtained. By calculating the tunneling probability using the modified action, the GUP-induced modifications to the emitted particle's behavior are considered, resulting in the derivation of the modified temperature of the SBHBGM. Finally, we study the quantum-corrected entropy of the SBHBGM and discuss the findings with possible future projects.

*Electronic address: izzet.sakalli@emu.edu.tr

†Electronic address: esra.yoruk@emu.edu.tr

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Contents

I. INTRODUCTION

Hawking radiation, a groundbreaking theoretical prediction introduced by Stephen Hawking [1–3], revolutionized our understanding of black holes and the fundamental interactions between gravity and quantum mechanics [4]. This phenomenon proposes that black holes are not entirely black; they emit radiation due to quantum effects near their event horizons, thus gradually losing mass and energy over time [5–15]. Hawking radiation challenges classical notions of black holes as inexorable gravitational sinks by highlighting the intricate interplay between quantum physics and general relativity in extreme gravitational environments [16].

To comprehend the implications of Hawking radiation fully and explore its various facets, a multitude of calculation methods have been developed by physicists [17–22]. These methods provide distinct perspectives on the underlying mechanisms, enabling us to decipher the enigmatic nature of black hole evaporation. In this discourse, we delve into the concept of Hawking radiation, followed by an exploration of the diverse methods employed to quantify and understand this phenomenon. The conventional formulation of Hawking radiation emerges from the principles of quantum field theory in curved spacetime. This approach considers virtual particle-antiparticle pairs [23] that momentarily appear near the event horizon. While one of these particles may fall into the black hole, the other escapes to infinity as real Hawking radiation. The energy needed to create these particles is borrowed from the black hole's mass, ultimately leading to its evaporation [24–26].

Several calculation methods have been proposed to derive the properties of Hawking radiation and elucidate its intricate details (see Ref. [22] and references therein). One notable avenue involves tortoise coordinate transformations, as explored by Damour, Ruffini, and Sannan [27–34]. This method facilitates the analysis of particle trajectories near the event horizon, allowing for a comprehensive understanding of how particles escape the black hole's gravitational grasp. Additionally, researchers such as Chandrasekhar, Bonner, and Vaidya demonstrated the separation of the Dirac and Maxwell equations in stationary spacetimes [35]. Such separations help us understand quantum behavior near event horizons and contribute to our knowledge of Hawking radiation. Another significant approach, pioneered by Parikh and Wilczek [36], interprets Hawking radiation as a quantum tunneling process. This method, known as the null geodesic method, draws parallels with particle tunneling through classically forbidden energy barriers. The utilization of the Hamilton-Jacobi method [37–43] further enriches the exploration of particle tunneling and provides a distinctive perspective on the mechanisms underlying Hawking radiation. Moreover, the advent of the GUP has spurred investigations into the effects of quantum gravity on Hawking radiation. The incorporation of GUP into the analysis necessitates innovative calculation techniques, as emphasized in some remarkable studies [44–50]. This avenue opens up new paths for understanding the interplay between quantum mechanics and gravity in the context of Hawking radiation.

In this work, we consider the SBHBGM spacetime, which was derived by Casana et al [51]. The addition of the bumblebee field complicates the equations for gravitational fields [52]. The bumblebee field affects the geometry of spacetime, leading to deviations from the classical Schwarzschild solution. Besides, this solution allows researchers to study how the bumblebee field modifies the physical properties around the black hole [53–57]. We then embark on a comprehensive journey through the phenomenon of Hawking radiation and some of its calculation methods. By examining these various approaches, we aim to deepen our grasp of the intricate processes occurring near black hole event horizons, ultimately advancing our comprehension of the profound interplay between quantum phenomena and the fabric of spacetime. To this end, we first compute the Hawking radiation of SBHBGM with classical methods: methods of Killing vectors and standard (without GUP) Hamilton-Jacobi method, respectively. When using the classical Hamilton-Jacobi method, three additional and regular coordinate systems, the Painlevé-Gullstrand (PG), ingoing Eddington-Finkelstein (IEF), and Kruskal-Szekeres (KS) coordinates are considered alongside the naive coordinates (see Refs. [58, 59] and references therein). These alternative coordinate choices in general relativity provide valuable insights into the behavior of gravitational fields, particularly around massive objects like black holes. Those regular coordinate systems are distinct from the standard Schwarzschild coordinates and are often used to gain a clearer understanding of the physics involved particularly near event horizons. The PG coordinates were introduced as an attempt to make the time coordinate more physically intuitive. In the Schwarzschild metric, the time coordinate is the same as the Schwarzschild time, which is not the "proper time" experienced by an observer falling into a black hole [60]. The PG coordinates address this issue by defining the time coordinate in such a way that it corresponds to the proper time experienced by a freely falling observer. The IEF coordinates take into account the one-way nature of light propagation and describe the radial position of light rays as they move toward the black hole [26, 61]. The metric in these coordinates remains regular at the event horizon, which makes it convenient for studying the behavior of particles and light as they cross the horizon. Devised independently by Martin Kruskal and George Szekeres [62, 63], KS coordinate system offers a perspective that simplifies the mathematical representation of the complex spacetime curvature near a black hole's event horizon. By transforming the conventional Schwarzschild coordinates, the KS coordinates unveil the intriguing properties of black hole interiors and exteriors, allowing for a clearer understanding of phenomena like gravitational time dilation, trapped surfaces, and the path of light. Then, we incorporate the GUP modification into the Hamilton-Jacobi equation [64], which yields a modified equation that describes the behavior of particles near the SBHBGM's event horizon. Using the modified action, we calculate the tunneling probability for particles to escape the event horizon. This probability takes into account the GUP-induced modifications to the emitted particle's behavior near the horizon. Since the tunneling probability is related to the Hawking temperature and radiation spectrum of the black hole, by considering the GUP effects, we derive the modified temperature of the SBHBGM. While the Bekenstein-Hawking formula [26] successfully relates the entropy to the black hole's macroscopic properties, it does not account for quantum effects that occur near the event horizon. As black holes can emit Hawking radiation due to quantum fluctuations, these quantum effects are expected to modify the entropy and other thermodynamic quantities. Quantum-corrected (QC) entropy [65, 66] attempts to incorporate these quantum corrections into the expression for entropy. Various approaches, including loop quantum gravity and string theory, have explored these corrections [67, 68]. These modifications to the entropy formula are often subtle and may depend on the specific quantum gravity theory being considered. The concept of quantum-corrected entropy is not limited to black hole physics. It has broader applications in the context of the holographic principle and the AdS/CFT correspondence [69–71], where it suggests a deep connection between gravitational physics and quantum field theories. We also study the QC entropy of the SBHBGM and explore how quantum fields near the event horizon impact the black hole's entropy. This investigation contributes to our understanding of the intricate connection between quantum mechanics and gravitational physics in extreme environments.

The paper is organized as follows: In Sec. II, we provide a brief introduction to the SBHBGM and examine its fundamental characteristics. In Sec. III, we focus on calculating the classical (without considering the GUP effects) Hawking radiation of the SBHBGM via the Hamilton-Jacobi method. We also attempt to demonstrate the coordinate independence of the Hawking radiation obtained through quantum tunneling by extending our findings to regular coordinates, which are PG, IEF and KS coordinate systems. Section IV is devoted to the GUP-modified Hawking radiation of the SBHBHM. We analyze the QC entropy of the SBHBGM in Sec. V. Finally, in Sec. VI, we present our concluding remarks. (Throughout the paper, we use geometrized units: $c = G = \hbar = k_B = 1$.)

II. SBHBGM GEOMETRY AND ITS PHYSICAL FEATURES

According to extended Einstein field equations of bumblebee gravity theory [51, 72, 73], we have

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G_N T_{\mu\nu} \tag{1}$$

where G_N is the Newtonian constant, $G_{\mu\nu}$ represents the Einstein tensor, while $T_{\mu\nu}$ corresponds to the overall energy-momentum tensor originating from both the matter sector's contribution ($T^{\rm M}_{\mu\nu}$) and the effects of the bumblebee field ($T^{B}_{\mu\nu}$): $T_{\mu\nu} = T^{\rm M}_{\mu\nu} + T^{B}_{\mu\nu}$. A reader can find the detailed derivation of the field equations and their corresponding metric solution for the SBHBGM in Ref. [51], which serves the following spherically symmetric vacuum solution:

$$ds^2 = -f dt^2 + \Xi f^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{2}$$

where $\Xi = 1 + \ell$ in which ℓ is the positive Lorentz symmetry-breaking parameter [51]. Metric (2) represents a purely radial Lorentz-violating solution outside a spherical body characterizing a modified black hole solution. The metric function (f) is given by $f = 1 - \frac{2M}{r}$ in which $r_h = 2M$ represents the event horizon and M denotes the mass. This solution for a black hole describes a situation where Lorentz violation occurs exclusively in the radial direction beyond a spherical object, defining a modified black hole solution. As the parameter ℓ approaches zero, it is evident that the conventional Schwarzschild metric is regained. In the context of the metric labeled as Eq. (2), the Kretschmann scalar [26] can be computed as follows:

$$\mathcal{K} = R^{\mu\nu\lambda\sigma} R_{\mu\nu\lambda\sigma} = \frac{4\left(12M^2 + 4\ell M r + \ell^2 r^2\right)}{r^6 \Xi^2}, \tag{3}$$

which is distinct from the Kretschmann scalar of a Schwarzschild black hole. This indicates that none of the coordinate transformations establish a connection between metric (2) and the usual Schwarzschild black hole metric. When r is equal to $2M$, the curvature of spacetime remains finite, implying that a proper coordinate transformation can eliminate the coordinate singularity. However, in the scenario where r equals 0, the physical singularity cannot be eliminated. Therefore, it can be observed that the characteristics of the physical singularity at $r = 0$ and the coordinate singularity at $r = r_h = 2M$ (event horizon) of the Schwarzschild black hole remain intact in the SBHBGM solution. On the other hand, the Hawking temperature [1] can be computed as

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi} f'(r_h) = \frac{1}{8\pi M\sqrt{\Xi}}, \tag{4}$$

where κ denotes the surface gravity [26]:

$$\kappa = \nabla_\mu \chi^\mu \nabla_\nu \chi^\nu = \frac{1}{4M\sqrt{\Xi}}, \tag{5}$$

by which χ^μ is the timelike Killing vector field and the prime (dash) symbol in Eq. (4) is used to denote the derivative of a function with respect to its argument. As can be seen from Fig. 1, the non-zero Lorentz symmetry breaking parameter ℓ has the effect of reducing the Hawking the temperature of a Schwarzschild black hole solution.

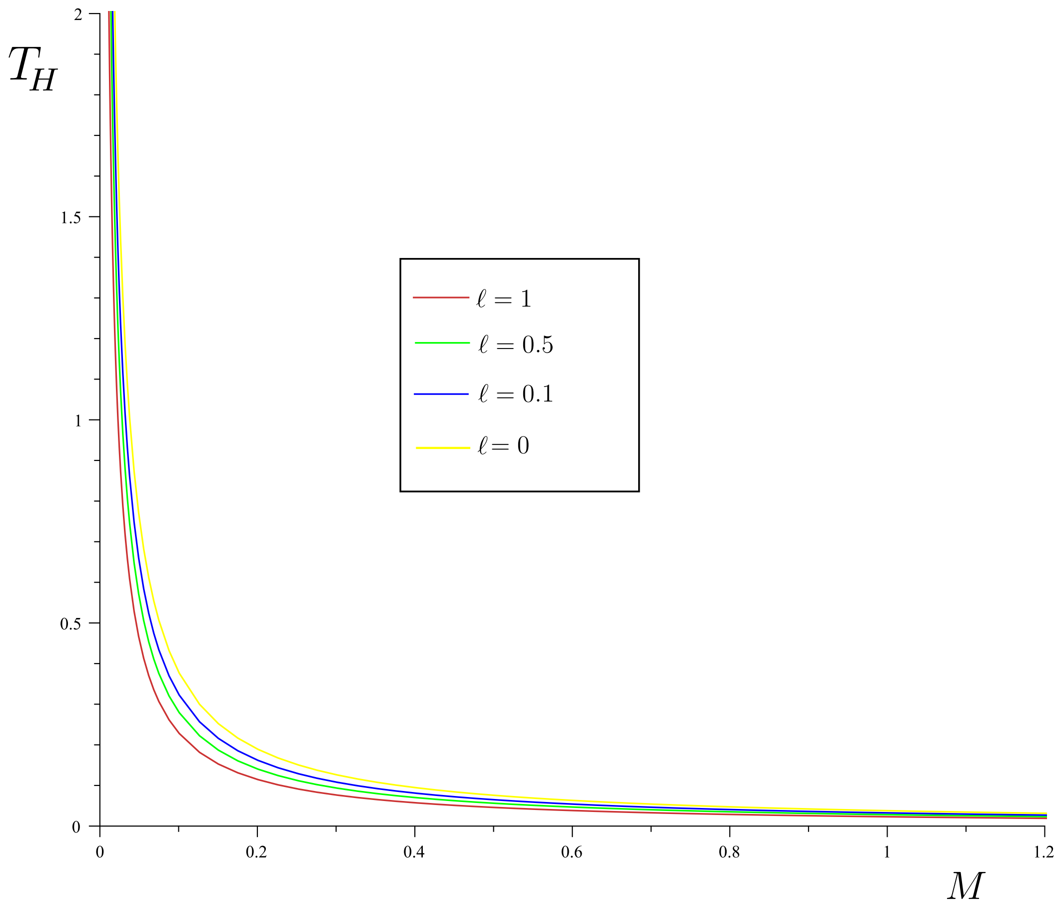

FIG. 1: Graph of T_H versus mass M. The considered Lorentz symmetry-breaking parameters ℓ are depicted with different colors. Plots are governed by Eq. (4)

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III. HAWKING RADIATION OF SBHBGM VIA HAMILTON-JACOBI METHOD: SEMI-CLASSICAL APPROACH (WITHOUT GUP)

The section focuses on elucidating the process of deriving Hawking radiation for black holes resembling the Schwarzschild metric within the framework of bumblebee gravity, which introduces the Lorentz-violating term. Through the utilization of the Hamilton-

Jacobi method, the section outlines the step-by-step mathematical procedure to uncover the radiation emitted by these modified black holes. To this end, let us first consider the Hamilton-Jacobi equation [37] :

$$g^{\mu\nu}\left(\frac{\partial P}{\partial x^{\mu}}\right)\left(\frac{\partial P}{\partial x^{\nu}}\right)+m^2=0, \tag{6}$$

in which m and $g^{\mu\nu}$ are the mass of the particle and the inverse metric tensor, respectively. Besides, P is the classical action of a relativistic particle that satisfies the Hamilton-Jacobi equation (6). Setting [38–41]

$$L^2=g^{\theta\theta}(\partial_\theta P)^2+g^{\phi\phi}(\partial_\phi P)^2, \tag{7}$$

which is a constant associated with the particle's angular momenta. Thus, we get

$$-\frac{1}{f}\left(\frac{\partial P}{\partial t}\right)^2+\frac{f}{\Xi}\left(\frac{\partial P}{\partial r}\right)^2+L^2+m^2=0. \tag{8}$$

Taking the Killing vectors of SBHBGM spacetime (2) into account, one can set

$$P(r,t)=-\omega t+W(r), \tag{9}$$

where ω is the particle energy measured by an observer located at spatial infinity and $W(r)$ is the time-independent function, which is called Hamilton's characteristic function. After some manipulations, one can obtain:

$$W^{\pm}(r)=\pm\int\frac{\omega\sqrt{\Xi}}{f}dr. \tag{10}$$

With the help of residue theory, the near-horizon solution yields

$$W^{\pm}(r_h)=\pm 2i\pi\omega M\sqrt{\Xi}. \tag{11}$$

Using the tunneling probability ($\mathcal{P}$) with the Boltzmann formula [22], we get

$$\mathcal{P}=\frac{\Gamma^{out}}{\Gamma^{in}}==\frac{\exp\left(-2ImW^{+}(r_h)\right)}{\exp\left(-2ImW^{-}(r_h)\right)}=\exp\left(-8\pi M\omega\sqrt{\Xi}\right)=\exp\left(\frac{-\omega}{T}\right), \tag{12}$$

which yields the surface temperature of the SBHBGM as follows:

$$T_{Sr}=\frac{1}{8M\pi\sqrt{\Xi}}, \tag{13}$$

which is nothing but the statistical Hawking temperature obtained in Eq. (4): $T_{Sr}=T_H$.

Besides its naive coordinates, we also consider two regular coordinate systems: PG and IEF coordinates. Detailed quantum tunneling calculations will be reevaluated using the HJ method within these coordinates, and the preservation of T_H invariance will be examined within each coordinate system in the following section.

A. Hawking Radiation of SBHBGM within PG Coordinate System

The PG coordinate system [58] is a specific coordinate system used in the study of black hole physics, particularly in the context of general relativity. It was introduced to provide a more intuitive and physically transparent description of the spacetime geometry around a spherically symmetric black hole compared to the commonly used Schwarzschild coordinates. In the Schwarzschild coordinate system, which is often used to describe the geometry of a non-rotating (static) black hole, the coordinate singularity at the event horizon makes it difficult to interpret the physical behavior of particles falling into the black hole. The PG coordinates were designed to address this issue. In the PG coordinates, the metric is chosen in such a way that the radial coordinate follows the motion of a freely-falling observer. This means that the coordinate system is adapted to an observer who is "riding" along with a falling particle. As a result, the coordinate singularity at the event horizon is removed, and the metric becomes regular at the horizon. The PG coordinates have the following properties:

Regular Horizon: The event horizon of the black hole appears as a regular surface in these coordinates, making it easier to analyze the behavior of particles and light near the horizon.

Non-Static Behavior: Unlike the Schwarzschild coordinates, the PG coordinates exhibit non-static behavior. This makes it easier to analyze the infall of matter into the black hole and the associated effects.

Negative Energy Particles: These coordinates can accommodate negative energy particles that move outward from the black hole, which can provide insights into the dynamics of black hole evaporation.

In this section, we shall use the PG coordinates for the SBHBGM as a regular coordinate system in the HJ equation and show how it gives the true Hawking temperature. Let us start with the following transformation:

$$dr \to \sqrt{\Xi} d\tilde{r}, \tag{14}$$

$$dt \to d\tilde{t} + \frac{\sqrt{1+f}}{f} dr, \tag{15}$$

$$\tilde{f} = 1 - \frac{2M}{\tilde{r}}. \tag{16}$$

Thus, metric (2) transforms into its PG form as

$$ds^2 = -\tilde{f}d\tilde{t}^2 + 2\sqrt{1-\tilde{f}}d\tilde{t}d\tilde{r} + d\tilde{r}^2 + \tilde{r}^2 d\Omega^2. \tag{17}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2, \tag{18}$$

which is the metric on a unit two-sphere S^2. Employing the Hamilton-Jacobi equation (6) with ansatz $P(\tilde{r}, \tilde{t}) = -\tilde{\omega}\tilde{t} + \tilde{W}(\tilde{r})$ and making some straightforward calculations, one can get two near-horizon solutions for Hamilton's characteristic function $\tilde{W}(\tilde{r})$:

$$\tilde{W}^-(\tilde{r}_h) = 0, \tag{19}$$

$$\tilde{W}^+(\tilde{r}_h) = +4i\pi\tilde{\omega}M\sqrt{\Xi}. \tag{20}$$

Therefore, the tunneling probability of radiating particles from the black hole geometry (17) is found as;

$$\tilde{\mathcal{P}} = \frac{\exp\left(-2Im\tilde{W}^+(\tilde{r}_h)\right)}{\exp\left(-2Im\tilde{W}^-(\tilde{r}_h)\right)} = \exp\left(-8\pi M\tilde{\omega}\sqrt{\Xi}\right) = \exp\left(\frac{-\tilde{\omega}}{\tilde{T}}\right). \tag{21}$$

Thus, one can read the black hole temperature of SBHBGM defined in PG coordinates as follows

$$\tilde{T} = \frac{1}{8M\pi\sqrt{\Xi}}, \tag{22}$$

which equals to the statistical Hawking temperature (4).

B. Hawking Radiation of SBHBGM within IEF Coordinate System

The IEF coordinates [74] offer a unique perspective on the geometry of spacetime surrounding a black hole, particularly in the context of ingoing particles such as photons. These coordinates are meticulously designed to maintain regularity at the black hole's event horizon, simplifying the analysis of particles as they approach this boundary and providing insight into the behavior of matter and radiation near the event horizon.

To pass to the IEF coordinate system, let us use the following transformation [59]:

$$dt = d\nu - dr^*, \tag{23}$$

where ν is a new null coordinate, the so-called advanced time and r^* denotes the tortoise coordinate:

$$dr^* = \frac{\sqrt{\Xi}dr}{f}. \tag{24}$$

Thus, the SBHBGM metric (2) recasts in

$$ds^2 = -fd\nu^2 + 2d\nu dr + r^2 d\Omega^2. \tag{25}$$

In this coordinate system, the scalar particle's energy can be measured by an observer as $E = -\partial_v P$, due to the Killing vector field of $\xi^\mu = \partial_v$ in metric (25).

$$P(\nu, r, \theta, \phi) = -\omega\nu + W_{IEF}(r) + J\left(x^i\right), \tag{26}$$

in which $\partial_i P = J_i$'s are constants and $i = 1, 2$ labels the angular coordinates θ and ϕ, respectively. Employing the Hamilton-Jacobi equation (6) for the metric (25), the final result for $W_{IEF}(r)$ can be found as

$$W^{\pm}_{IEF}(r) = \omega \int \frac{\Xi}{f}\left(1 \pm \sqrt{1 - \frac{\rho f}{\omega^2}}\right) dr, \tag{27}$$

where

$$\rho = m^2 + \frac{J_\theta^2}{r^2} + \frac{J_\phi^2}{r^2 \sin^2\theta}. \tag{28}$$

The expression for $W_{IEF}(r)$ simplifies to the following expression in the vicinity of the event horizon:

$$W^{\pm}_{IEF}(r) = \omega \int \frac{\Xi}{f}(1 \pm 1) dr. \tag{29}$$

Thus, one gets

$$W^{-}_{IEF}(r_h) = 0, \tag{30}$$

$$W^{+}_{IEF}(r_h) = +4i\pi\omega M\sqrt{\Xi}. \tag{31}$$

In the sequel, we find out the tunneling probability of the emitted quanta from the SBHBGM defined in the IEF coordinates:

$$\mathcal{P}_{IEF} = \frac{\exp\left(W^{+}_{IEF}(r_h)\right)}{\exp\left(W^{-}_{IEF}(r_h)\right)} = \exp\left(-8\pi M\omega\sqrt{\Xi}\right) = exp\left(\frac{-\omega}{T_{IEF}}\right). \tag{32}$$

As a result, we get

$$T_{IEF} = \frac{1}{8M\pi\sqrt{\Xi}}, \tag{33}$$

which fully agrees with Eq. (4).

C. Hawking Radiation of SBHBGM within KS Coordinate System

Introduced by Martin Kruskal and George Szekeres [62, 63], the KS coordinate system offers a unique perspective for understanding the geometry and behavior of black holes, particularly those described by the family of Schwarzschild solutions. By transforming the standard metric into a new set of coordinates, the KS coordinates unveil the underlying structure of spacetime around a black hole, enabling insights into phenomena such as event horizons and the nature of singularities. This coordinate system proves indispensable in simplifying the mathematical representation of these complex gravitational systems and illuminating their intriguing properties. In this section, we will use the Hamilton-Jacobi equation to represent how to obtain T_H through the KS form of the SBHGBGM. To this end, let us rewrite the metric (2) in the following form:

$$ds^2 = -f\left(dt^2 - \frac{\Xi}{f^2}dr^2\right) + r^2 d\Omega^2, \tag{34}$$

which can be transformed to

$$ds^2 = -f \ \mathrm{d}u \ \mathrm{d}v + r^2 \ \mathrm{d}\Omega^2, \tag{35}$$

by the following coordinate transformations:

$$\mathrm{d}u = \mathrm{d}t - \mathrm{d}r^*, \qquad \mathrm{d}v = \mathrm{d}t + \mathrm{d}r^*. \tag{36}$$

Recall that the definition of tortoise coordinate was given r^* in Eq. (24), which can be written explicitly as follows:

$$r^* = \sqrt{\Xi}\left(r + rh \ \ln\left(\frac{r}{rh} - 1\right)\right). \tag{37}$$

After establishing new coordinates (U, V) that are determined by the surface gravity (5):

$$U = -\mathrm{e}^{-\kappa u}, \quad V = \mathrm{e}^{\kappa v}, \tag{38}$$

one can redefine metric (34) in the KS coordinate system:

$$ds^2 = -\mathcal{L} \ \mathrm{d}U \ \mathrm{d}V + r^2 \ \mathrm{d}\Omega^2, \tag{39}$$

in which

$$\mathcal{L} = -\frac{f}{\kappa^2 UV} = \frac{4\Xi r_h^3 \mathbf{e}^{\left(-\frac{r}{rh}\right)}}{r}. \tag{40}$$

With the exception of the physical singularity $r = 0$, this metric is regular everywhere. Alternatively, metric (39) can be changed into

$$ds^2 = -\mathcal{L}\left(\,\mathrm{d}\mathcal{T}^2 - \mathrm{d}\mathcal{R}^2\right) + r^2\,\mathrm{d}\Omega^2, \tag{41}$$

which can be made by the following transformations

$$\mathcal{T} = \frac{1}{2}(V+U) = \mathbf{e}^{\sqrt{\Xi}r}\left(\frac{r}{r_h}-1\right)^{\sqrt{\Xi}r_h}\sinh\left(\frac{t}{2r_h\sqrt{\Xi}}\right), \tag{42}$$

$$\mathcal{R} = \frac{1}{2}(V-U) = \mathbf{e}^{\sqrt{\Xi}r}\left(\frac{r}{r_h}-1\right)^{\sqrt{\Xi}r_h}\cosh\left(\frac{t}{2r_h\sqrt{\Xi}}\right). \tag{43}$$

One can observe directly from above that

$$\mathcal{R}^2 - \mathcal{T}^2 = \mathbf{e}^{2\sqrt{\Xi}r}\left(\frac{r}{r_h}-1\right)^{2\sqrt{\Xi}r_h} \tag{44}$$

which indicates that the future and past horizons are represented by $\mathcal{R} = \pm\mathcal{T}$. In this case, on the other hand, $\partial_{\mathcal{T}}$ is not a timelike Killing vector for the metric (39). Therefore, it is advantageous to take into account the metric's timelike Killing vector in the following form:

$$\partial_T = \mathcal{N}\left(\mathcal{R}\partial_{\mathcal{T}} + \mathcal{T}\partial_{\mathcal{R}}\right), \tag{45}$$

where $\mathcal{N}$ denotes the normalization constant. The determination of the normalization constant $\mathcal{N}$ is crucial in calculating the norm of the Killing vector, which attains a negative unity value at either spatial infinity or the position of the observer measuring the temperature of the SBHBGMM. Therefore, at spatial infinity, the normalization constant is found to be

$$\mathcal{N} = \left.\frac{1}{2r_h\sqrt{\Xi}f}\right|_{r=\infty} = \kappa. \tag{46}$$

Without loss of generality, one might consider the (1+1)-dimensional form of the KS metric (41) which is

$$ds^2 = -\mathcal{L}\left(\,\mathrm{d}\mathcal{T}^2 - \mathrm{d}\mathcal{R}^2\right). \tag{47}$$

In this situation, the Hamilton-Jacobi method's calculations become simpler. The above metric's Hamilton-Jacobi equation (6) is as follows:

$$-\mathcal{L}^{-1}\left[(\partial_{\mathcal{T}}P)^2 - (\partial_{\mathcal{R}}P)^2\right] + m^2 = 0. \tag{48}$$

According to this equation, the ansatz for the P could be expressed as

$$P = \rho(y) + J\left(x^i\right), \tag{49}$$

where $y = \mathcal{R} - \mathcal{T}$ and $\rho(y)$ is a function to be determined. To make things even simpler, we can set $J(x_i) = 0$ and $m = 0$. The energy is now described as

$$E = -\partial_T P = -\kappa \left(\mathcal{R} \partial_{\mathcal{T}} P + \mathcal{T} \partial_{\mathcal{R}} P \right). \tag{50}$$

Using the above equation with ansatz (49), one derives the following expression

$$\rho(y) = \int \frac{2 E r_h \sqrt{\Xi}}{y} \, \mathrm{d}y. \tag{51}$$

The above expression has a divergence at the horizon $y_h = 0$, namely $\mathcal{R} = \mathcal{T}$. Thus, it leads to a pole at the horizon which could be overcome by doing a semi-circular contour of integration in the complex plane. The result is found to be

$$\mathrm{Im}\, \rho(y_h) = 2\pi E r_h \sqrt{\Xi} = \frac{\pi E}{\kappa}. \tag{52}$$

which leads to the following tunneling probability of the emitted quanta from the SBHBGM defined in the KS coordinates (39):

$$\mathcal{P}_{KS} = \exp\left(-2\, \mathrm{Im}\, \rho(y_h)\right) = \exp\left(\frac{-E}{T_{KS}}\right), \tag{53}$$

which results in

$$T_{KS} = \frac{1}{4\pi r_h \sqrt{\Xi}}. \tag{54}$$

Equation (54) is nothing but the Hawking temperature seen in Eq. (4). Namely, we have impeccably recovered the T_H in the background of the KS metric of the SBHBGM.

IV. HAWKING RADIATION OF SBHBGM VIA HAMILTON-JACOBI METHOD: SEMI-CLASSICAL APPROACH (WITH GUP)

In this section, our focus revolves around the intricate interplay between the GUP and the Hawking radiation of the SBHBGM, employing the highly insightful Hamilton-Jacobi method. Quantum gravity theories, spanning a spectrum of perspectives, consistently postulate the intriguing concept of a minimal length, which finds its roots in the very fabric of the quantum realm. Within this theoretical framework, the realization of this minimal length manifests through various avenues. Among these, a particularly noteworthy route involves the application of the GUP. To embark on our exploration, we commence by delving into the modified Hamilton-Jacobi equation [64] tailored for scalar particles, as elegantly articulated below:

$$g^{0j}\left(\partial_0 S\right)\left(\partial_j S\right)+\left[g^{kk}\left(\partial_k S\right)^2+m^2\right]\times\left\{1-2\beta\left[g^{jj}\left(\partial_j S\right)^2+m^2\right]\right\}=0, \tag{55}$$

where $k,j=1,2,3$, β is the GUP parameter, and Eq. (55) yields the following expression with the use of metric (2) and ansatzes (7) and (9):

$$-\frac{2f^2\beta(d_rW)^4}{\Xi^2}-\frac{f(4\beta L^2+4\beta m^2-1)(d_rW)^2}{\Xi}-2(L^2+m^2)^2\beta+m^2+\frac{-\omega^2+fL^2}{f}=0. \tag{56}$$

In Eq. (56), only the leading orders of β are considered and the higher order terms of β are neglected since they have negligibly low values. In the sequel, if one solves that equation for W_0 and W_1, the following expressions are obtained around the horizon:

$$W_0=\pm 2i\pi\omega M\sqrt{\Xi}, \tag{57}$$

$$W_1=\pm 4i\pi\omega^3 M\sqrt{\Xi}, \tag{58}$$

which belong to

$$W^{\pm}_{GUP}=W_0+\beta W_1=\pm 2i\pi\omega M\sqrt{\Xi}\left(1+2\beta\omega^2\right). \tag{59}$$

We calculate the tunneling probability as;

$$\mathcal{P}_{\mathcal{GUP}}=\frac{exp\left(-2ImW^{+}_{GUP}\right)}{exp\left(-2ImW^{-}_{GUP}\right)}=\frac{exp\left(-4\pi\omega M\sqrt{\Xi}(1+2\beta\omega^2)\right)}{exp\left(4\pi\omega M\sqrt{\Xi}(1+2\beta\omega^2)\right)}=exp\left(\frac{-\omega}{T_{GUP}}\right), \tag{60}$$

whence the GUP-modified Hawking temperature of the SBHBGM can be found to be

$$T_{GUP}=\frac{1}{8M\pi\sqrt{\Xi}(1+2\beta\omega^2)}. \tag{61}$$

The term $\frac{1}{1+2\beta\omega^2}$ in the expression implies that as the frequency and/or β increase, the modification due to GUP becomes more prominent, leading to a reduction in the modified Hawking temperature compared to the conventional value of T_H presented in Eq. (4). This alteration can be seen as an intricate consequence of the underlying quantum gravitational effects, where the minimal length scale encoded by β influences the radiation process. Below is the correlation between the conventional Hawking temperature and the modified version:

$$T_{GUP}=\frac{T_H}{1+2\beta\omega^2}\approx T_H(1-2\beta\omega^2)+\mathcal{O}(\beta^2). \tag{62}$$

The approximated form of the expression, $T_H(1-2\beta\omega^2)+\mathcal{O}(\beta^2)$, further simplifies the relationship, highlighting the primary influence of the GUP-induced modification. The first term $T_H(1-2\beta\omega^2)$ denotes the dominant effect of GUP on the modified temperature, showcasing how the presence of the minimal length scale affects the energy emission. The additional term $\mathcal{O}(\beta^2)$ accounts for higher-order corrections stemming from the GUP, contributing to the refinement of the temperature modification.

V. QC ENTROPY OF SBHBGM

A fundamental aspect of black hole thermodynamics lies in the concept of entropy, which plays a central role in connecting the macroscopic behavior of black holes with the underlying microscopic degrees of freedom. The QC-induced modifications give rise to a revision of the black hole entropy expression, incorporating corrections that diverge from the traditional Bekenstein-Hawking formula [26]. The underlying physics driving these corrections is rooted in the profound changes to the density of states for quantum states near the Planck scale, thereby influencing the counting of microscopic configurations responsible for the black hole entropy. Consequently, the QC-modified entropy not only offers a tantalizing link between quantum gravity phenomena and black hole thermodynamics but also holds the potential to address long-standing issues such as the black hole information paradox [75–77].

In this section, we delve into the QC entropy within the framework of the SBHBGM. We begin by outlining the fundamental principles of entropy and the first law of thermodynamics. Building upon this foundation, we proceed to derive the QC-modified expression for the black hole entropy. Through a careful analysis of the QC-modified entropy formula, we aim to illuminate the role of quantum gravity effects in reshaping the thermodynamic properties of the SBHBGM and to uncover potential avenues for testing these modifications through astrophysical observations and experimental scenarios.

The Bekenstein-Hawking entropy formula, a cornerstone of modern theoretical physics, provides a simple yet powerful expression for the entropy of a black hole, denoted as S_{BH}:

$$S_{BH}=\frac{A_H}{4}, \tag{63}$$

which recasts in the following expression for the SBHBGM:

$$S_{BH}=\pi r_H^2=4\pi M^2. \tag{64}$$

Let us now take a closer look at the first law of thermodynamics:

$$dE = T_H dS_{BH}, \tag{65}$$

which yields

$$dS_{BH} = 8\pi M dM. \tag{66}$$

Recalling the Hawking temperature (4), Eq. (65) becomes

$$dE = \frac{1}{8\pi M\sqrt{\Xi}} 8\pi M dM = \frac{dM}{\sqrt{\Xi}}. \tag{67}$$

By integrating Eq. (67), we get

$$E = \frac{M}{\sqrt{\Xi}}, \tag{68}$$

which demonstrates that the total thermal energy of the black hole, E, is proportional to the inverse square root of the quantity Ξ, which is influenced by the bumblebee parameter ℓ. This implies that deviations from Lorentz invariance, introduced by the parameter Ξ, have a direct impact on the scaling of the thermal energy with respect to the black hole's mass. After combining Eqs. (64) and (68), one obtains

$$S_{BH} = 4\pi\Xi E^2. \tag{69}$$

Equation (69) highlights the LIV-modified Bekenstein-Hawking entropy S_{BH} that accounts for the influence of the LIV parameter ℓ. The existence of Ξ parameter in Eq. (69) signifies a departure from the traditional entropy formula ($S_{BH} = 4\pi E^2$) due to the LIV parameter, resulting in a deviation from the standard black hole thermodynamics dictated by general relativity. In summary, the provided expressions elucidate the intricate interplay between black hole thermodynamics, Lorentz invariance violation through the parameter ℓ, and the resulting modifications to the black hole's thermal energy and entropy. These modifications introduce a departure from conventional expectations, underscoring the potential influence of new physics, such as deviations from Lorentz invariance, on the behavior of black holes and their thermodynamic attributes.

On the other hand, In the framework of string theory and loop quantum gravity, the concept of quantum corrected entropy S_{QG} emerges as a pivotal element. This quantum correction is described by the following expression [67, 68]:

$$S_{QG} = S_{BH} + \alpha \ln(4S_{BH}), \tag{70}$$

where S_{BH} is the Bekenstein-Hawking entropy and α signifies a parameter linked to quantum corrections. This expression captures the interplay between the intrinsic entropy of the black hole and quantum modifications arising from these theories. The shift in quantum-corrected entropy is then given by:

$$\begin{aligned}\Delta S_{QG} &= S_{QG}(E-\omega) - S_{QG}(E) \\ &= 4\pi\Xi(E-\omega)^2 + \alpha\ln\left(16\pi\Xi(E-\omega)^2\right) - 4\pi E^2\Xi - \alpha\ln(16\pi\Xi E^2). \quad (71)\end{aligned}$$

This equation portrays the change in quantum-corrected entropy due to variations in energy E and angular frequency ω, incorporating both the bumblebee factor Ξ and the quantum correction parameter α. Expanding Eq. (71) with a Taylor series with respect to ω, by keeping the leading order, yields:

$$\Delta S_{QG} \approx -\left(8\pi\Xi E + \frac{2\alpha}{E}\right)\omega = -\left(\frac{1}{T_H} + \frac{2\alpha}{E}\right)\omega. \quad (72)$$

This equation unveils a connection between the modified factors due to quantum corrections and the Hawking temperature T_H, highlighting the intricate interplay between thermodynamics and quantum effects.

The quantum-corrected tunneling rate, denoted as Γ_{QG}, is described by:

$$\Gamma_{QG} \sim \Delta S_{QG} = -\frac{\omega}{T_{QG}}, \quad (73)$$

whence the modified temperature T_{QG}, at the leading order of α, reads

$$T_{QG} \approx .T_H\left(1 - \frac{2\alpha T_H}{E}\right). \quad (74)$$

Remarkably, as the quantum correction parameter α approaches zero, T_{QG} converges towards the standard Hawking temperature T_H representing the classical regime. The expression (74) elucidates the dependence of the modified temperature T_{QG} on the bumblebee parameter ℓ and the quantum correction parameter α, highlighting their joint influence on the temperature deviation from the classical value T_H.

In summary, the presented equations intricately interweave the quantum-corrected entropy, modified tunneling rates, and temperature deviations, offering a glimpse into the profound relationship between black hole thermodynamics, quantum gravity theories, and the underlying microscopic quantum effects that potentially reshape our understanding of these enigmatic cosmic objects.

VI. CONCLUSION

In conclusion, the features of the SBHBGM spacetime were explored within the context of bumblebee gravity theory. The extended Einstein field equations, incorporating the effects of the bumblebee field, were introduced. The modified black hole solution (SBHBGM) is characterized by a radial metric function $g_{rr} = \frac{\Xi}{1-\frac{2M}{r}}$, showcased the influence of the Lorentz symmetry-breaking or the bumblebee parameter ℓ on the spacetime curvature. The Kretschmann scalar(3) demonstrated the distinct nature of the metric compared to the conventional Schwarzschild black hole.

By employing the Hamilton-Jacobi method, Hawking radiation for the SBHBGM was investigated. Different regular coordinate systems — PG, IEF, and KS coordinates — were considered to analyze the radiation process. In each coordinate system, the tunneling probability and temperature of the black hole were computed. Remarkably, the Lorentz symmetry-breaking parameter ℓ influenced the Hawking temperature, with non-zero ℓ leading to a reduction in temperature compared to the Schwarzschild case. The effects of Lorentz invariance violation were emphasized, introducing intriguing deviations from the standard Hawking temperature. Thus, we have also demonstrated that Hawking radiation possesses an independent invariant physical property regardless of the coordinate system.

Furthermore, the impact of the GUP was integrated into the analysis. The GUP-modified Hamilton-Jacobi equation was employed to derive tunneling probabilities and temperatures, unveiling the intricate interaction between the GUP, Lorentz symmetry violation, and Hawking radiation. The GUP-induced modifications were introduced with a new parameter β, which played a role in reducing the modified Hawking temperature. The relationship between the conventional Hawking temperature and the GUP-modified temperature was elucidated, offering insights into how quantum gravitational effects influence the radiation process.

Finally, the concept of QC entropy was introduced within the SBHBGM framework. The interplay between black hole thermodynamics and quantum gravity effects, incorporating both the Lorentz symmetry-breaking parameter ℓ and a quantum correction parameter α, was explored. The expressions for QC entropy and modified temperature underscored the intricate connection between quantum corrections, thermodynamic properties, and the underlying microscopic quantum behavior.

In conclusion, the investigation of the SBHBGM spacetime within the framework of bumblebee gravity, along with the incorporation of quantum gravity effects, provided a comprehensive understanding of the modifications introduced by Lorentz invariance violation and the GUP. These modifications have the potential to manifest as observable deviations from classical black hole thermodynamics, offering a unique opportunity to probe the effects of quantum gravity at astrophysical scales. In the future, this research can be extended to investigate the behavior of rotating black holes within the bumblebee gravity theory [73, 78]. The study of rotating black holes introduces new complexities due to frame-dragging effects and angular momentum considerations. The analysis could involve exploring the modifications to the metric, coordinate systems, and thermodynamic properties induced by the presence of rotation. Moreover, the exploration of quantum-corrected entropy and tunneling rates in the context of rotating black holes could provide valuable insights into the impact of quantum gravity effects on these objects. All of these are within the scope of our near-future work agenda.

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